gravity. However, the story of Newton's law rests heavily on Greek work on the ellipse and Kepler's analysis of observational data.

A sub-theme of the book is the practical uses of mathematics. Here I have provided a very eclectic sample of applications, both past and present. Again, the omission of any topic does not indicate that it lacks importance.

Mathematics has a long, glorious, but somewhat neglected history, and the subject's influence on the development of human culture has been immense. If this book conveys a tiny part of that story, it will have achieved what I set out to do.

Coventry, May 2007

CHAPTER 1

Tokens, Tallies and Tablets

The birth of numbers

Mathematics began with numbers, and numbers are still fundamental, even though the subject is no longer limited to numerical calculations. By building more sophisticated concepts on the basis of numbers, mathematics has developed into a broad and varied area of human thought, going far beyond anything that we encounter in a typical school syllabus. Today's mathematics is more about structure, pattern and form than it is about numbers as such. Its methods are very general, and often abstract. Its applications encompass science, industry, commerce – even the arts. Mathematics is universal and ubiquitous.

It started with numbers

Over many thousands of years, mathematicians from many different cultures have created a vast superstructure on the foundations of number: geometry, calculus, dynamics, probability, topology, chaos, complexity and so on. The journal *Mathematical Reviews*, which keeps track of every new mathematical publication, classifies the subject into nearly a hundred major areas, subdivided into several thousand specialities. There are more than 50,000 research mathematicians in the world, and they publish more than a million pages of new mathematics every year. Genuinely new mathematics, that is, not just small variations on existing results.

Mathematicians have also burrowed into the logical foundations of their subject, discovering concepts even more fundamental than numbers – mathematical logic, set theory. But again the main motivation, the starting point from which all else flows, is the concept of number.

Numbers seem very simple and straightforward, but appearances are deceptive. Calculations with numbers can be hard; getting the *right* number can be difficult. And even then, it is much easier to use numbers than to specify what they really are. Numbers count things, but they are not things, because you can pick up two cups, but you can't pick up the number 'two'. Numbers are denoted by symbols, but different cultures use different symbols for the same number. Numbers are abstract, yet we base our society on them and it would not function without them. Numbers are some kind of mental construct, yet we feel that they would continue to have meaning even if humanity were wiped out by a global catastrophe and there were no minds left to contemplate them.

Writing numbers

The history of mathematics begins with the invention of written symbols to denote numbers. Our familiar system of digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to represent all conceivable numbers, however large, is a relatively new invention; it came into being about 1500 years ago, and its extension to decimals, which lets us represent numbers to high precision, is no more than 450 years old. Computers, which have embedded mathematical calculations so deeply into our culture that we no longer notice their presence, have been with us for a mere 50 years; computers powerful enough and

fast enough to be useful in our homes and offices first became widespread about 20 years ago.

Without numbers, civilization as we now know it could not exist. Numbers are everywhere, hidden servants that scurry around behind the scenes – carrying messages, correcting our spelling when we type, scheduling our holiday flights to the Caribbean, keeping track of our goods, ensuring that our medicines are safe and effective. And, for balance, making nuclear weapons possible, and guiding bombs and missiles to their targets. Not every application of mathematics has improved the human condition.

How did this truly enormous numerical industry arise? It all began with little clay tokens, 10,000 years ago in the Near East.

Even in those days, accountants were keeping track of who owned what, and how much – even though writing had not then been invented, and there were no symbols for numbers. In place of number symbols, those ancient accountants used small clay tokens. Some were cones, some were spheres and some were shaped like eggs. There were cylinders, discs and pyramids. The archaeologist Denise Schmandt-Besserat deduced that these tokens represented basic staples of the time. Clay spheres represented bushels of grain, cylinders stood for animals, eggs for jars of oil. The earliest tokens date back to 8000 BC, and they were in common use for 5000 years.

As time passed, the tokens became more elaborate and more specialized. There were decorated cones to represent loaves of bread, and diamond-shaped slabs to represent beer. Schmandt-Besserat realized that these tokens were much more than an accountancy device. They were a vital first step along the path to number symbols, arithmetic and mathematics. But that initial step was rather strange, and it seems to have happened by accident.

