CHAPTER 8

The System of the World

The invention of calculus

The most significant single advance in the history of mathematics was calculus, invented independently around 1680 by Isaac Newton and Gottfried Leibniz. Leibniz published first, but Newton – egged on by over-patriotic friends – claimed priority and portrayed Leibniz as a plagiarist. The row soured relations between English mathematicians and those of continental Europe for a century, and the English were the main losers.

The system of the world

Even though Leibniz probably deserves priority, Newton turned calculus into a central technique of the budding subject of mathematical physics, humanity's most effective known route to the understanding of the natural world. Newton called his theory 'The System of the World'. This may not have been terribly modest, but it was a pretty fair description. Before Newton, human understanding of patterns in nature consisted mainly of the ideas of Galileo about moving bodies, in particular the parabolic trajectory of an object such

as a cannonball, and Kepler's discovery that Mars follows an ellipse through the heavens. After Newton, mathematical patterns governed almost everything in the physical world: the movement of terrestrial and heavenly bodies, the flow of air and water, the transmission of heat, light and sound, and the force of gravity.

Curiously, though, Newton's main publication on the mathematical laws of nature, his *Principia Mathematica*, does not mention calculus at all; instead, it relies on the clever application of geometry in the style of the ancient Greeks. But appearances are deceptive: unpublished documents known as the *Portsmouth Papers* show that when he was working on the *Principia*, Newton already had the main ideas of calculus. It is likely that Newton *used* the methods of calculus to make many of his discoveries, but chose not to present them that way. His version of calculus was published after his death in the *Method of Fluxions* of 1732.

Calculus

What is calculus? The methods of Newton and Leibniz are more easily understood if we preview the main ideas. Calculus is the mathematics of instantaneous rates of change – how rapidly is some particular quantity changing *at this very instant*? For a physical example: a train is moving along a track: how fast is it going *right now*? Calculus has two main branches. *Differential calculus* provides methods for calculating rates of change, and it has many geometric applications, in particular finding tangents to curves. *Integral calculus* does the opposite: given the rate of change of some quantity, it specifies the quantity itself. Geometric applications of integral calculus include the computation of areas and volumes. Perhaps the most significant discovery is this unexpected connection between two apparently unrelated classical geometric questions: finding tangents to a curve and finding areas.

Calculus is about *functions*: procedures that take some general number and calculate an associated number. The procedure is usually specified by a formula, assigning to a given number x (possibly in some specific range) an associated number $f(x)$. Examples include the square root function $f(x) = \sqrt{x}$ (which requires x to be positive) and the square function $f(x) = x^2$ (where there is no restriction on x).

The first key idea of calculus is *differentiation*, which obtains the *derivative* of a function. The derivative is the rate at which $f(x)$ is changing, compared to how x is changing – the *rate of change* of $f(x)$ with respect to x.

Geometrically, the rate of change is the slope of the tangent to the graph of f at the value x. It can be approximated by finding the slope of the *secant* – a line that cuts the graph of f at two nearby points, corresponding to x and $x + h$, respectively, where h is small. The slope of the secant is

$$\frac{f(x+h) - f(x)}{h}$$

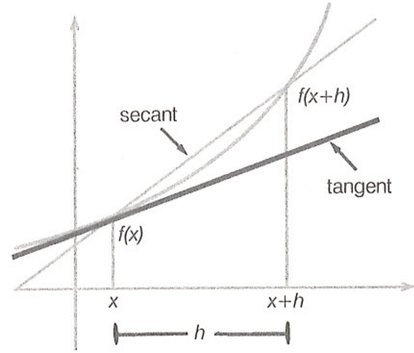
Now suppose that h becomes very small. Then the secant approaches the tangent to the graph at x. So in some sense the required slope – the derivative of f at x – is the *limit* of this expression as h becomes arbitrarily small.

Let's try this calculation with a simple example, $f(x) = x^2$. Now

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2hx + h^2 - x^2}{h} = 2x + h$$

As h becomes very, very small, the slope $2x + h$ gets closer and closer to $2x$. So the derivative of f is the function g for which $g(x) = 2x$. The main conceptual issue here is to define what we mean by limit. It took more than a century to find a logical definition.

