where

$$d_1 = \frac{\ln(S_0/K_2) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The price in this formula is greater than the price given by the Black–Scholes–Merton formula for a regular call option with strike price K_2 by

$$(K_2 - K_1)e^{-rT}N(d_2)$$

To understand this difference, note that the probability that the option will be exercised is $N(d_2)$ and, when it is exercised, the payoff to the holder of the gap option is greater than that to the holder of the regular option by $K_2 - K_1$.

For a gap put option, the payoff is $K_1 - S_T$ when $S_T < K_2$. The value of the option is

$$K_1e^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) \tag{25.2}$$

where d_1 and d_2 are defined as for equation (25.1).

Example 25.1

An asset is currently worth $500,000. Over the next year, it is expected to have a volatility of 20%. The risk-free rate is 5%, and no income is expected. Suppose that an insurance company agrees to buy the asset for $400,000 if its value has fallen below $400,000 at the end of one year. The payout will be $400{,}000 - S_T$ whenever the value of the asset is less than $400,000. The insurance company has provided a regular put option where the policyholder has the right to sell the asset to the insurance company for $400,000 in one year. This can be valued using equation (14.21), with $S_0 = 500{,}000$, $K = 400{,}000$, $r = 0.05$, $\sigma = 0.2$, $T = 1$. The value is $3,436.

Suppose next that the cost of transferring the asset is $50,000 and this cost is borne by the policyholder. The option is then exercised only if the value of the asset is less than $350,000. In this case, the cost to the insurance company is $K_1 - S_T$ when $S_T < K_2$, where $K_2 = 350{,}000$, $K_1 = 400{,}000$, and S_T is the price of the asset in one year. This is a gap put option. The value is given by equation (25.2), with $S_0 = 500{,}000$, $K_1 = 400{,}000$, $K_2 = 350{,}000$, $r = 0.05$, $q = 0$, $\sigma = 0.2$, $T = 1$. It is $1,896. Recognizing the costs to the policyholder of making a claim reduces the cost of the policy to the insurance company by about 45% in this case.

25.4 FORWARD START OPTIONS

Forward start options are options that will start at some time in the future. Sometimes employee stock options, which were discussed in Chapter 15, can be viewed as forward start options. This is because the company commits (implicitly or explicitly) to granting at-the-money options to employees in the future.

Consider a forward start at-the-money European call option that will start at time T_1 and mature at time T_2. Suppose that the asset price is S_0 at time zero and S_1 at time T_1.

To value the option, we note from the European option pricing formulas in Chapters 14 and 16 that the value of an at-the-money call option on an asset is proportional to the asset price. The value of the forward start option at time T_1 is therefore cS_1/S_0, where c is the value at time zero of an at-the-money option that lasts for $T_2 - T_1$. Using risk-neutral valuation, the value of the forward start option at time zero is

$$e^{-rT_1}\hat{E}\left[c\frac{S_1}{S_0}\right]$$

where $\hat{E}$ denotes the expected value in a risk-neutral world. Since c and S_0 are known and $\hat{E}[S_1] = S_0e^{(r-q)T_1}$, the value of the forward start option is ce^{-qT_1}. For a non-dividend-paying stock, $q = 0$ and the value of the forward start option is exactly the same as the value of a regular at-the-money option with the same life as the forward start option.

25.5 CLIQUET OPTIONS

A cliquet option (which is also called a ratchet or strike reset option) is a series of call or put options with rules for determining the strike price. Suppose that the reset dates are at times $\tau, 2\tau, \ldots, (n-1)\tau$, with $n\tau$ being the end of the cliquet's life. A simple structure would be as follows. The first option has a strike price K (which might equal the initial asset price) and lasts between times 0 and τ; the second option provides a payoff at time 2τ with a strike price equal to the value of the asset at time τ; the third option provides a payoff at time 3τ with a strike price equal to the value of the asset at time 2τ; and so on. This is a regular option plus $n - 1$ forward start options. The latter can be valued as described in Section 25.2.

Some cliquet options are much more complicated than the one described here. For example, sometimes there are upper and lower limits on the total payoff over the whole period; sometimes cliquets terminate at the end of a period if the asset price is in a certain range. When analytic results are not available, Monte Carlo simulation can often be used for valuation.

25.6 COMPOUND OPTIONS

Compound options are options on options. There are four main types of compound options: a call on a call, a put on a call, a call on a put, and a put on a put. Compound options have two strike prices and two exercise dates. Consider, for example, a call on a call. On the first exercise date, T_1, the holder of the compound option is entitled to pay the first strike price, K_1, and receive a call option. The call option gives the holder the right to buy the underlying asset for the second strike price, K_2, on the second exercise date, T_2. The compound option will be exercised on the first exercise date only if the value of the option on that date is greater than the first strike price.