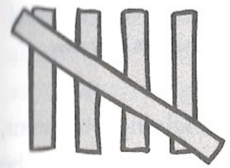
It came about because the tokens were used to keep records, perhaps for tax purposes or financial ones, or as legal proof of ownership. The advantage of tokens was that the accountants could quickly arrange them in patterns, to work out how many animals or how much grain someone owned or owed. The disadvantage was that tokens could be counterfeited. So to make sure that no one interfered with the accounts, the accountants wrapped the tokens in clay envelopes – in effect, a kind of seal. They could quickly find out how many tokens were inside any given envelope, and of what kind, by breaking the envelope open. They could always make a new envelope for later storage.

However, repeatedly breaking open an envelope and renewing it was a rather inefficient way to find out what was inside, and the bureaucrats of ancient Mesopotamia thought of something better. They inscribed symbols on the envelope, listing the tokens that it contained. If there were seven spheres inside, the accountants would draw seven pictures of spheres in the wet clay of the envelope.

At some point the Mesopotamian bureaucrats realized that once they had drawn the symbols on the outside of the envelope, they didn't actually need the contents, so they didn't need to break open the envelope to find out which tokens were inside it. This obvious but crucial step effectively created a set of written number symbols, with different shapes for different classes of goods. All other number symbols, including those we use today, are the intellectual descendants of this ancient bureaucratic device. In fact, the replacement of tokens by symbols may also have constituted the birth of writing itself.

Tally marks

These clay marks were by no means the earliest examples of number-writing, but all earlier instances are little more than scratches, tally marks, recording numbers as a series of strokes – such as ||||||||||||| to represent the number 13. The oldest known marks of this kind – 29 notches carved on a baboon's leg bone – are about 37,000 years old. The bone was found in a cave in the Lebombo mountains, on the border between Swaziland and



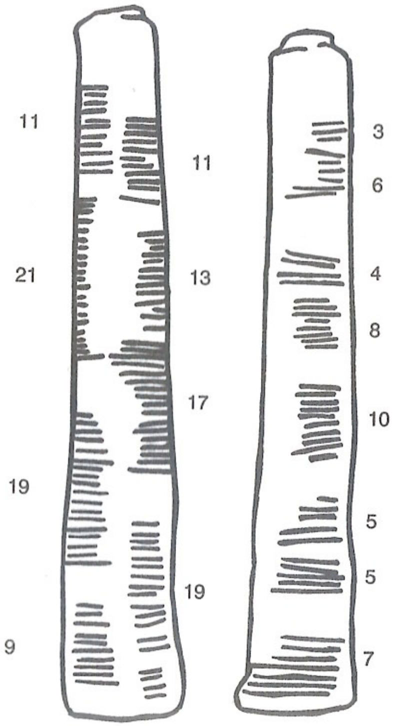
Tally marks have the advantage that they can be built up one at a time, over long periods, without altering or erasing previous marks. They are still in use today, often in groups of five, with the fifth stroke cutting diagonally across the previous four.

The presence of tally marks can still be seen in modern numerals. Our symbols 1, 2, 3 are derived from a single stroke, two horizontal strokes linked by a sloping line, and three horizontal strokes linked by a sloping line.

South Africa, so the cave is known as the Border Cave, and the bone is the Lebombo bone. In the absence of a time machine, there is no way to be certain what the marks represented, but we can make informed guesses. A lunar month contains 28 days, so the notches may have related to the phases of the Moon.

There are similar relics from ancient Europe. A wolf bone found in former Czechoslovakia has 57 marks arranged in eleven groups of five with two left over, and is about 30,000 years old. Twice 28 is 56, so this might perhaps be a two-month long lunar record. Again, there seems to be no way to test this suggestion. But the marks look deliberate, and they must have been made for some reason.

Another ancient mathematical inscription, the Ishango bone from Zaire, is 25,000 years old (previous estimates of 6000–9000 years were revised in 1995). At first sight the marks along the edge of the bone seem almost random, but there may be hidden patterns. One row contains the prime numbers between 10 and 20, namely 11, 13, 17 and 19, whose sum is 60. Another row contains 9, 11, 19 and 21, which also sum to 60. The third row resembles a method sometimes used to multiply two numbers together by repeated



The Ishango bone showing the patterns of marks and the numbers they may represent.

doubling and halving. However, the apparent patterns may just be coincidental, and it has also been suggested that the Ishango bone is a lunar calendar.