The other key idea in calculus is that of *integration*. This is most easily viewed as the reverse process to differentiation. Thus the integral of g, written



Geometry of approximations to the derivative

$$\int g(x)\,dx$$

is whichever function $f(x)$ has derivative $g(x)$. For instance, because the derivative of $f(x) = x^2$ is $g(x) = 2x$, the integral of $g(x) = 2x$ is $f(x) = x^2$. In symbols,

$$\int 2x\,dx = x^2$$

The need for Calculus

Inspiration for the invention of calculus came from two directions. Within pure mathematics, differential calculus evolved from methods for finding tangents to curves, and integral calculus evolved from methods for calculating the areas of plane shapes and the volumes of solids. But the main stimulus towards calculus came from physics – the growing realization that nature has patterns. For reasons we still do not really understand, many of the fundamental patterns in nature involve rates of change. So they make sense, and can be discovered, only through calculus.

Prior to the Renaissance, the most accurate model of the motion of the Sun, Moon and planets was that of Ptolemy. In his model, the Earth was fixed, and everything else – in particular, the Sun – revolved around it on a series of (real or imaginary, depending on taste) circles. The circles originated as spheres in the work of the Greek astronomer Hipparchus; his spheres spun about gigantic axles, some of which were attached to other spheres and moved along with them. This kind of compound motion seemed necessary to model the complex motions of the planets. Recall that some planets, such as Mercury, Venus and Mars, seemed to travel along complicated paths that included loops. Others – Jupiter and Saturn were the only other planets known at that time – behaved more sedately, but even these bodies exhibited strange irregularities, known since the time of the Babylonians.

We have already met Ptolemy's system, known as *epicycles*, which replaced the spheres by circles, but retained the compound motion. Hipparchus's model was not terribly accurate, compared to observations, but Ptolemy's model fitted the observations very accurately indeed, and for over a thousand years it was seen as the last word on the topic. His writings, translated into Arabic as the *Almagest*, were used by astronomers in many cultures.

God v science

Even the *Almagest*, however, failed to agree with all planetary movements. Moreover, it was rather complicated. Around the year 1000, a few Arabian and European thinkers began to wonder whether the daily motion of the Sun might be explained by a rotating Earth, and some of them also played with the idea that the Earth revolves round the Sun. But little came of these speculations at the time.

In Renaissance Europe, however, the scientific attitude began to take root, and one of the first casualties was religious dogma. At that time, the Roman Catholic Church exerted substantial control over its adherents' view of the universe. It wasn't just that the existence of the universe, and its daily unfolding, were credited to the Christian God. The point was that the nature of the universe was believed to correspond to a very literal reading of the Bible.

The Earth was therefore seen as the centre of all things, the solid ground around which the heavens revolved. And human beings were the pinnacle of creation, the reason for the universe's existence.

No scientific observation can ever disprove the existence of some invisible, unknowable creator. But observations can – and did – debunk the view of the Earth as the centre of the universe. And this caused a huge fuss, and got a lot of innocent people killed, sometimes in hideously cruel ways.

Copernicus

The fat hit the fire in 1543, when the Polish scholar Nicholas Copernicus published an astonishing, original and somewhat heretical book: *On the Revolutions of the Heavenly Spheres*. Like Ptolemy, he used epicycles for accuracy. Unlike Ptolemy, he placed the Sun at the centre, while everything else, including the Earth, but excluding the Moon, turned around the Sun. The Moon alone revolved around the Earth.

Copernicus's main reason for this radical proposal was pragmatic: it replaced Ptolemy's 77 epicycles by a mere 34. Among the epicycles envisaged by Ptolemy, there were many repetitions of a particular circle: circles with that specific size, and speed of rotation, kept appearing, associated with many distinct celestial bodies. Copernicus realized that if all these epicycles were transferred to the Earth, only one of them would be needed. We now interpret this in terms of the motion of the planets relative to the Earth. If we mistakenly assume the Earth is fixed, as it seems to be to a naive observer, then the motion of the Earth round the Sun becomes transferred to all of the planets as an additional epicycle.

Another advantage of Copernicus's theory was that it treated all the planets in exactly the same manner. Ptolemy needed different mechanisms to explain the inner planets and the outer ones. Now, the only difference was that the inner planets were closer to the Sun than the Earth was, while the outer planets were further away. It all made excellent sense – but on the whole it was rejected, for a variety of reasons, not all of them religious.

