M6140 Topology Exercises - 8th Week (2020)

1 Mapping Spaces

Definition 1. By $\mathcal{C}(X, Y)$ we denote the set of all continuous maps from a topological space X to a topological space Y.

Exercise 1. Suppose that X is a non-empty topological space and Y is a topological space. Show that $\mathcal{C}(X, Y)$ is a singleton iff Y is a singleton.

Exercise 2. Suppose that X is a non-empty topological space and Y is a topological space. Prove that the cardinality of Y is less or equal to the cardinality of $\mathcal{C}(X, Y)$.

Definition 2. Let $A \subseteq X$, $B \subseteq Y$ be subsets in topological spaces X, Y. Define

$$W(A, B) := \{f \in \mathcal{C}(X, Y) \mid f(A) \subseteq B\}.$$

Exercise 3. Suppose that X is a non-empty topological space and Y is a topological space. Prove that

$$\mathscr{S}^{\mathrm{pw}} := \{W(a, U) \mid a \in X, U \text{ is an open set in } Y\}$$

is a cover of $\mathcal{C}(X, Y)$.

Definition 3. The set $\mathscr{S}^{\mathrm{pw}}$ can be viewed as a subbase, the generated topology is called the *topology of pointwise convergence*, and the resulting topological space is denoted by $\mathcal{C}^{\mathrm{pw}}(X, Y)$.

Exercise 4. Suppose that X is a non-empty topological space and Y is a topological space. Prove that

$$\mathscr{S}^{\mathrm{co}} := \{W(K, U) \mid K \text{ is compact in } X, U \text{ is an open set in } Y\}$$

is a cover of $\mathcal{C}(X, Y)$.

Definition 4. The set $\mathscr{S}^{\mathrm{co}}$ can be viewed as a subbase, the generated topology is called the *compact-open topology*, and the resulting topological space is denoted simply by $\mathcal{C}(X, Y)$.

Exercise 5. Prove that $\mathcal{T}(\mathcal{C}^{\mathrm{pw}}(X, Y)) \subseteq \mathcal{T}(\mathcal{C}(X, Y))$ for each pair of topological spaces X, Y, where X is non-empty.

Exercise 6. Show that $\mathcal{T}(\mathcal{C}^{\mathrm{pw}}(I, I)) \neq \mathcal{T}(\mathcal{C}(I, I))$.

Exercise 7. Suppose that X is a non-empty topological space and Y is a T_2 topological space. Prove that $\mathcal{C}^{\mathrm{pw}}(X, Y)$ and $\mathcal{C}(X, Y)$ are both T_2.

Exercise 8. Suppose that X, Y, X', Y' are topological spaces and $g\colon X' \to X$, $h\colon Y \to Y'$ are continuous maps. Show that $\varphi\colon \mathcal{C}(X, Y) \to \mathcal{C}(X', Y')$ given by $\varphi(f) := h \circ f \circ g$ is continuous.

Exercise 9. Prove that $\mathcal{C}^{\mathrm{pw}}(I, I)$, $\mathcal{C}(I, I)$ aren't homeomorphic.

Exercise 10. Suppose that X is a T_2 space, Y is a T_2 locally compact space, and Z is a topological space. Show that there exists a homeomorphism $\mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, \mathcal{C}(Y, Z))$.