When the usual geometric Brownian motion assumption is made, European-style compound options can be valued analytically in terms of integrals of the bivariate normal distribution.[2] With our usual notation, the value at time zero of a European call

[2] See R. Geske, "The Valuation of Compound Options," *Journal of Financial Economics*, 7 (1979): 63–81; M. Rubinstein, "Double Trouble," *Risk*, December 1991/January 1992: 53–56.

APPENDIX

DETERMINING IMPLIED RISK-NEUTRAL DISTRIBUTIONS FROM VOLATILITY SMILES

The price of a European call option on an asset with strike price K and maturity T is given by

$$c = e^{-rT} \int_{S_T=K}^{\infty} (S_T - K)\, g(S_T)\, dS_T$$

where r is the interest rate (assumed constant), S_T is the asset price at time T, and g is the risk-neutral probability density function of S_T. Differentiating once with respect to K gives

$$\frac{\partial c}{\partial K} = -e^{-rT} \int_{S_T=K}^{\infty} g(S_T)\, dS_T$$

Differentiating again with respect to K gives

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K)$$

This shows that the probability density function g is given by

$$g(K) = e^{rT} \frac{\partial^2 c}{\partial K^2} \tag{19A.1}$$

This result, which is from Breeden and Litzenberger (1978), allows risk-neutral probability distributions to be estimated from volatility smiles.[9] Suppose that c_1, c_2, and c_3 are the prices of T-year European call options with strike prices of $K - \delta$, K, and $K + \delta$, respectively. Assuming δ is small, an estimate of $g(K)$, obtained by approximating the partial derivative in equation (19A.1), is

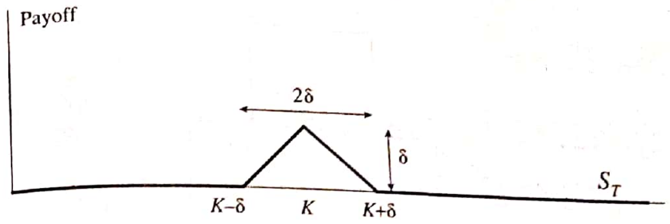
$$e^{rT} \frac{c_1 + c_3 - 2c_2}{\delta^2}$$

For another way of understanding this formula, suppose you set up a butterfly spread with strike prices $K - \delta$, K, and $K + \delta$, and maturity T. This means that you buy a call with strike price $K - \delta$, buy a call with strike price $K + \delta$, and sell two calls with strike price K. The value of your position is $c_1 + c_3 - 2c_2$. The value of the position can also be calculated by integrating the payoff over the risk-neutral probability distribution, $g(S_T)$, and discounting at the risk-free rate. The payoff is shown in Figure 19A.1. Since δ is small, we can assume that $g(S_T) = g(K)$ in the whole of the range $K - \delta < S_T < K + \delta$, where the payoff is nonzero. The area under the "spike" in Figure 19A.1 is $0.5 \times 2\delta \times \delta = \delta^2$. The value of the payoff (when δ is small) is therefore $e^{-rT} g(K)\delta^2$. It follows that

$$e^{-rT} g(K)\delta^2 = c_1 + c_3 - 2c_2$$

[9] See D. T. Breeden and R. H. Litzenberger, "Prices of State-Contingent Claims Implicit in Option Prices," *Journal of Business*, 51 (1978), 621–51.

Figure 19A.1 Payoff from butterfly spread.



which leads directly to

$$g(K) = e^{rT} \frac{c_1 + c_3 - 2c_2}{\delta^2} \tag{19A.2}$$

Example 19A.1

Suppose that the price of a non-dividend-paying stock is \$10, the risk-free interest rate is 3%, and the implied volatilities of 3-month European options with strike prices of \$6, \$7, \$8, \$9, \$10, \$11, \$12, \$13, \$14 are 30%, 29%, 28%, 27%, 26%, 25%, 24%, 23%, 22%, respectively. One way of applying the above results is as follows. Assume that $g(S_T)$ is constant between $S_T = 6$ and $S_T = 7$, constant between $S_T = 7$ and $S_T = 8$, and so on. Define:

$$\begin{aligned}
g(S_T) &= g_1 \quad \text{for} \quad 6 \leqslant S_T < 7 \\
g(S_T) &= g_2 \quad \text{for} \quad 7 \leqslant S_T < 8 \\
g(S_T) &= g_3 \quad \text{for} \quad 8 \leqslant S_T < 9 \\
g(S_T) &= g_4 \quad \text{for} \quad 9 \leqslant S_T < 10 \\
g(S_T) &= g_5 \quad \text{for} \quad 10 \leqslant S_T < 11 \\
g(S_T) &= g_6 \quad \text{for} \quad 11 \leqslant S_T < 12 \\
g(S_T) &= g_7 \quad \text{for} \quad 12 \leqslant S_T < 13 \\
g(S_T) &= g_8 \quad \text{for} \quad 13 \leqslant S_T < 14
\end{aligned}$$

