Basics of radiation hydrodynamics

Let there be light

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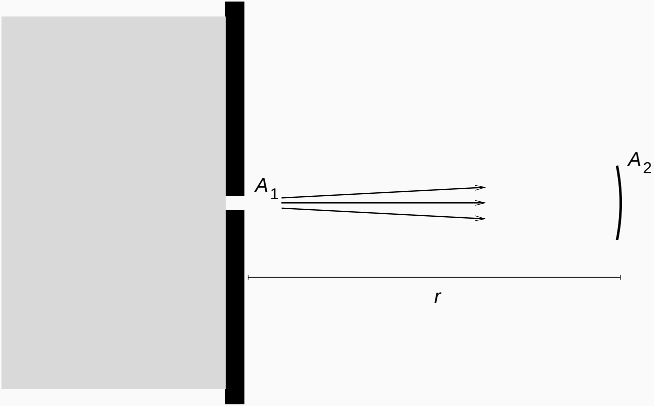
Desription of radiation

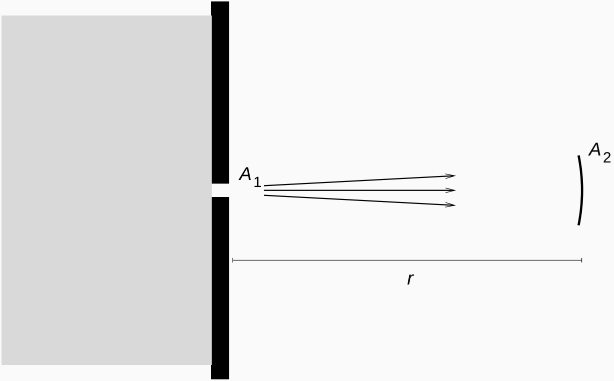
Definition of specific intensity

The *specific intensity* of radiation can be defined using an ideal apparatues (a pinhole camera). The energy collected by the detector during time Δt in a bandwidth $\Delta \nu$ is

$$\Delta E = I_\nu \frac{A_1 A_2}{r^2} \Delta t \Delta \nu.$$



Intricate dependences

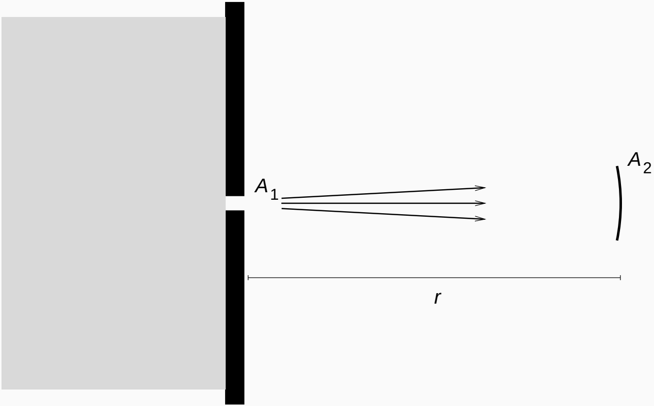


The specific intensity $I(\boldsymbol{r}, \boldsymbol{n}, \nu, t)$ depends on

- $\boldsymbol{r}$: location of the pinhole,
- t: time,
- $\boldsymbol{n}$: orientation of the screen,
- ν: frequency.

} tangent spaces $(\boldsymbol{n}, \nu)$ attached to fourdimensional spacetime manifold $(\boldsymbol{r}, t)$.

Radial dependency



$$\Delta E = I_\nu \frac{A_1 A_2}{r^2} \Delta t \Delta \nu.$$

Here $\Delta\Omega = A_2/r^2$ is the solid angle subtended by A_2 at the aperture. This means that the intensity does not depend on the location of the detector. Intensity does not change as the bundle of radiation moves.