Copernicus's theory was complicated, unfamiliar and his book was difficult to read. Tycho Brahe, one of the best astronomical observers of the period, found discrepancies between Copernicus's heliocentric theory and some subtle observations, which also disagreed with Ptolemy's theory; he tried to find a better compromise.

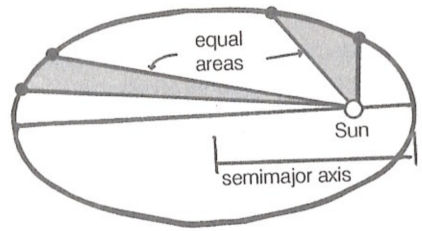
Kepler

When Brahe died, his papers were inherited by Kepler, who spent years analysing the observations, looking for patterns. Kepler was something of a mystic, in the Pythagorean tradition, and he tended to impose artificial patterns on observational data. The most famous of these abortive attempts to find regularities in the heavens was his beautiful but utterly misguided explanation of the spacing of the planets in terms of the regular solids. In his day, the known planets were six in number: Mercury, Venus, Earth, Mars, Jupiter and Saturn. Kepler wondered whether their distances from the Sun had a geometric pattern. Moreover, he wondered why there were six planets. He noticed that six planets leave room for five intervening shapes, and since there were exactly five regular solids, this would explain the limit of six planets. He came up with a series of six spheres lying one inside the other, each one bearing the orbit of a planet around its equator. Between the spheres, nestling tightly outside one sphere and inside the next, he placed the five solids, in the order

Mercury
Octahedron
Venus
Icosahedron

Earth
Dodecahedron
Mars
Tetrahedron
Jupiter
Cube
Saturn

The numbers fitted reasonably well, especially given the limited accuracy of observations at that time. But there are 120 different ways to rearrange the five solids, which between them give an awful lot of different spacings. It is hardly surprising that one of these was in reasonably close agreement with reality. The later discovery of more planets knocked this particular piece of pattern-seeking firmly on the head, consigning it to the waste bin of history.



Planet moves over given time interval

Along the way, though, Kepler discovered some patterns that we still recognize as genuine, now called *Kepler's Laws of Planetary Motion*. He extracted them, over some twenty years of calculation, from Brahe's observations of Mars. The laws state:

(i) Planets move round the Sun in elliptical orbits.
(ii) Planets sweep out equal areas in equal times.
(iii) The square of the period of revolution of any planet is proportional to the cube of its average distance from the Sun.

Johannes Kepler
1571–1630

Kepler was the son of a mercenary and an innkeeper's daughter. As a child, he lived with his mother in his grandfather's inn after the death of his father, probably in a war between the Netherlands and the Holy Roman Empire. He was mathematically precocious, and in 1589 he studied astronomy under Michael Maestlin at the University of Tübingen. Here he came to grips with the Ptolemaic system. Most astronomers of the period were more concerned with calculating orbits than in asking how the planets really moved, but from the beginning Kepler was interested in the precise paths followed by the planets, rather than the proposed system of epicycles. He became acquainted with the Copernican system, and was quickly convinced that it was literally true, and not just a mathematical trick.

In 1596 he made his first attempt to find patterns in the movements of the planets, via his *Mysterium Cosmographicum* (Mystery of the Cosmos), with its strange model based on regular solids. This model did not agree well with observations, so Kepler wrote to a leading observational astronomer, Tycho Brahe. Kepler became Brahe's mathematical assistant, and was set to work on the orbit of Mars. After Brahe's death, he continued to work on the problem. Brahe had left a wealth of data, and Kepler struggled to fit a sensible orbit to them. The surviving calculations occupy nearly 1000 pages, which Kepler referred to as 'my war with Mars'. His final orbit was so precise that the only difference from modern data arises from minute drifting of the orbit over the intervening centuries.

1611 was a bad year. Kepler's son died aged seven. Next, his wife died. Then Emperor Rudolf, who tolerated Protestants, abdicated, and Kepler was forced to leave Prague. In 1613

Kepler remarried, and a problem that occurred to him during their wedding celebrations led to the writing of his *New Stereometry of Wine Barrels* of 1615.

