Exercises 4

(1) Consider the functor cat $[A,B]$, whose objects are functors and arrows are natural transformations. Prove that $\eta : F \to G \in [A,B]$ is an isomorphism $\iff$ $\eta_x : Fx \to Gx$ is an iso in B for all $x \in A$.

(2) Let $PX = \{U : U \subseteq X\}$ be the set of subsets of X.
Extend P to a functor $Set \xrightarrow{P} Set$ & describe a natural transformation $1 \Rightarrow P$.

(3) Consider the adjoint functors

$$\begin{tikzcd} Vect \arrow[r, shift left, "U"'] \arrow[r, phantom, "\perp"] & Set \arrow[l, shift left, "F"'] \end{tikzcd}$$

Describe a natural transformation $\varepsilon : FU \Longrightarrow Id$.

(4) - Let $\mathcal{A}, \mathcal{B}$ be categories. The product category $\mathcal{A} \times \mathcal{B}$ has objects (a,b) where $a \in \mathcal{A}$, $b \in \mathcal{B}$ & morphisms $(a,b) \xrightarrow{(f,g)} (c,d)$ where $f: a \to c$ & $g: b \to d$.

- Let $F, G : A \rightrightarrows B$ be functors, and $J = \{0 \to 1\}$.

Show that natural transformations $F \Rightarrow G$ are the same thing as functors

$$H : J \times A \to B$$

such that the diagram

$$
\begin{tikzcd}
a \arrow[d, mapsto] & A \arrow[d, "i_0"] \arrow[dr, "F"] & \\
(0,a) & J \times A \arrow[r, "H"] & B \\
(1,a) & A \arrow[u, "i_1"'] \arrow[ur, "G"'] & \\
a \arrow[u, mapsto] & &
\end{tikzcd}
$$

commutes.