The first numerals

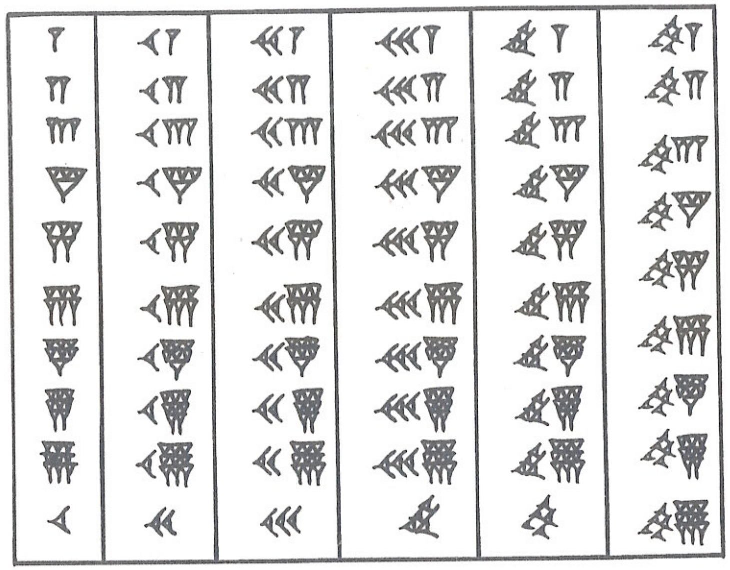
The historical path from accountants' tokens to modern numerals is long and indirect. As the millennia passed, the people of Mesopotamia developed agriculture, and their nomadic way of life gave way to permanent settlements, in a series of city states – Babylon, Eridu, Lagash, Sumer, Ur. The early symbols inscribed on tablets of wet clay mutated into pictographs – symbols that represent words by simplified pictures of what the words mean – and the pictographs were further simplified by assembling them from a small number of wedge-shaped marks, impressed into the clay using a dried reed with a flat, sharpened end. Different types of wedge could be made by holding the reed in different ways. By 3000 BC, the Sumerians had developed an elaborate form of writing, now called *cuneiform* or wedge-shaped.

The history of this period is complicated, with different cities becoming dominant at different times. In particular, the city of Babylon came to prominence, and about a million Babylonian clay tablets have been dug from the Mesopotamian sands. A few

hundred of these deal with mathematics and astronomy, and they show that the Babylonian knowledge of both subjects was extensive. In particular, the Babylonians were accomplished astronomers, and they developed a systematic and sophisticated symbolism for numbers, capable of representing astronomical data to high precision.

The Babylonian number symbols go well beyond a simple tally system, and are the earliest known symbols to do so. Two different types of wedge are used: a thin vertical wedge to represent the number 1, and a fat horizontal one for the number 10. These wedges are arranged in groups to indicate the numbers 2–9 and 20–50. However, this pattern stops at 59, and the thin wedge then takes on a second meaning, the number 60.

The Babylonian number system is therefore said to be 'base 60', or sexagesimal. That is, the value of a symbol may be some number, or 60 times such a number, or 60 times 60 times such a



Babylonian symbols for the numbers 1–59

number, depending on the symbol's position. This is similar to our familiar decimal system, in which the value of a symbol is multiplied by 10, or by 100, or by 1000, depending on its position. In the number 777, for instance, the first 7 means 'seven hundred', the second means 'seventy' and the third means 'seven'. To a Babylonian, a series of three repetitions  of the symbol for '7' would have a different meaning, though based on a similar principle. The first symbol would mean $7 \times 60 \times 60$, or 25,200; the second would mean $7 \times 60 = 420$; the third would mean 7. So the group of three symbols would mean $25{,}200 + 420 + 7$, which is 25,627 in our notation. Relics of Babylonian base-60 numbers can still be found today. The 60 seconds in a minute, 60 minutes in an hour and 360 degrees in a full circle, all trace back to ancient Babylon.