In 1619 he published *Harmonices Mundi* (The Harmony of the World), a sequel to his Mystery of the Cosmos. The book contained a wealth of new mathematics, including tiling patterns and polyhedra. It also formulated the third law of planetary motion. While he was writing the book, his mother was accused of being a witch. With the aid of the law faculty at Tübingen, she was eventually freed, partly because the prosecutors had not followed the correct legal procedures for torture.

The most unorthodox feature of Kepler's work is that he discarded the classical circle (allegedly the most perfect shape possible) in favour of the ellipse. He did so with some reluctance, saying himself that he only settled on the ellipse when everything else had been eliminated. There is no particular reason to expect these three laws to bear any closer relation to reality than the hypothetical arrangement of regular solids, but as it happened, the three laws were of real scientific significance.

Galileo

Another major figure of the period was Galileo Galilei, who discovered mathematical regularities in the movement of a pendulum and in falling bodies. In 1589, as professor of mathematics at the University of Pisa, he carried out experiments on bodies rolling down an inclined plane, but did not publish his results. It was at this time that he realized the importance of controlled experiments in the study of natural phenomena, an idea that is now fundamental to all science. He took up astronomy, making a series of fundamental discoveries, which eventually led him to espouse the Copernican theory of the Sun as the centre of

Galileo Galilei
1564–1642

Galileo was the son of Vincenzo Galilei, a music teacher who had performed experiments with strings to validate his musical theories. At the age of ten Galileo went to a monastery at Vallombrosa to be educated, with a view to becoming a doctor. But Galileo wasn't really interested in medicine, and spent his time doing mathematics and natural philosophy – what we now call science.

In 1589 Galileo became professor of mathematics at the University of Pisa. In 1591 he took up a better-paid position in Padua, where he taught Euclidean geometry and astronomy to medical students. At that time doctors made use of astrology in their treatment of patients, so these topics were a necessary part of the curriculum.

Learning about the invention of the telescope, Galileo built one for himself and became so proficient that he gave his methods to the Venetian Senate, granting them sole rights in their use in return for a salary increase. In 1609 Galileo observed the heavens, making discovery after discovery: four of Jupiter's moons, individual stars within the Milky Way, mountains on the Moon. He presented a telescope to Cosimo de Medici, Grand Duke of Tuscany, and soon became the Duke's chief mathematician.

He discovered the existence of sunspots and published this observation in 1612. By now his astronomical discoveries had convinced him of the truth of Copernicus's heliocentric theory, and in 1616 he made his views explicit in a letter to the Grand Duchess Christina, saying that the Copernican theory represents physical reality and is not just a convenient way to simplify calculations.

At this time, Pope Paul V ordered the Inquisition to decide on the truth or falsity of the heliocentric theory, and they declared it

false. Galileo was instructed not to advocate the theory, but a new pope was elected, Urban VIII, who seemed more relaxed about the issue, so Galileo did not take the prohibition very seriously. In 1623 he published *Il Saggiatore* (*The Assayer*), dedicating it to Urban. In it, he made a famous statement that the universe 'is written in the language of mathematics, and its characters are triangles, circles and other geometric figures, without which it is humanly impossible to understand a single word of it'.

In 1630 Galileo asked permission to publish another book, *Dialogue Concerning the Two Chief Systems of the World*, about the geocentric and heliocentric theories. In 1632, when permission arrived from Florence (but not Rome), he went ahead. The book claimed to prove that the Earth moves, the main evidence being the tides. In fact Galileo's theory of tides was completely wrong, but the Church authorities saw the book as theological dynamite and the Inquisition banned it, summoning Galileo to Rome to be tried for heresy. He was found guilty, but escaped with a sentence of life imprisonment, in the form of house arrest. In this he fared better than many other heretics, for whom being burnt at the stake was a common punishment. While under house arrest he wrote the *Discourses*, explaining his work on moving bodies to the outside world. It was smuggled out of Italy and published in Holland.

the solar system. This set him on a collision course with the Church, and he was eventually tried for heresy and placed under house arrest.