The value of g_1 can be calculated by interpolating to get the implied volatility for a 3-month option with a strike price of \$6.5 as 29.5%. This means that options with strike prices of \$6, \$6.5, and \$7 have implied volatilities of 30%, 29.5%, and 29%, respectively. From DerivaGem their prices are \$4.045, \$3.549, and \$3.055, respectively. Using equation (19A.2), with $K = 6.5$ and $\delta = 0.5$, gives

$$g_1 = \frac{e^{0.03\times0.25}(4.045 + 3.055 - 2 \times 3.549)}{0.5^2} = 0.0057$$

Similar calculations show that

$$g_2 = 0.0444, \quad g_3 = 0.1545, \quad g_4 = 0.2781$$

$$g_5 = 0.2813, \quad g_6 = 0.1659, \quad g_7 = 0.0573, \quad g_8 = 0.0113$$

Some exotic options are easier to hedge than the corresponding regular options; others are more difficult. In general, Asian options are easier to hedge because the payoff becomes progressively more certain as we approach maturity. Barrier options can be more difficult to hedge because delta is discontinuous at the barrier. One approach to hedging an exotic option, known as static options replication, is to find a portfolio of regular options whose value matches the value of the exotic option on some boundary. The exotic option is hedged by shorting this portfolio.

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Practice Questions (Answers in Solutions Manual)

25.1. Explain the difference between a forward start option and a chooser option.

25.2. Describe the payoff from a portfolio consisting of a floating lookback call and a floating lookback put with the same maturity.

25.3. Consider a chooser option where the holder has the right to choose between a European call and a European put at any time during a 2-year period. The maturity dates and strike prices for the calls and puts are the same regardless of when the choice is made. Is it ever optimal to make the choice before the end of the 2-year period? Explain your answer.

25.4. Suppose that c_1 and p_1 are the prices of a European average price call and a European average price put with strike price K and maturity T, c_2 and p_2 are the prices of a European average strike call and European average strike put with maturity T, and c_3 and p_3 are the prices of a regular European call and a regular European put with strike price K and maturity T. Show that $c_1 + c_2 - c_3 = p_1 + p_2 - p_3$.

25.5. The text derives a decomposition of a particular type of chooser option into a call maturing at time T_2 and a put maturing at time T_1. Derive an alternative decomposition into a call maturing at time T_1 and a put maturing at time T_2.

25.6. Section 25.8 gives two formulas for a down-and-out call. The first applies to the situation where the barrier, H, is less than or equal to the strike price, K. The second applies to the situation where $H \geqslant K$. Show that the two formulas are the same when $H = K$.

25.7. Explain why a down-and-out put is worth zero when the barrier is greater than the strike price.

25.8. Suppose that the strike price of an American call option on a non-dividend-paying stock grows at rate g. Show that if g is less than the risk-free rate, r, it is never optimal to exercise the call early.

25.9. How can the value of a forward start put option on a non-dividend-paying stock be calculated if it is agreed that the strike price will be 10% greater than the stock price at the time the option starts?

25.10. If a stock price follows geometric Brownian motion, what process does $A(t)$ follow where $A(t)$ is the arithmetic average stock price between time zero and time t?

25.11. Explain why delta hedging is easier for Asian options than for regular options.

25.12. Calculate the price of a 1-year European option to give up 100 ounces of silver in exchange for 1 ounce of gold. The current prices of gold and silver are $380 and $4, respectively; the risk-free interest rate is 10% per annum; the volatility of each commodity price is 20%; and the correlation between the two prices is 0.7. Ignore storage costs.

25.13. Is a European down-and-out option on an asset worth the same as a European down-and-out option on the asset's futures price for a futures contract maturing at the same time as the option?

25.14. Answer the following questions about compound options:
(a) What put–call parity relationship exists between the price of a European call on a call and a European put on a call? Show that the formulas given in the text satisfy the relationship.
(b) What put–call parity relationship exists between the price of a European call on a put and a European put on a put? Show that the formulas given in the text satisfy the relationship.

25.15. Does a floating lookback call become more valuable or less valuable as we increase the frequency with which we observe the asset price in calculating the minimum?