CVIČENÍ 5

5.1

1. $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = x^{-1}\left(-\frac{1}{1}\right) + c = -x^{-1} + c = -\frac{1}{x} + c$

2. $\int x^{12}\,dx = \frac{x^{13}}{13} + c$

e^{+a}

$(e^{+})^{a} = e^{ax}$

3. $\int \frac{3x}{4}\,dx = \frac{3}{4}\int x\,dx = \frac{3}{4}\,\frac{x^2}{2} + c = \frac{3x^2}{8} + c$

4. $\int \frac{\cos^3 x - 0.8}{\cos^2 x}\,dx = \int \cos x\,dx - 0.8\int \frac{1}{\cos^2 x}\,dx = \sin x - 0.8\tan x + c$

5. $\int \frac{4x - 2\sqrt{x}}{x}\,dx = \int 4\,dx - 2\int \frac{x^{\frac{1}{2}}}{x}\,dx = \int 4\,dx - 2\int x^{-\frac{1}{2}}\,dx = 4x - 2x^{\frac{1}{2}}\cdot 2 + c = 4x - 4\sqrt{x} + c$

$= 4(x - \sqrt{x}) + c$

6. $\int (8\cos x - 3\sin x)\,dx = 8\int \cos x\,dx - 3\int \sin x\,dx = 8\sin x + 3\cos x + c$

7. $\int \left(\frac{4x}{\sqrt{3x}} + (3-2x)^2\right)dx = \frac{4}{\sqrt{3}}\int \frac{x}{\sqrt{x}}\,dx + \int 9 - 12x + 4x^2\,dx =$

$= \frac{4}{\sqrt{3}}\int \sqrt{x}\,dx + \int 9\,dx - 12\int x\,dx + 4\int x^2\,dx =$

$= \frac{4}{\sqrt{3}}\,x^{\frac{3}{2}}\cdot\frac{2}{3} + 9x - 12\,x^2\cdot\frac{1}{2} + 4\,\frac{x^3}{3} + c = \frac{8}{3\sqrt{3}}\sqrt{x^3} + 9x - 6x^2 + \frac{4x^3}{3} + c$

$= \frac{8\sqrt{3}}{3\cdot 3}\sqrt{x^3} + 9x - 6x^2 + \frac{4x^3}{3}\,dx = \frac{8}{9}\sqrt{3x^3} + 9x - 6x^2 + \frac{4}{3}x^3 + c$

8. $\int x(2x-5)\,dx = \int 2x^2 - 5x\,dx = 2\int x^2\,dx - 5\int x\,dx = 2\cdot\frac{x^3}{3} - 5\frac{x^2}{2} + c = \frac{2}{3}x^3 - \frac{5}{2}x^2 + c$

9. $\int \left(x^3 - \frac{1}{x} + \frac{\sqrt[4]{x}}{2}\right)dx = \frac{x^4}{4} - \ln x + \frac{1}{2}\int x^{\frac{1}{4}}\,dx + c = \frac{x^4}{4} - \ln x + \frac{1}{2}\,x^{\frac{5}{4}}\cdot\frac{4}{5} + c =$

$= \frac{x^4}{4} - \ln x + \frac{2}{5}\sqrt[4]{x^5} + c = \frac{x^4}{4} - \ln x + \frac{2}{5}\,x\sqrt[4]{x} + c$

10. $\int \frac{e^{2x} - 1}{e^x}\,dx = \int e^x\,dx - \int \frac{1}{e^x}\,dx = \int e^x\,dx - \int e^{-x}\,dx = e^x + e^{-x} + c$

5.2

1. $$\int 3e^{-3x+1}\,dx = \left| \begin{matrix} t = -3x+1 \\ dt = -3\,dx \end{matrix} \right| = -\int e^{t}\,dt = -e^{t} + k = -e^{-3x+1} + c$$

2. $$\int \cos x \sqrt{\sin x}\,dx = \left| \begin{matrix} t = \sin x \\ dt = \cos x\,dx \end{matrix} \right| = \int \sqrt{t}\,dt = \int t^{\frac{1}{2}}\,dt = t^{\frac{3}{2}} \cdot \frac{2}{3} + k = \frac{2}{3}\sqrt{t^{3}} + k = \frac{2}{3}\sqrt{\sin^{3} x} + c$$

3. $$\int \sqrt[3]{5-6x}\,dx = \left| \begin{matrix} t = 5-6x \\ dt = -6\,dx \end{matrix} \right| = -\frac{1}{6}\int \sqrt[3]{t}\,dt = -\frac{1}{6}\int t^{\frac{1}{3}}\,dt = -\frac{1}{6} t^{\frac{4}{3}} \cdot \frac{3}{4} + k = -\frac{1}{8} t^{\frac{4}{3}} + k = \frac{-(5-6x)^{\frac{4}{3}}}{8} + c$$