Radiation transport equation

Intensity does not change as the bundle of radiation moves over time τ:

$$\Delta I_\nu = I_\nu(\boldsymbol{r} + \boldsymbol{n}c\tau, \boldsymbol{n}, t + \tau) - I_\nu(\boldsymbol{r}, \boldsymbol{n}, t) = 0.$$

Taylor-expanding the left-hand side and discarding terms of order τ^2 and higher we obtain radiation transport equation in an empty space

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \boldsymbol{n} \cdot \nabla I_\nu = 0.$$

Change of the intensity due to absorption over time τ: $\Delta I_\nu = -k_\nu c\tau I_\nu$.
Change of the intensity due to emission over time τ: $\Delta I_\nu = j_\nu c\tau$.

Summing all the contributions we arive at the (nonrelativistic) radiation transport equation in the form of

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \boldsymbol{n} \cdot \nabla I_\nu = j_\nu - k_\nu I_\nu.$$

Radiation transport equation: the quantities

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \boldsymbol{n} \cdot \nabla I_\nu = j_\nu - k_\nu I_\nu$$

- I_ν is the *specific intensity*, which is connected with the phase space density of photons

$$f(\boldsymbol{r}, \boldsymbol{p}, t) = \frac{c^2}{h^4 \nu^3} I_\nu,$$

  where $f(\boldsymbol{r}, \boldsymbol{p}, t)$ appears in the Boltzmann equation. (Note: radiation transport equation can be derived from the Boltzmann equation.)
- k_ν is the *absorption* (extinction) *coefficient*
- j_ν is the *emission coefficient* (emissivity)

Coupling with hydrodynamics

First moment of the radiation transport equation

Taking advantage of the constancy of $\boldsymbol{n}$:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \nabla \cdot (\boldsymbol{n} I_\nu) = j_\nu - k_\nu I_\nu.$$

Integrating over the angles:

$$\frac{1}{c}\frac{\partial}{\partial t}\oint I_\nu \, \mathrm{d}\Omega + \nabla \cdot \oint \boldsymbol{n} I_\nu \, \mathrm{d}\Omega = \oint (j_\nu - k_\nu I_\nu) \, \mathrm{d}\Omega.$$

Denoting the radiation energy density E_ν and the vector flux $\boldsymbol{F}_\nu$

$$E_\nu = \frac{1}{c}\oint I_\nu \, \mathrm{d}\Omega \qquad\qquad \boldsymbol{F}_\nu = \oint \boldsymbol{n} I_\nu \, \mathrm{d}\Omega$$

we rewrite the first moment of the radiation transport equation as

$$\frac{\partial E_\nu}{\partial t} + \nabla \cdot \boldsymbol{F}_\nu = \oint (j_\nu - k_\nu I_\nu) \, \mathrm{d}\Omega.$$

First moment of the radiation transport equation

Integrating the momentum equation

$$\frac{\partial E_\nu}{\partial t} + \nabla \cdot \boldsymbol{F}_\nu = \oint (j_\nu - k_\nu I_\nu)\, \mathrm{d}\Omega.$$

over frequencies we obtain

$$\frac{\partial E}{\partial t} + \nabla \cdot \boldsymbol{F} = \int \mathrm{d}\nu \oint (j_\nu - k_\nu I_\nu)\, \mathrm{d}\Omega,$$

where frequency-integrated radiation energy density is $E = \int \mathrm{d}\nu\, E_\nu$ and the vector flux is $\boldsymbol{F} = \int \mathrm{d}\nu \boldsymbol{F}_\nu$.

Derived equation represents the equation of energy conservation. The terms on the left-hand side represent the conservation law with energy density and energy flux. The terms on the right-hand side describe the rates of gain (due to emission) and loss (due to absorption) of radiation energy per unit of volume.

