Lecture 8. Poincaré duality

Last time we proved that if M is an oriented manifold of dim n and $A \subseteq M$ compact that

- $H_i(M, M\setminus A) = 0$ for $i > n$
- there is a class $\mu_A \in H_n(M, M\setminus A)$ such that
$$(\rho_x)_*(\mu_A) = \mu_x \quad \text{for all } x \in A,$$
where $\mu_x \in H_n(M, M\setminus x)$ and $\rho_x : (M, M\setminus A) \longrightarrow (M, M\setminus x)$.

To formulate Poincaré duality we need another product called cap product
$$\cap : H_n(X;R) \otimes H^k(X;R) \longrightarrow H_{n-k}(X;R)$$
defined on chains and cochains by the formula
$$\sigma \cap \varphi = \underbrace{\varphi(\sigma/[v_0, \dots v_k])}_{R} \cdot \underbrace{\sigma/[v_k, \dots v_n]}_{C_{n-k}(X)}$$

One can prove that
$$\partial(\sigma \cap \varphi) = (-1)^k (\partial\sigma \cap \varphi - \sigma \cap \delta\varphi)$$
It enables to define cap product on the level of homologies and cohomologies:

$$\begin{aligned}
\cap \quad : \quad & H_n(X) \otimes H^k(X) \longrightarrow H_{n-k}(X) \\
& H_n(X,A) \otimes H^k(X) \longrightarrow H_{n-k}(X,A) \\
& H_n(X,A) \otimes H^k(X,A) \longrightarrow H_{n-k}(X) \\
& H_n(X, A\cup B) \otimes H^k(X,A) \longrightarrow H_{n-k}(X,B)
\end{aligned}$$
for A, B open in X.

Naturality For $f : X \to Y$ we get
$$f_*(a \cap f^*(b)) = f_*(a) \cap b$$

Theorem (Poincaré duality)

Let M be a closed (= compact) R-oriented manifold of dim n. Then the map

$$D : H^k(M; R) \longrightarrow H_{n-k}(M; R)$$
$$D(\varphi) = [M] \cap \varphi$$

is an isomorphism.

As for the proof. It shows up that it is better to formulate a more general statement (without assumption that M is compact) and prove this. („more difficult" is sometimes easier.) To it we need the notion of cohomology with compact support

Consider a space X with a directed system of compact sets (ordering by inclusions). For each pair $K \subseteq L$, the inclusion

$$(X, X-L) \hookrightarrow (X, X \setminus K)$$

induces in cohomology homomorphism

$$H^k(X, X \setminus K) \longrightarrow H^k(X, X-L).$$

So we can define cohomology groups of compact support as

$$H^k_c(X) = \varinjlim_{K} H^k(X, X-K)$$

If X is compact then

$$H^k_c(X) = H^k(X).$$

Example:

$$H^k_c(\mathbb{R}^n; \mathbb{Z}) = \lim_{r \to \infty} H^k(\mathbb{R}^n, \mathbb{R}^n \setminus D(0,r))$$

where $D(0,r) = \{x \in \mathbb{R}^n, \|x\| \leq r\}$.

Then $H_c^n(\mathbb{R}^n) = \lim\limits_{\substack{r\to\infty \\ r\in\mathbb{N}}} (\mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to \dots) = \mathbb{Z}$

$k \neq n$ $H_c^k(\mathbb{R}^n) = \lim\limits_{\substack{r\to\infty \\ r\in\mathbb{N}}} (0 \to 0 \to 0 \to \dots) = 0.$

Generalized Poincaré duality

Let M be an R-oriented manifold of dimension n. Let $K \subseteq M$ be compact. Let $\mu_K \in H_n(M, M \setminus K; R)$ is a class such that $(\rho_x)_* \mu_K = \mu_x$ for all $x \in K$. Then we define

$$D_K : H^k(M, M \setminus K; R) \longrightarrow H_{n-k}(M; R),$$
$$D_K(\varphi) = \mu_K \cap \varphi.$$

If $K \subseteq L$ are two compact sets, we can prove that

$$D_L(\rho^* \varphi) = D_K(\varphi) \qquad \varphi \in H^k(M, M \setminus K; R)$$

for $\rho : (M, M \setminus L) \hookrightarrow (M, M \setminus K)$.