What numbers did for them

The Babylonians used their number system for day-to-day commerce and accountancy, but they also used it for a more sophisticated purpose: astronomy. Here the ability of their system to represent fractional numbers with high accuracy was essential. Several hundred tablets record planetary data. Among them is a single, rather badly damaged, tablet which details the daily motion of the planet Jupiter over a period of about 400 days. It was written in Babylon itself, around 163 BC. A typical entry from the tablet lists the numbers

126 8 16;6,46,58 -0;0,45,18
-0;0,11,42 +0;0,0,10,

which correspond to various quantities employed to calculate the planet's position in the sky. Note that the numbers are stated to three sexagesimal places – slightly better than to five decimal places.

Because it is awkward to typeset cuneiform, scholars write Babylonian numerals using a mixture of our base-10 notation and their base-60 notation. So the three repetitions of the cuneiform symbol for 7 would be written as 7,7,7. And something like 23,11,14 would indicate the Babylonian symbols for 23, 11 and 14 written in order, with the numerical value $(23 \times 60 \times 60) + (11 \times 60) + 14$, which comes to 83,474 in our notation.

Symbols for small numbers

Not only do we use ten symbols to represent arbitrarily large numbers: we also use the same symbols to represent arbitrarily small ones. To do this, we employ the decimal point, a small dot. Digits to the left of the dot represent whole numbers; those to the right of the dot represent fractions. Decimal fractions are multiples of one tenth, one hundredth, and so on. So 25.47, say, means 2 tens plus 5 units plus 4 tenths plus 7 hundredths.

The Babylonians knew this trick, and they used it to good effect in their astronomical observations. Scholars denote the Babylonian equivalent of the decimal point by a semicolon (;), but this is a sexagesimal point and the numbers to its right are multiples of $1/60$, $(1/60 \times 1/60) = 1/3600$, and so on. As an example, the list of numbers 12,59;57,17 means

$$12 \times 60 + 59 + 57/60 + 17/3600$$

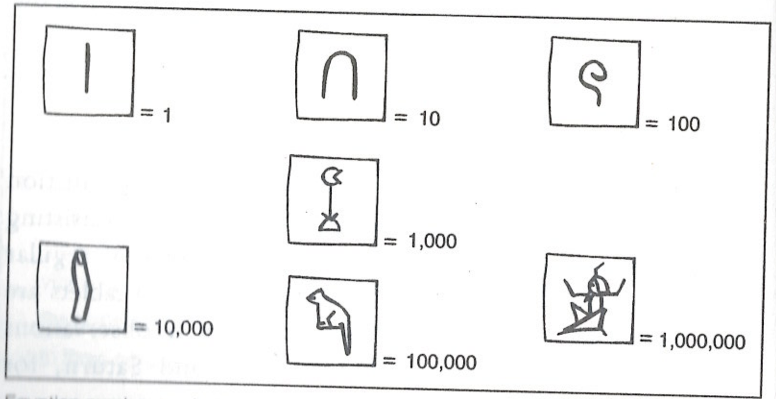
which is roughly 779.955.

Nearly 2000 Babylonian tablets with astronomical information are known, though many of these are fairly routine, consisting of descriptions of ways to predict eclipses, tables of regular astronomical events and shorter extracts. About 300 tablets are more ambitious and more exciting; they tabulate observations of the motion of Mercury, Mars, Jupiter and Saturn, for example.

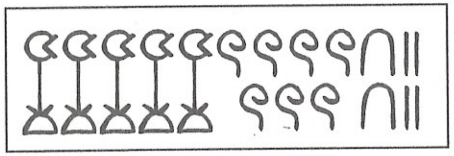
Fascinating as it may be, Babylonian astronomy is somewhat tangential to our main story, which is Babylonian pure mathematics. But it seems likely that the application to astronomy was a spur to the pursuit of the more cerebral areas of that subject. So it is a good idea to recognize just how accurate the Babylonian astronomers were when it came to observing heavenly events. For example, they found the orbital period of Mars (strictly, the time between successive appearances in the same position in the sky) to be 12,59;57,17 days in their notation – roughly 779.955 days, as noted above. The modern figure is 779.936 days.

