Exercise 12 - Algebra 3

- Given a comm ring R, the tensor product $M \otimes N$ of R-modules is the univ. bilin map $M \times N \xrightarrow{\otimes} M \otimes N$.

- It is freely generated $m \otimes n$ for $m \in M$, $n \in N$ sub to relo: $r(m \otimes n) = rm \otimes n = m \otimes rn$

$(a+b) \otimes n = a \otimes n + b \otimes n$

$m \otimes (a+b) = m \otimes a + m \otimes b$.

1. Prove that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong 0$ if n, m are coprime integers.

2. Let $R[x_1, \ldots, x_n]$ be the ring of polynomials (elements are R-linear comb of monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$)

   Show there is an isomorphism

   $R[x_1, \ldots, x_n] \otimes_R R[y_1, \ldots, y_m] \cong R[x_1, \ldots, x_n, y_1, \ldots, y_m]$.

③ • Let M, N be R-modules. A "pure tensor" in $M \otimes N$ is an element of the form $m \otimes n$ for $m \in M$, $n \in N$.

Show that if N is cyclic (generated by a single element) then each tensor in $M \otimes N$ is pure.