It enables us to define

$$D_M : H_c^k(M; R) \longrightarrow H_{n-k}(M; R)$$
$$D(\varphi) = \mu_K \cap \varphi$$

since every $\varphi \in H_c^k(M; R)$ comes from an element in $H^k(M, M \setminus K; R)$ for some $K \subseteq M$ compact.

THEOREM (Poincaré duality for all manifolds)

If M is an R-oriented manifold of dimension n, then

$$D_M : H_c^k(M; R) \longrightarrow H_{n-k}(M; R)$$

is an isomorphism.

The proof is based on: If $M = U \cup V$ where U and V are open, then the following diagram with LES commutes:

$$\begin{tikzcd}
H_c^k(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H_c^k(U) \oplus H_c^k(V) \arrow[r] \arrow[d, "D_U \oplus D_V"] & H_c^k(M) \arrow[r] \arrow[d, "D_M"] & H_c^{k+1}(U \cap V) \arrow[d, "D_{U\cap V}"] \\
H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U \cap V)
\end{tikzcd}$$

From this diagram we can prove

(A) If $M = U \cup V$, U, V open and $D_U, D_V, D_{U\cap V}$ are isos, then D_M is an iso.

Using definition of cohomology with compact support we can prove:

(B) If $M = \bigcup_{i=1}^{\infty} U_i$ where U_i are open, $U_1 \subset U_2 \subset U_3 \subset \ldots$ and D_{U_i} are isos, then D_M is an iso.

The proof of duality itself can be carried out in 4 steps.

(1) $M = \mathbb{R}^n$ We have $H_c^k(\mathbb{R}^n) \cong H^k(\Delta^n, \partial\Delta^n)$

$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \Delta^n) \cong H_n(\Delta^n, \partial\Delta^n)$

The generator $\mu \in H_n(\Delta^n, \partial\Delta^n)$ is represented by $id: \Delta^n \to \Delta^n$. Take $\varphi \in H^n(\Delta^n, \partial\Delta^n) \cong Hom(H_n(\Delta^n, \partial\Delta^n), R)$. Then

$$\mu \cap \varphi = \varphi(\mu) \cdot 1 = \pm 1.$$

For $\varphi \in H^k(\Delta^n, \partial\Delta^n) = 0$, $k \neq n$, the statement is trivial.

(2) $M \subseteq \mathbb{R}^n$ open. M is a union of countably many open convex sets which are homeomorphic to $\mathbb{R}^n$. The statement follows from (A) and (B)

(3) M is a manifold which is a countable union of open sets homeomorphic to $\mathbb{R}^n$. Use again (A) and (B).

(4) General M (see Hatcher, page 248). $\square$

<u>Corollary</u>: Euler characteristic of odd dimensional orientable manifold is zero.

Euler characteristic of even dimensional oriented manifold is even number.

<u>Proof</u>: $\operatorname{rank} H_{n-k}(M;\mathbb{Z}) = \operatorname{rank} H^k(M;\mathbb{Z}) =$
$= \operatorname{rank} \operatorname{Hom}(H_k(M), \mathbb{Z})$
$= \operatorname{rank} H_k(M)$.

Then $\sum_{i=0}^{n} (-1)^i H_i(M;\mathbb{Z})$ $= 0$ for n odd
$\in 2\mathbb{Z}$ for n even

<u>Example</u>: Real projective spaces of even dimensions are not orientable.

We have computed that $H_n(\mathbb{R}P^n;\mathbb{Z}) \cong 0$ but $H^0(\mathbb{R}P^n;\mathbb{Z}) \cong \mathbb{Z}$. So $\mathbb{R}P^n$ for n even cannot satisfy assumption of Poincaré duality theorem. $\square$

Duality and cup product:

For $c \in C_n(X;R)$ and $\varphi \in C^k(X;R)$ and $\psi \in C^{n-k}(X;R)$ we have

$$\psi(c \frown \varphi) = (\varphi \smile \psi)(c)$$

Left hand side is

$$\psi(\varphi(c/[v_0,\dots v_k]) \cdot c/[v_k,\dots v_n]) = \varphi(c/[v_0\dots v_k]) \cdot \psi(c/[v_k,\dots v_n])$$

For closed R-oriented manifolds we define bilinear form

$$(*) \qquad \begin{aligned} H^k(M;R) \otimes H^{n-k}(M;R) &\longrightarrow R \\ \varphi \otimes \psi &\longmapsto (\varphi \smile \psi)[M] \end{aligned}$$

The form $A \otimes B \longrightarrow R$ is regular if induced maps

$$A \longrightarrow \mathrm{Hom}(B,R)$$
$$B \longrightarrow \mathrm{Hom}(A,R)$$

are isomorphisms.