During the last few years of his life, his health failing, he wrote *Discourses and Mathematical Demonstrations Concerning the Two New Sciences*, explaining his work on the motion of bodies on inclined planes. He stated that the distance an initially stationary body moves under uniform acceleration is proportional to the square of the time. This law is the basis of his earlier discovery that a projectile follows a parabolic path. Together with Kepler's laws of planetary motion, it brought into being a new subject: *mechanics*, the mathematical study of moving bodies.

That's the physical astronomical background that led up to calculus. Next, we'll look at the mathematical background.

The invention of calculus

The invention of calculus was the outcome of a series of earlier investigations of what seem to be unrelated problems, but which possess a hidden unity. These included calculating the instantaneous velocity of a moving object from the distance it has travelled at any given time, finding the tangent to a curve, finding the length of a curve, finding the maximum and minimum values of a variable quantity, finding the area of some shape in the plane and the volume of some solid in space. Some important ideas and examples were developed by Fermat, Descartes and the more obscure Englishman, Isaac Barrow, but the methods remained special to particular problems. A general method was needed.

Leibniz

The first real breakthrough was made by Gottfried Wilhelm Leibniz, a lawyer by profession, who devoted much of his life to mathematics, logic, philosophy, history and many branches of science. Around 1673 he began work on the classical problem of finding the tangent to a curve, and noticed that this was in effect the inverse problem to that of finding areas and volumes. The latter boiled down to finding a curve given its tangents; the former problem was exactly the reverse.

Leibniz used this connection to define what, in effect, were integrals, using the abbreviation, *omn* (an abbreviation of *omnia*, the Latin word for 'all'). Thus we find, in his manuscript papers, formulas such as

$$\text{omn } x^2 = \frac{x^3}{3}$$

By 1675 he had replaced *omn* by the symbol $\int$ still used today,

which is an old-fashioned elongated letter s, standing for sum. He worked in terms of small increments dx and dy to the quantities x and y, and used their ratio dy/dx to determine the rate of change of y as a function of x. Essentially, if f is a function then Leibniz wrote

$$dy = f(x + dx) - f(x)$$

so that

$$\frac{dy}{dx} = \frac{f(x + dx) - f(x)}{dx}$$

which is the usual secant approximation to the slope of the tangent.

Leibniz recognized that this notation has its problems. If dy and dx are non-zero then dy/dx is not the instantaneous rate of change of y, but an approximation. He tried to circumvent this problem by assuming dx and dy to be infinitesimally small. An *infinitesimal* is a non-zero number that is smaller than any other non-zero number. Unfortunately, it is easy to see that no such number can exist (half an infinitesimal is also non-zero, but smaller) so this approach does little more than displace the problem elsewhere.

By 1676 Leibniz knew how to integrate and differentiate any power of x, writing the formula

$$dx^n = nx^{n-1}dx$$

which we would now write as

$$\frac{d}{dx} x^n = nx^{n-1}$$

In 1677 he derived rules for differentiating the sum, product and quotient of two functions, and by 1680 he had obtained the formula for the length of an arc of a curve, and the volume of a solid of revolution, as integrals of various related quantities.

Although we know these facts, and the associated dates, from his unpublished notes, he first published his ideas on calculus rather later, in 1684. Jakob and Johann Bernoulli found this paper rather obscure, describing it as 'an enigma rather than an explanation'. With hindsight, we see that by that time Leibniz had discovered a significant part of basic calculus, with applications to complicated curves like the cycloid, and a sound grasp of concepts such as curvature. Unfortunately, his writings were fragmented and virtually unreadable.

Newton

The other creator of calculus was Isaac Newton. Two of his friends, Isaac Barrow and Edmond Halley, came to recognize his remarkable abilities, and encouraged him to publish his work. Newton disliked being criticized, and when in 1672 he published his ideas about light, his work provoked a storm of criticism, which reinforced his reluctance to commit his thoughts to print. Nevertheless, he continued to publish sporadically, and wrote two books. In private he continued developing his ideas about gravity, and in 1684 Halley tried to convince Newton to publish the work. But aside from Newton's general misgivings about criticism, there was a technical obstacle. He had been forced to model planets as point particles, with non-zero mass but zero size, which he felt was unrealistic and would invite criticism. He wanted to replace these unrealistic points by solid spheres, but he could not prove that the gravitational attraction of a sphere is the same as that of a point particle of the same mass.