4. $$\int \sin x \cos^{5} x\,dx = \left| \begin{matrix} t = \cos x \\ dt = -\sin x\,dx \end{matrix} \right| = -\int t^{5}\,dt = -\frac{t^{6}}{6} + k = -\frac{\cos^{6} x}{6} + c$$

5. $$\int (2x+1)^{3}\,dx = \left| \begin{matrix} t = 2x+1 \\ dt = 2\,dx \end{matrix} \right| = \frac{1}{2}\int t^{3}\,dt = \frac{1}{2} \cdot \frac{t^{4}}{4} + k = \frac{(2x+1)^{4}}{8} + c$$

6. $$\int \frac{\cos x}{3\sin^{2/3} x}\,dx = \left| \begin{matrix} t = \sin x \\ dt = \cos x\,dx \end{matrix} \right| = \frac{1}{3}\int \frac{1}{t^{2/3}}\,dt = \frac{1}{3}\int t^{-\frac{2}{3}}\,dt = \frac{1}{3} t^{\frac{1}{3}} \cdot (3) + k = t^{\frac{1}{3}} + k =$$
$$= \sin^{1/3} x + c = \sqrt[3]{\sin x} + c$$

7. $$\int 6x \sin 3x^{2}\,dx = \left| \begin{matrix} t = 3x^{2} \\ dt = 6x\,dx \end{matrix} \right| = \int \sin t\,dt = -\cos t + k = -\cos 3x^{2} + c$$

8. $$\int \frac{8x^{2}}{\sqrt[3]{(8x^{3}+27)^{2}}}\,dx = \left| \begin{matrix} t = 8x^{3}+27 \\ dt = 24x^{2}\,dx \end{matrix} \right| = \frac{1}{3}\int \frac{1}{\sqrt[3]{t^{2}}}\,dt = \frac{1}{3}\int t^{-\frac{2}{3}}\,dt = \frac{1}{3} t^{\frac{1}{3}} \cdot 3 + k = t^{\frac{1}{3}} + k =$$
$$= (8x^{3}+27)^{1/3} + c = \sqrt[3]{(8x^{3}+27)} + c$$

9. $$\int 6\tan 3x\,dx = \int 6\frac{\sin 3x}{\cos 3x}\,dx = \left| \begin{matrix} t = \cos 3x \\ dt = -\sin 3x \cdot 3\,dx \\ = -3\sin 3x\,dx \end{matrix} \right| = \int -2\frac{1}{t}\,dt = -2\int t^{-1}\,dt = -2\ln|t| + k =$$
$$= -2\ln|\cos 3x| + c$$

10. $$\int \frac{3x}{(x^{2}+1)^{2}}\,dx = \left| \begin{matrix} t = x^{2}+1 \\ dt = 2x\,dx \end{matrix} \right| = \frac{3}{2}\int \frac{1}{t^{2}}\,dt = \frac{3}{2}\int t^{-2}\,dt = \frac{3}{2} t^{-1}(-1) + k = -\frac{3}{2t} + k =$$
$$= -\frac{3}{2(x^{2}+1)} + c$$

5.3

1. $\int_0^4 \sqrt{x}\,dx = \int_0^4 x^{\frac{1}{2}} dx = \left[x^{\frac{3}{2}} \cdot \frac{2}{3}\right]_0^4 = \left[\frac{2}{3}\sqrt{x^3}\right]_0^4 = \frac{2}{3}\left(\sqrt{4^3} - \sqrt{0^3}\right) = \frac{2}{3}\left(\sqrt{64} - 0\right) = \frac{2 \cdot 8}{3} = \frac{16}{3}$

2. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = \left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) = 1 - (-1) = 2$

3. $\int_0^3 e^{\frac{x}{3}} dx = \left| \begin{matrix} t = \frac{x}{3} \\ dt = \frac{1}{3} dx \end{matrix} \right| = \int_0^1 3e^t dt = \left[3e^t\right]_0^1 = 3(e^1 - e^0) = 3e - 3 = 3(e-1)$

$= \left[3e^{\frac{x}{3}}\right]_0^3 = 3\left(e^{\frac{3}{3}} - e^{\frac{0}{3}}\right) = 3e^1 - 3e^0 = 3e - 3 = 3(e-1)$