The Ancient Egyptians

Perhaps the greatest of the ancient civilizations was that of Egypt, which flourished on the banks of the Nile and in the Nile Delta between 3150 BC and 31 BC, with an extensive earlier 'pre-dynastic' period stretching back to 6000 BC, and a gradual tailing off under the Romans from 31 BC onwards. The Egyptians were accomplished builders, with a highly developed system of religious beliefs and ceremonies, and they were obsessive record-keepers. But their mathematical achievements were modest compared to the heights scaled by the Babylonians.

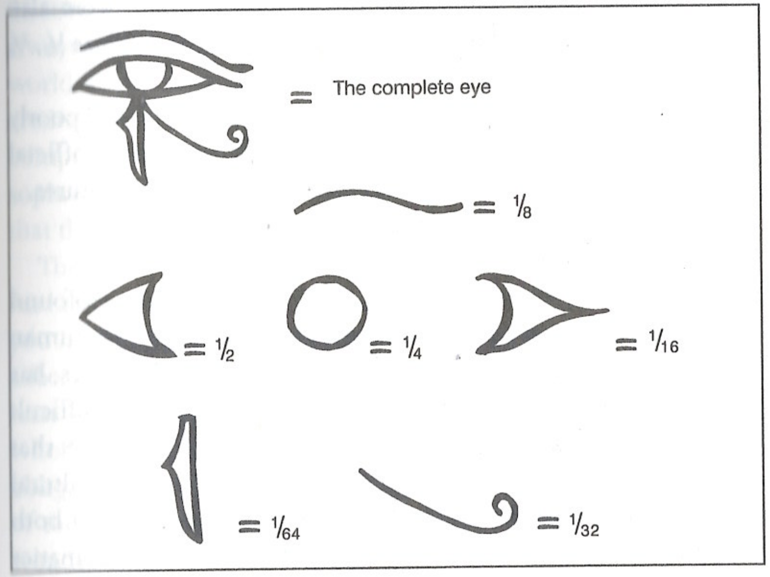


Egyptian number symbols

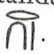

The number 5724 in Egyptian hieroglyphs

The ancient Egyptian system for writing whole numbers is simple and straightforward. There are symbols for the numbers 1, 10, 100, 1000 and so on. Repeating these symbols up to nine times, and then combining the results, can represent any whole number. For example, to write the number 5724, the Egyptians would group together five of their symbols for 1000, seven symbols for 100, two symbols for 10 and four symbols for 1.

Fractions caused the Egyptians severe headaches. At various periods, they used several different notations for fractions. In the Old Kingdom (2700–2200 BC), a special notation for our fractions, $1/2$, $1/4$, $1/8$, $1/16$, $1/32$ and $1/64$, was obtained by repeated halving. These symbols used parts of the 'eye of Horus' or 'wadjet-eye' hieroglyph.



Special fractions formed from parts of the wadjet eye

The best known Egyptian system for fractions was devised during the Middle Kingdom (2000–1700 BC). It starts with a notation for any fraction of the form $1/n$, where n is a positive integer. The symbol  (the hieroglyph for the letter R) is written over the top of the standard Egyptian symbols for n. So, for example, $^1/_{11}$ is written . Other fractions are then expressed by adding together several of these 'unit fractions'. For instance, $^5/_6 = {}^1/_2 + {}^1/_3$.

Special symbols for special fractions

Interestingly, the Egyptians did not write $^2/_5$ as $^1/_5 + {}^1/_5$. Their rule seems to have been: use *different* unit fractions. There were also different notations for some of the simpler fractions, such as $^1/_2$, $^2/_3$ and $^3/_4$.

The Egyptian notation for fractions was cumbersome and poorly adapted to calculation. It served them well enough in official records, but it was pretty much ignored by subsequent cultures.

Numbers and people

Whether you like arithmetic or not, it is hard to deny the profound effects that numbers have had on the development of human civilization. The evolution of culture, and that of mathematics, has gone hand in hand for the last four millennia. It would be difficult to disentangle cause and effect – I would hesitate to argue that mathematical innovation drives cultural change, or that cultural needs determine the direction of mathematical progress. But both of those statements contain a grain of truth, because mathematics and culture co-evolve.