In 1686 he succeeded in filling the gap, and the *Principia* saw the light of day in 1687. It contained many novel ideas. The most important were mathematical laws of motion, extending the work of Galileo, and gravity, based on the laws found by Kepler.

Newton's main law of motion (there are some subsidiary ones) states that the acceleration of a moving body, multiplied by its mass, is equal to the force that acts on the body. Now velocity is the derivative of position, and acceleration is the derivative of velocity.

Isaac Newton
1642–1727

Newton lived on a farm in the tiny village of Woolsthorpe, in Lincolnshire. His father had died two months before he was born, and his mother managed the farm. He was educated in very ordinary local schools, and exhibited no special talent of any kind, except for a facility with mechanical toys. He once made a hot air balloon and tested it out with the family cat as pilot; the balloon and the cat were never seen again. He went to Trinity College at Cambridge University, having done reasonably well in most of his examinations – except geometry. As an undergraduate, he made no great impact.

The Plague

Then, in 1665, the great plague began to devastate London and the surrounding area, and the students were sent home before the same thing happened in Cambridge. Back in the family farmhouse, Newton began thinking much more deeply about scientific and mathematical matters.

Gravity

During 1665–66 he devised his law of gravity to explain planetary motion, developed the laws of mechanics to explain and analyse any kind of moving body or particle, invented both differential and integral calculus, and made major advances in optics. Characteristically, he published none of this work, returning quietly to Trinity to take his master's degree and being elected a fellow of the college. Then he secured the position of Lucasian Professor of Mathematics, when the incumbent, Barrow, resigned in 1669. He gave very ordinary lectures, rather badly, and very few undergraduates went to them.

So even to *state* Newton's Law we need the *second derivative* (the derivative of the derivative) of position with respect to time, nowadays written

$$\frac{d^2x}{dt^2}$$

Newton wrote two dots over the top of *x* instead ($\ddot{x}$).

The law of gravity states that any two particles of matter attract each other with a force that is proportional to their masses, and inversely proportional to the square of the distance between them. So, for example, the force attracting the Earth to the Moon would become one quarter as great if the Moon were removed to twice its distance, or one ninth as great if its distance were tripled. Again, because this law is about forces, it involves the second derivative of position.

Newton deduced this law from Kepler's three laws of planetary motion. The published deduction was a masterpiece of classical Euclidean geometry. Newton chose this style of presentation because it involved familiar mathematics, and so could not easily be criticized. But many aspects of the *Principia* owed their genesis to Newton's unpublished invention of calculus.

Among his earliest work on the topic was a paper titled *On Analysis by Means of Equations with an Infinite Number of Terms*, which he circulated to a few friends in 1669. In modern terminology, he asked what the equation of a function f(x) is, if the area under its graph is of the form x^m. (Actually he asked something slightly more general, but let's keep it simple.) He deduced, to his own satisfaction, that the answer is $f(x) = mx^{m-1}$.

Newton's approach to calculating derivatives was much like that of Leibniz, except that he used *o* in place of dx, so his method suffers from the same logical problem: it seems to be only approximate. But Newton could show that by assuming *o* to be very small, the approximation would become ever better. In the limit, when *o*

becomes as small as we please, the error vanishes. So, Newton maintained, his end result was *exact*. He introduced a new word, *fluxion*, to capture the main idea – that of a quantity flowing towards zero but not actually getting there.

In 1671 he wrote a more extensive treatment, the *Method of Fluxions and Infinite Series*. The first book on calculus was not published until 1711; the second appeared in 1736. It is clear that by 1671 Newton possessed most of the basic ideas of calculus.

Objectors to this procedure, notably Bishop George Berkeley in his 1734 book *The Analyst, a Discourse Addressed to an Infidel Mathematician*, pointed out that it is illogical to divide numerator and denominator by *o* when later *o* is set to 0. In effect, the procedure conceals the fact that the fraction is actually 0/0, which is well known to be meaningless. Newton responded that he was not actually setting *o* equal to 0; he was working out what happened when *o* became as close as we wish to 0 *without actually getting there*. The method was about fluxions, not numbers.