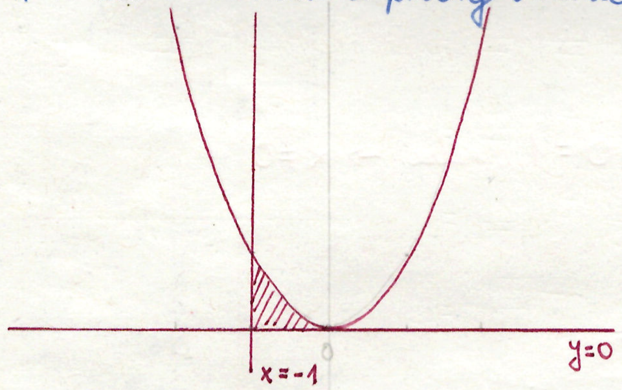
4. $\int_0^{\pi} \frac{2\sin x}{5 + 4\cos x} dx = \left| \begin{matrix} t = 5 + 4\cos x \\ dt = -4\sin x\,dx \end{matrix} \right| = \int_9^1 -\frac{1}{2}\frac{1}{t} dt = \int_1^9 \frac{1}{2} t^{-1} dt = \left[\frac{1}{2}\ln|t|\right]_1^9 = \frac{1}{2}(\ln 9 - \ln 1) =$

$= \frac{1}{2}(\ln 9 - 0) = \frac{1}{2}\ln 3^2 = \frac{2}{2}\ln 3 = \ln 3$

5. $\int_{-1}^1 2x^3 dx = \left[2 \cdot \frac{x^4}{4}\right]_{-1}^1 = \left[\frac{x^2}{2}\right]_{-1}^1 = \frac{1}{2} - \frac{(-1)^2}{2} = \frac{1}{2} - \frac{1}{2} = 0$

5.4

1. Určete obsah rovinné plochy ohraničené křivkami $y = 0$, $x = -1$ a $y = x^2$.

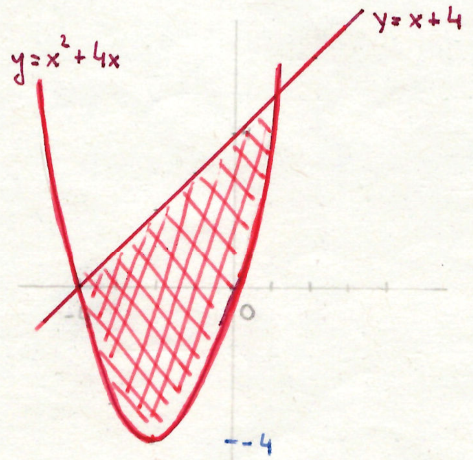


$\int_{-1}^0 x^2 - 0\,dx = \left[\frac{x^3}{3}\right]_{-1}^0 = \frac{0^3}{3} - \frac{(-1)^3}{3} = 0 - \frac{-1}{3} = \frac{1}{3}$

2. Určete obsah rovinné plochy ohraničené křivkami $y = x^2 + 4x$ $\quad y = x + 4$

$y = x(x+4)$ $\quad 0 = x + 4 \rightarrow x = -4$

$x = 0 \quad x = -4$



$\int_{-4}^1 (x+4) - (x^2 + 4x)\,dx = \int_{-4}^1 x + 4 - x^2 - 4x\,dx = \int_{-4}^1 -x^2 - 3x + 4\,dx =$

$= \left[-\frac{x^3}{3} - \frac{3x^2}{2} + 4x\right]_{-4}^1 = -\frac{1}{3} - \frac{3}{2} + 4 - \left(-\frac{(-4)^3}{3} - \frac{3(-4)^2}{2} + 4(-4)\right) =$

$= \frac{-2 - 9 + 24}{6} - \left(\frac{64}{3} - \frac{3 \cdot 16}{2} - 16\right) = \frac{13}{6} - \frac{128 - 9 \cdot 16 - 96}{6} =$

$= \frac{13 - 128 + 144 + 96}{6} = \frac{13 + 16 + 96}{6} = \frac{125}{6}$

CVIČENÍ 6

6.1

1. $y' = e^x - \sin 2x$

$\frac{dy}{dx} = e^x - \sin 2x$

$dy = e^x - \sin 2x \, dx$

$\int dy = \int e^x - \sin 2x \, dx$ $\quad \begin{vmatrix} t = 2x \\ dt = 2dx \end{vmatrix} = \int \frac{\sin t}{2} dt = \frac{1}{2}\int \sin t \, dt = -\frac{1}{2}\cos t + k = -\frac{1}{2}\cos 2x + c$