Theorem (Modified Poincaré duality)

Let M be a closed R-orientable manifold and let R be a field. Then the bilinear form $(*)$ is regular.

Let M be a closed $\mathbb{Z}$-orientable manifold. Then the bilinear form

$$H^k(M;\mathbb{Z})/\mathrm{Tor}\, H^k(M;\mathbb{Z}) \otimes H^{n-k}(M;\mathbb{Z})/\mathrm{Tor}\, H^{n-k}(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

is regular.

Example: Using the theorem above we will prove that as a graded ring

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[w]/\langle w^{n+1}\rangle$$

where $w \in H^2(\mathbb{C}P^n;\mathbb{Z})\cong\mathbb{Z}$ is a generator.

Proof by induction: $n=1$, $\mathbb{C}P^1 = e_0 \cup e_2 = S^2$ and hence

$$H^*(\mathbb{C}P^1;\mathbb{Z}) \cong \mathbb{Z}[w]/\langle w^2\rangle.$$

Let $n \geq 2$. Then the inclusion

$$\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$$

induces isomorphism $H^k(\mathbb{C}P^{n-1};\mathbb{Z}) \to H^k(\mathbb{C}P^n;\mathbb{Z})$ for $k \leq 2n-2$.

So using inductive assumption we get that $H^{2k}(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$ is generated by w^k where $w \in H^2(\mathbb{C}P^n;\mathbb{Z})$, for $k \leq n-1$.

It suffices to show that w^n is a generator of $H^{2n}(\mathbb{C}P^n;\mathbb{Z})$. $\mathbb{C}P^n$ is orientable (it is simply connected), so

$$H^2(\mathbb{C}P^n)\otimes H^{2n-2}(\mathbb{C}P^n) \longrightarrow \mathbb{Z}$$

is regular, which means that

$$w \cup w^{n-1}[\mathbb{C}P^n] = \pm 1$$

And this means that w^n is a generator of $H^{2n}(\mathbb{C}P^n)$.

Poincaré duality for manifolds with boundary

Let M be a compact R-orientable manifold with boundary $\partial M = A \cup B$, where A, B are $(n-1)$-dim manifolds such that $\partial A = \partial B = A \cap B$. Then the cap product defines Poincaré duality isomorphism

$$D : H^k(M, A; R) \longrightarrow H_{n-k}(M, B; R).$$

ALEXANDER DUALITY

Let $K \subsetneq S^n$ be compact subset of S^n which is a deform. retract of an open neighbourhood. Then

$$\tilde{H}_i(S^n \setminus K;\mathbb{Z}) \cong \tilde{H}^{n-i-1}(K;\mathbb{Z})$$

Proof: For $i>0$ and a neighbourhood U of K

$$\begin{aligned} H_i(S^n \setminus K) &\cong H_c^{n-i}(S^n\setminus K) && \text{by Poincaré duality} \\ &\cong \varinjlim_U H^{n-i}(S^n\setminus K, U\setminus K) && \text{by definition} \\ &\cong \varinjlim_U H^{n-i}(S^n, U) && \text{by excision} \\ &\cong \varinjlim_U H^{n-i-1}(U) && \text{by connecting homomorphism} \\ &\cong H^{n-i-1}(K) && K \text{ is a def retract of some small } U \end{aligned}$$

not true for $i=0$

For $i=0$ we can use the first three isos:

$$\begin{aligned} \tilde{H}_0(S^n\setminus K) &\cong \operatorname{Ker}(H_0(S^n\setminus K) \to H_0(pt)) \\ &\cong \operatorname{Ker}(H_0(S^n\setminus K) \to H_0(S^n)) \\ &\cong \operatorname{Ker}(\varinjlim H^n(S^n,U) \to H^n(S^n)) \\ &\cong \varinjlim(\operatorname{Ker}(H^n(S^n,U) \to H^n(S^n))) \\ &\cong \varinjlim H^{n-1}(U) \cong H^{n-1}(K) \end{aligned}$$