The mathematicians sought refuge in physical analogies – Leibniz referred to the 'spirit of finesse' as opposed to the 'spirit of logic' – but Berkeley was perfectly correct. It took more than a century to find a good answer to his objections, by defining the intuitive notion of 'passing to a limit' in a rigorous manner. Calculus then turned into a more subtle subject, *analysis*. But for a century after the invention of calculus, nobody except Berkeley worried much about its logical foundations, and calculus flourished despite this flaw.

It flourished because Newton was right, but it would take nearly 200 years before his concept of a fluxion was formulated in a logically acceptable way, in terms of limits. Fortunately for mathematics, progress was not halted until a decent logical foundation was discovered. Calculus was too useful, and too important, to be held up over a few logical quibbles. Berkeley was incensed, maintaining that the method seemed to work only because various errors cancelled each other out. He was right – but he failed to enquire *why* they always cancelled out. Because if that were the case, they weren't really errors at all!

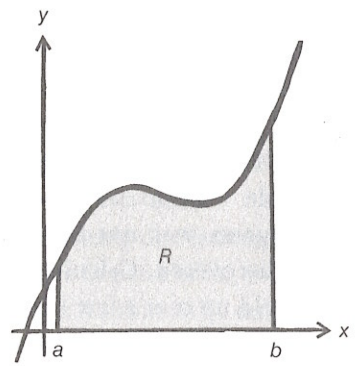
Associated with differentiation is the reverse process, *integration*. The integral of $f(x)$, written, $\int f(x)dx$, is whichever function yields $f(x)$ when it is differentiated. Geometrically it represents the area under the graph of the function f. The *definite integral* $\int_a^b f(x)dx$ is the area under this graph between the values $x = a$ and $x = b$.

Derivatives and integrals solved problems that had taxed the ingenuity of previous mathematicians. Velocities, tangents, maxima and minima could all be found using differentiation. Lengths, areas and volumes could be calculated by integration. But there was more. Surprisingly, it seemed that the patterns of nature were written in the language of the calculus.

The English get left behind

As the importance of the calculus became ever clearer, greater prestige became attached to its creator. But who *was* the creator?

We have seen that Newton began thinking about calculus in 1665, but did not publish anything on the topic until 1687. Leibniz, whose ideas ran along roughly similar lines to Newton's, had started working on calculus in 1673, and published his first papers on the topic in 1684. The two worked independently, but Leibniz *might* have learned about Newton's work when he visited Paris in 1672 and London in 1673; Newton had sent a copy of *On Analysis* to Barrow in 1669, and Leibniz talked to several people who also knew Barrow and so might have known about this work.



The definite integral

What calculus did for them

An early use of calculus to understand natural phenomena was the question of the shape of a hanging chain. The question was controversial; some mathematicians thought the answer was a parabola, others disagreed. In 1691 Leibniz, Christiaan Huygens and Joahnn Bernoulli all published proposed solutions. The clearest was Bernoulli's. He wrote down a differential equation to describe the position of the chain, based on Newtonian mechanics and Newton's laws of motion.

The solution, it turned out, was not a parabola, but a curve known as a catenary, with equation

$$y = k(e^x + e^{-x})$$

for constant *k*.

The cables of suspension bridges, however, are parabolic. The difference arises because these cables carry the weight of the bridge, as well as their own weight. Again, this can be demonstrated using calculus.

When Leibniz published his work in 1684, some of Newton's friends took umbrage – probably because Newton had been pipped to the publication post and they all belatedly realized what was at stake – and accused Leibniz of stealing Newton's ideas. The continental mathematicians, especially the Bernoullis, leaped to Leibniz's defence, suggesting that it was Newton, not Leibniz, who was guilty of plagiarism. In point of fact, both men had made their discoveries pretty much independently, as their unpublished manuscripts show; to muddy the waters, both had leaned heavily on previous work of Barrow, who probably had better grounds for grievance than either of them.

The accusations could easily have been withdrawn, but instead the dispute grew more heated; John Bernoulli extended his distaste from Newton to the entire English nation. The end result was a disaster for English mathematics, because the English doggedly stuck with Newton's geometric style of thinking, which was difficult to use, whereas the continental analysts employed Leibniz's more formal, algebraic methods, and pushed the subject ahead at a rapid pace. Most of the payoff in mathematical physics therefore went to the French, Germans, Swiss and Dutch, while English mathematics languished in a backwater.