$y = e^x + \frac{1}{2}\cos 2x + c \qquad x \in \mathbb{R},\ c \in \mathbb{R}$

2. $\frac{1}{y} y' = -4 \qquad y \neq 0 \qquad k = 0 \rightarrow y = 0 \rightarrow \frac{1}{0} 0' = -4$ ... nelze $\rightarrow k \neq 0$

$\frac{1}{y}\frac{dy}{dx} = -4$

$\frac{dy}{y} = -4dx$

$\int \frac{1}{y} dy = \int -4dx$

$\ln|y| = -4x + c$

$y = e^{-4x+c} = e^{-4x} \cdot \overbrace{e^c}^{k}$

$y = k \cdot e^{-4x} \qquad k \neq 0,\ x \in \mathbb{R}$

3. $2xy' = 1 - x^2 \qquad x = 0 \rightarrow 2 \cdot 0 \cdot y' = 1 - 0^2 \rightarrow 0 = 1$ ... nelze $\rightarrow x \neq 0$

$\frac{dy}{dx} = \frac{1-x^2}{2x} \qquad x \neq 0$

$dy = \frac{1}{2x} - \frac{x^2}{2x} dx$

$\int dy = \int \frac{1}{2}\frac{1}{x} - \frac{x}{2} dx$

$y = \frac{1}{2}\ln|x| - \frac{1}{2}\frac{x^2}{2} + c$

$y = \frac{1}{2}\ln|x| - \frac{x^2}{4} + c \qquad x \neq 0 \quad c \in \mathbb{R}$

4. $2y - x^3 y' = 0$ $\qquad$ $k=0 \rightarrow y=0 \rightarrow 0-0'=0$ ... OK $\rightarrow k \in \mathbb{R}$

$2y = x^3 \frac{dy}{dx}$ $\qquad$ $x=0 \rightarrow y = k \cdot e^{-\frac{1}{x^2}}$ ... nejde $\rightarrow x \neq 0$

$x^3 \frac{dy}{dx} = 2y$

$\frac{1}{y} dy = \frac{2}{x^3} dx \qquad y \neq 0 \quad x \neq 0$

$\int \frac{1}{y} dy = 2 \int x^{-3} dx$

$\ln|y| = 2 \frac{x^{-2}}{-2} + c$

$\ln|y| = -\frac{1}{x^2} + c$

$y = e^{-1/x^2 + c}$

$y = k \cdot e^{-\frac{1}{x^2}} \qquad \cancel{k \neq 0} \qquad k \in \mathbb{R},\ x \neq 0$

5. $y^2 y' = x - 2$

$y^2 \frac{dy}{dx} = x - 2$

$y^2 dy = x - 2 dx$

$\int y^2 dy = \int x - 2 dx$

$\frac{y^3}{3} = \frac{x^2}{2} - 2x + c$

$y^3 = \frac{3x^2}{2} - 6x + C$

$y = \sqrt[3]{\frac{3x^2}{2} - 6x + C} \qquad x \in \mathbb{R},\ C \in \mathbb{R}$

6. $y' = 3\sqrt[3]{y^2}$ $\qquad$ $c = x \rightarrow y = 0 \rightarrow 0' = 3 \cdot \sqrt[3]{0} \rightarrow 0 = 0$ ... OK $\rightarrow x \in \mathbb{R},\ c \in \mathbb{R}$

$\frac{dy}{dx} = 3 y^{\frac{2}{3}}$

$y^{-\frac{2}{3}} dy = 3 dx \qquad y \neq 0$

$\int y^{-\frac{2}{3}} dy = \int 3 dx$

$y^{\frac{1}{3}} \cdot 3 = 3x + c$

$\sqrt[3]{y} = x + c$

$y = (x + c)^3 \qquad c \overset{?}{\neq} x, \quad x \in \mathbb{R};\ c \in \mathbb{R}$

7. $\frac{y'}{y} = \frac{1}{x-1}$ $\quad y \neq 0 \quad x \neq 1$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x-1}$

$\frac{1}{y}dy = \frac{dx}{x-1}$

$\int \frac{1}{y}dy = \int \frac{1}{x-1}dx$

$\ln|y| = \ln|x-1| + c$

$y = e^{\ln|x-1|+c}$

$y = (x-1)\cdot k \qquad x \neq 1 \quad k \neq 0$

6.2

1. $y' = \frac{1}{(x-6)^2} \qquad y(7)=1$

$\frac{dy}{dx} = \frac{1}{(x-6)^2} \qquad x \neq 6$

$dy = \frac{1}{(x-6)^2}dx$

$\int dy = \int \frac{1}{(x-6)^2}dx \qquad \left| \begin{array}{l} t = x-6 \\ dt = dx \end{array} \right| = \int \frac{1}{t^2}dt = \int t^{-2}dt = t^{-1}(-1) + k = -\frac{1}{t} + k = -\frac{1}{x-6} + c$

$y = -\frac{1}{x-6} + c \qquad x \neq 6$

Part. řešení pro $y(7)=1$: $\quad 1 = -\frac{1}{7-6} + c$

$1 = -\frac{1}{1} + c$

$1+1 = c$

$c = 2$

Partikulární řešení pro $y(7)=1$ je $y = -\frac{1}{x-6} + 2$, $x \neq 6$.

