Tutorial 6—Global Analysis

1. Suppose $M = \mathbb{R}^2$ with coordinates (x, y). Consider the vector fields $\xi(x, y) = y\frac{\partial}{\partial x}$ and $\eta(x, y) = \frac{x^2}{2}\frac{\partial}{\partial y}$ on M. We computed in class their flows and saw that they are complete. Compute $[\xi, \eta]$ and its flow? Is $[\xi, \eta]$ complete?

2. Let M be a (smooth) manifold and $\xi, \eta \in \mathfrak{X}(M)$ two vector fields on M. Show that
   (a) $[\xi, \eta] = 0 \iff (\mathrm{Fl}^\xi_t)^*\eta = \eta$, whenever defined $\iff \mathrm{Fl}^\xi_t \circ \mathrm{Fl}^\eta_s = \mathrm{Fl}^\eta_s \circ \mathrm{Fl}^\xi_t$, whenever defined.
   (b) If N is another manifold, $f : M \to N$ a smooth map, and ξ and η are f-related to vector fields $\tilde{\xi}$ resp. $\tilde{\eta}$ on N, then $[\xi, \eta]$ is f-related to $[\tilde{\xi}, \tilde{\eta}]$.

3. Suppose α^i_j for $i = 1, ..., k$ and $j = 1, ..., n$ are smooth real-valued functions defined on some open set $U \subset \mathbb{R}^{n+k}$ satisfying
$$\frac{\partial \alpha^i_j}{\partial x^k} + \alpha^\ell_k \frac{\partial \alpha^i_j}{\partial z^\ell} = \frac{\partial \alpha^i_k}{\partial x^j} + \alpha^\ell_j \frac{\partial \alpha^i_k}{\partial z^\ell},$$
   where we write $(x, z) = (x^1, ..., x^n, z^1, ..., z^k)$ for a point in $\mathbb{R}^{n+k}$. Show that for any point $(x_0, z_0) \in U$ there exists an open neighbourhood V of x_0 in $\mathbb{R}^n$ and a unique C^∞-map $f : V \to \mathbb{R}^k$ such that
$$\frac{\partial f^i}{\partial x^j}(x^1, ..., x^n) = \alpha^i_j(x^1, ..., x^n, f^1(x), ..., f^k(x)) \quad \text{and} \quad f(x_0) = z_0.$$
   In the class/tutorial we proved this for $k = 1$ and $j = 2$.

4. Which of the following systems of PDEs have solutions $f(x, y)$ (resp. $f(x, y)$ and $g(x, y)$) in an open neighbourhood of the origin for positive values of $f(0, 0)$ (resp. $f(0, 0)$ and $g(0, 0)$)?
   (a) $\frac{\partial f}{\partial x} = f \cos y$ and $\frac{\partial f}{\partial y} = -f \log f \tan y$.
   (b) $\frac{\partial f}{\partial x} = e^{xf}$ and $\frac{\partial f}{\partial y} = xe^{yf}$.
   (c) $\frac{\partial f}{\partial x} = f$ and $\frac{\partial f}{\partial y} = g$; $\frac{\partial g}{\partial x} = g$ and $\frac{\partial g}{\partial y} = f$.

5. Suppose $E \to M$ is a (smooth) vector bundle of rank k over a manifold M. Then E is called *trivializable*, if it isomorphic to the trivial vector bundle $M \times \mathbb{R}^k \to M$.

(a) Show that $E \to M$ is trivializable $\iff$ $E \to M$ admits a global frame, i.e. there exist (smooth) sections $s_1, ..., s_k$ of E such that $s_1(x)$, ..., $s_k(x)$ span E_x for any $x \in M$.

(b) Show that the tangent bundle of any Lie group G is trivializable.

(c) Recall that $\mathbb{R}^n$ has the structure of a (not necessarily associative) division algebra over $\mathbb{R}$ for $n = 1, 2, 4, 8$. Use this to show that the tangent bundle of the spheres $S^1 \subset \mathbb{R}^2$, $S^3 \subset \mathbb{R}^4$ and $S^7 \subset \mathbb{R}^8$ is trivializable.