The differential equation

The most important single idea to emerge from the flurry of work on calculus was the existence, and the utility, of a novel kind of equation – the *differential equation*. Algebraic equations relate various powers of an unknown number. Differential equations are much grander: they relate various derivatives of an unknown function.

Newton's laws of motion tell us that if $y(t)$ is the height of a particle moving under gravity near the Earth's surface, then the second derivative d^2y/dt^2 is proportional to the force g that acts; specifically,

$$g = m\frac{d^2y}{dt^2}$$

where m is the mass of the particle. This equation does not specify the function y directly. Instead, it specifies a property of its second derivative. We must solve the differential equation to find y itself. Two successive integrations yield the solution

$$y = \frac{gt^2}{2m} + at + b$$

What calculus does for us

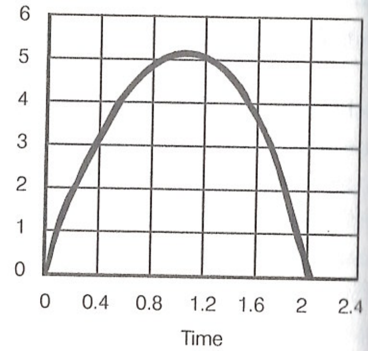
Differential equations abound in science: they are by far the commonest way to model natural systems. To choose one application at random, they are used routinely to calculate the trajectories of space probes, such as the Mariner mission to Mars, the two Pioneer craft that explored the solar system and gave us such wonderful images of Jupiter, Saturn, Uranus and Neptune, and the recent Mars Rovers *Spirit* and *Opportunity*, six-wheeled robot vehicles that explored the Red Planet.

The Cassini mission, currently exploring Saturn and its moons, is another example. Among its discoveries is the existence of lakes of liquid methane and ethane on Saturn's moon Titan. Of course, calculus is not the only technique used by space missions – but without it, these missions would literally never have got off the ground.

More practically, every aircraft that flies, every car that travels the road and every suspension bridge and earthquake-proof building owes its design in part to calculus. Even our understanding of how animal populations change size over time stems from differential equations. The same goes for the spread of epidemics, where calculus models are used to plan the most effective way to intervene and prevent disease spreading. A recent model of the foot and mouth disease epidemic in the UK has shown that the strategy adopted at the time was not the best available.

where b is the initial height of the particle and a is its initial velocity. The formula tells us that the graph of height y against time t is an upside-down parabola. This is Galileo's observation.

The pioneering efforts of Copernicus, Kepler, Galileo and other Renaissance scientists led to the discovery of mathematical patterns in the natural world. Some apparent patterns turned out to be



Parabolic trajectory of a projectile

spurious, and were discarded; others provided very accurate models of nature, and were retained and developed. From these early beginnings, the notion that we live in a 'clockwork universe', running according to rigid, unbreakable rules, emerged, despite serious religious opposition, mainly from the Church of Rome.

Newton's great discovery was that nature's patterns seem to manifest themselves not as regularities in certain quantities, but as relations among their *derivatives*. The laws of nature are written in the language of calculus; what matters are not the values of physical variables, but the rates at which they change. It was a profound insight, and it created a revolution, leading more or less directly to modern science, and changing our planet forever.

CHAPTER 9

Patterns in Nature

Formulating laws of physics

The main message in Newton's *Principia* was not the specific laws of nature that he discovered and used, but the idea that such laws exist – together with evidence that the way to model nature's laws mathematically is with differential equations. While England's mathematicians engaged in sterile vituperation over Leibniz's alleged (and totally fictitious) theft of Newton's ideas about calculus, the continental mathematicians were cashing in on Newton's great insight, making important inroads into celestial mechanics, elasticity, fluid dynamics, heat, light and sound – the core topics of mathematical physics. Many of the equations that they derived remain in use to this day, despite – or perhaps because of – the many advances in the physical sciences.

Differential equations

To begin with, mathematicians concentrated on finding explicit formulas for solutions of particular kinds of ordinary differential equation. In a way this was unfortunate, because formulas of this type usually fail to exist, so attention became focused on equations that could be solved by a formula rather than equations that