2. $y' = -2\frac{1}{x^3}$ $\qquad y(0)=2$

$$\frac{dy}{dx} = -2x^{-3} \qquad x \neq 0$$

$$dy = -2x^{-3}dx$$

$$\int dy = -2\int x^{-3}dx$$

$$y = -2\frac{x^{-2}}{-2} + c$$

$$y = x^{-2} + c$$

$$y = \frac{1}{x^2} + c \qquad x \neq 0,\ c \in \mathbb{R}$$

Part. řešení pro $y(0)=2$: $\quad 2 = \frac{1}{0^2} + c$

$2 = \frac{1}{0} + c$ … nejde … $\frac{1}{0}$ je neurčitý výraz.

Partikulární řešení pro $y(0)=2$ neexistuje.

3. $y' = -2\frac{1}{x^3}$ $\qquad y(-1) = 3$

$\vdots$

$$y = \frac{1}{x^2} + c \qquad x \neq 0,\ c \in \mathbb{R}$$

Part. řešení pro $y(-1)=3$: $\quad 3 = \frac{1}{(-1)^2} + c$

$$3 = \frac{1}{1} + c$$

$$c = 3 - 1$$

$$c = 2$$

Partikulární řešení pro $y(-1)=3$ je $y = \frac{1}{x^2} + 2$, $x \neq 0$.

4. $y' = e^x - \sin 2x$, $\quad y(0) = \frac{1}{2}$

$\vdots$

$$y = e^x + \frac{1}{2}\cos 2x + c,\ x \in \mathbb{R},\ c \in \mathbb{R}$$

Partikulární řešení pro $y(0) = \frac{1}{2}$: $\quad \frac{1}{2} = e^0 + \frac{1}{2}\cos 2\cdot 0 + c$

$$\frac{1}{2} = 1 + \frac{1}{2} + c$$

$$\frac{1}{2} = \frac{3}{2} + c$$

$$\frac{1}{2} - \frac{3}{2} = c$$

$$-1 = c$$

Partikulární řešení pro $y(0) = \frac{1}{2}$ je $y = e^x + \frac{1}{2}\cos 2x - 1$, $x \in \mathbb{R}$.

5. $\frac{1}{y} y' = -4, \qquad y(-\frac{1}{4}) = 1$

$\vdots$

$y = k \cdot e^{-4x}, \quad k \neq 0, \; x \in \mathbb{R}$

Part. řešení pro $y(-\frac{1}{4}) = 1$: $\quad 1 = k \cdot e^{-4(-\frac{1}{4})}$

$1 = k \cdot e^{1}$

$1 = k \cdot e$

$k = \frac{1}{e} = e^{-1}$

Partikulární řešení pro $y(-\frac{1}{4}) = 1$ je $y = \frac{1}{e} e^{-4x}, \; x \in \mathbb{R}$.

6. $2xy' = (1 - x^2), \qquad y(1) = -1$

$\vdots$

$y = \frac{1}{2} \ln|x| - \frac{x^2}{4} + c, \quad x \neq 0, \; c \in \mathbb{R}$

Partikulární řešení pro $y(1) = -1$: $\quad -1 = \frac{1}{2} \ln|1| - \frac{1}{4} + c$

$-1 = 0 - \frac{1}{4}$

$-\frac{3}{4} = c$

Partikulární řešení pro $y(1) = -1$ je $y = \frac{1}{2} \ln|x| - \frac{x^2}{4} - \frac{3}{4}, \quad x \neq 0$.

7. $\frac{y'}{y} = \frac{1}{x-1}, \qquad y(1) = 4$

$\vdots$

$y = (x-1)k, \quad x \neq 1, \; k \neq 0$

Part. řešení pro $y(1) = 4$: $\quad 1 = (1-1) \cdot k$

$1 = 0 \cdot k$

$1 = 0 \longrightarrow$ spor $\longrightarrow$ Partikulární řešení pro $y(1) = 4$ neexistuje.