There is a significant difference, though. Many cultural changes are clearly apparent. New kinds of housing, new forms of transport, even new ways to organize government bureaucracies, are relatively obvious to every citizen. Mathematics, however, mostly takes place behind the scenes. When the Babylonians used their astronomical observations to predict solar eclipses, for instance, the average citizen was impressed by how accurately the priests forecast this astonishing event, but even the majority of the priesthood had little or no idea of the methods employed. They knew how to read tablets listing eclipse data, but what mattered was how to use them. How they had been constructed was an arcane art, best left to specialists.

Some priests may have had good mathematical educations – all trained scribes did, and trainee priests took much the same lessons as scribes, in their early years – but an appreciation of mathematics wasn't really necessary to *enjoy* the benefits that flowed from new discoveries in that subject. So it has ever been, and no doubt always will be. Mathematicians seldom get credit for changing our world. How many times do you see all kinds of modern miracles credited to computers, without the slightest appreciation that computers only work effectively if they are programmed to use sophisticated algorithms, that is procedures to solve problems, and that the basis of almost all algorithms is mathematics?

The main mathematics that does lie on the surface is arithmetic. And the invention of pocket calculators, tills that tot up how much you have to pay and tax accountants who do the sums for you, for a fee, are pushing even arithmetic further behind the scenes. But at least most of us are aware that the arithmetic is there. We are wholly dependent on numbers, be it for keeping track of legal obligations, levying taxes, communicating instantly with the far side of the planet, exploring the surface of Mars or assessing the latest wonder drug. All of these things trace back to ancient Babylon, and to the scribes and teachers who discovered effective ways to record

What numbers do for us

Most upmarket modern cars now come equipped with satnav – satellite navigation. Stand-alone satnav systems can be purchased relatively cheaply. A small device, affixed to your car, then tells you exactly where you are at any moment and displays a map – often in fancy colour graphics and perspective – showing the neighbouring roads. A voice system can even tell you where to go to reach a specified destination. If this sounds like something out of science fiction, it is. An essential component, not part of the small box attached to the car, is the Global Positioning System (GPS), which comprises 24 satellites orbiting the Earth, sometimes more as replacements are launched. These satellites send out signals, and these signals can be used to deduce the location of the car to within a few metres.

Mathematics comes into play in many aspects of the GPS network, but here we mention just one: how the signals are used to work out the location of the car.

Radio signals travel at the speed of light, which is roughly 300,000 kilometres per second. A computer on board the car – a chip in the box you buy – can work out the distance from your car to any given satellite if it knows how long the signal has taken to travel from the satellite to your car. This is typically about one tenth of a second, but precise time measurement is now easy. The trick is to structure the signal so that it contains information about timing.

In effect, the satellite and the receiver in the car both play the same tune, and compare its timing. The 'notes' coming from the satellite will lag slightly behind those produced in the car. In this analogy, the tunes might go like this:

CAR	... feet, in ancient times, walk upon England's ...
SATELLITE	... And did those feet, in ancient times, walk ...

Here the satellite's song is lagging some three words behind the same song in the car. Both satellite and receiver must generate the same 'song', and successive 'notes' must be distinctive, so that the timing difference is easy to observe.

Of course, the satnav system doesn't actually use a song. The signal is a series of brief pulses whose duration is determined by a 'pseudo-random code'. This is a series of numbers, which looks random but is actually based on some mathematical rule. Both the satellite and the receiver know the rule, so they can generate the same sequence of pulses.

numbers and calculate with them. They used their arithmetical skills for two main purposes: down-to-earth everyday affairs of ordinary human beings, such as land-measurement and accountancy, and highbrow activities like predicting eclipses or recording the movements of planets across the night-time sky.

We do the same today. We use simple mathematics, little more than arithmetic, for hundreds of tiny tasks – how much anti-parasite treatment to put into a garden fishpond, how many rolls of wallpaper to buy to paper the bedroom, whether it will save money to travel further for cheaper petrol. And our culture uses sophisticated mathematics for science, technology and increasingly for commerce too. The inventions of number notation and arithmetic rank alongside those of language and writing as some of the innovations that differentiate us from trainable apes.