Tutorial 9—Global Analysis

1. Suppose M is a manifold and $D_i : \Omega^k(M) \to \Omega^{k+r_i}(M)$ for $i = 1, 2$ a graded derivation of degree r_i of $(\Omega(M), \wedge)$.

   (a) Show that
   $$[D_1, D_2] := D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$$
   is a graded derivation of degree $r_1 + r_2$.

   (b) Suppose D is a graded derivation of $(\Omega(M), \wedge)$. Let $\omega \in \Omega^k(M)$ be a differential form and $U \subset M$ an open subset. Show that $\omega|_U = 0$ implies $D(\omega)|_U = 0$.

   Hint: Think about writing 0 as $f\omega$ for some smooth function f and use the defining properties of a graded derivation.

   (c) Suppose D and $\tilde{D}$ are two graded derivations such that $D(f) = \tilde{D}(f)$ and $D(df) = \tilde{D}(df)$ for all $f \in C^\infty(M, \mathbb{R})$. Show that $D = \tilde{D}$.

2. Suppose M is a manifold and $\xi, \eta \in \Gamma(TM)$ vector fields.

   (a) Show that the insertion operator $i_\xi : \Omega^k(M) \to \Omega^{k-1}(M)$ is a graded derivation of degree -1 of $(\Omega(M), \wedge)$.

   (b) Recall from class that $[d, d] = 0$. Verify (the remaining) graded-commutator relations between $d, \mathcal{L}_\xi, i_\eta$:

   (i) $[d, \mathcal{L}_\xi] = 0$.

   (ii) $[d, i_\xi] = d \circ i_\xi + i_\xi \circ d = \mathcal{L}_\xi$.

   (iii) $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]}$.

   (iv) $[\mathcal{L}_\xi, i_\eta] = i_{[\xi,\eta]}$.

   (v) $[i_\xi, i_\eta] = 0$.

   Hint: Use (c) from 2.

3. Prove the **Poincaré Lemma**: Suppose $\omega \in \Omega^k(\mathbb{R}^n)$ is a closed k-form, where $k \geq 1$. Show that there exists $\tau \in \Omega^{k-1}(\mathbb{R}^n)$ such that $d\tau = \omega$.

   Hint:
   Consider the vector field $\xi \in \Gamma(\mathbb{R}^n)$ on $\mathbb{R}^n$ given by $\xi(x) = x \in T_x\mathbb{R}^n \cong \mathbb{R}^n$ and let $\alpha : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the smooth map $\alpha(t, x) = \alpha_t(x) = tx$. Then the flow of ξ is given by $\mathrm{Fl}^\xi_t = \alpha(e^t, x)$.

- Show that $(\frac{1}{t}\alpha_t^* i_\xi \omega)(x)$ is smooth in (t, x) for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Hence, $t \mapsto \frac{1}{t}\alpha_t^* i_\xi \omega$ is a smooth family of $(k-1)$-forms on $\mathbb{R}^n$.
- Show that $\frac{d}{dt}\alpha_t^* \omega = d(\frac{1}{t}\alpha_t^* i_\xi \omega)$ and that $\omega = d\tau$, where $\tau = \int_0^1 \frac{1}{t}\alpha_t^* i_\xi \omega dt \in \Omega^{k-1}(\mathbb{R}^n)$.

4. Show that n-dimensional projective space $\mathbb{R}P^n$ is orientable $\iff$ n is odd.
5. Suppose $(M, g) \subset (\mathbb{R}^{n+1}, g_{\text{euc}} = \langle \cdot, \cdot \rangle)$ is a hypersurface. Show that M is orientable if and only if it admits a global unit normal vector field.