Second moment of the radiation transport equation

Multiplying the radiation transport equation

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \nabla \cdot (\boldsymbol{n} I_\nu) = j_\nu - k_\nu I_\nu$$

by $\boldsymbol{n}$ and integrating over the angles:

$$\frac{1}{c}\frac{\partial}{\partial t}\oint \boldsymbol{n} I_\nu \,\mathrm{d}\Omega + \nabla \cdot \oint \boldsymbol{n}\boldsymbol{n} I_\nu \,\mathrm{d}\Omega = \oint (\boldsymbol{n} j_\nu - k_\nu \boldsymbol{n} I_\nu)\,\mathrm{d}\Omega.$$

Denoting the vector flux $\boldsymbol{F}_\nu$ and the pressure tensor P_ν

$$\boldsymbol{F}_\nu = \oint \boldsymbol{n} I_\nu \,\mathrm{d}\Omega \qquad \mathrm{P}_\nu = \frac{1}{c}\oint \boldsymbol{n}\boldsymbol{n} I_\nu \,\mathrm{d}\Omega$$

and assuming isotropy of j_ν and k_ν we derive the second momentum equation

$$\frac{1}{c}\frac{\partial \boldsymbol{F}_\nu}{\partial t} + c\nabla \cdot \mathrm{P}_\nu = -k_\nu \boldsymbol{F}_\nu.$$

Second moment of the radiation transport equation

Integrating the second momentum equation

$$\frac{1}{c}\frac{\partial \boldsymbol{F}_\nu}{\partial t} + c\nabla\cdot \mathsf{P}_\nu = -k_\nu \boldsymbol{F}_\nu.$$

over frequencies wind dividing by c we obtain

$$\frac{1}{c^2}\frac{\partial \boldsymbol{F}}{\partial t} + \nabla\cdot \mathsf{P} = -\int \mathrm{d}\nu\, k_\nu c\frac{\boldsymbol{F}_\nu}{c^2},$$

where frequency-integrated vector flux is $\boldsymbol{F} = \int \mathrm{d}\nu \boldsymbol{F}_\nu$ and the pressure tensor is $\mathsf{P} = \int \mathrm{d}\nu \mathsf{P}_\nu$.

The radiation momentum density is $\boldsymbol{F}/c^2$ and the momentum flux is P. Therefore the terms on the left-hand side represent the conservation law with momentum density and momentum flux. The right-hand side represents the momentum lost per unit time.

Coupling with Euler's equaions

The continuity equation remains to be the same:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0.$$

The loss of photon momentum is the gain of momentum of matter. Therefore, the negative of the photom momentum loss rate is the radiative force that shall be included in the momentum equation

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \rho \boldsymbol{g} + \frac{1}{c} \int \mathrm{d}\nu \, k_\nu \boldsymbol{F}_\nu.$$

Similarly, the loss of radiation energy is the gain of energy of matter. As a result, the negative of the energy loss rate shall be included in the equation for energy

$$\frac{\partial}{\partial t} \left(\rho \epsilon + \frac{\rho v^2}{2} \right) + \nabla \cdot \left[\rho \boldsymbol{v} \left(\epsilon + \frac{v^2}{2} \right) + p \boldsymbol{v} \right] = \rho \boldsymbol{v} \boldsymbol{g} - \int \mathrm{d}\nu \oint (j_\nu - k_\nu I_\nu) \, \mathrm{d}\Omega.$$

Field criterion of thermal instability

Equilibrium between heating and cooling

In absence of macroscopic motion, there should be equilibrium between heating and cooling:

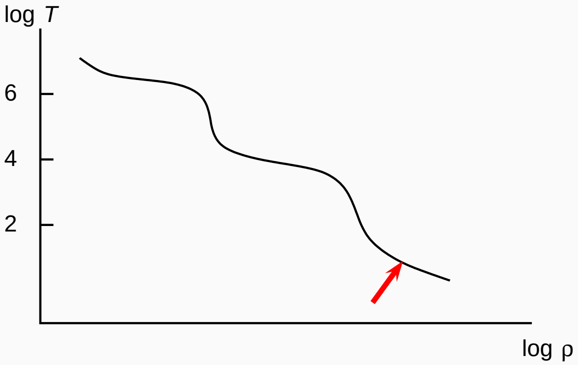
$$\int \mathrm{d}\nu \oint (j_\nu - k_\nu I_\nu)\,\mathrm{d}\Omega = 0.$$

For optically thin gas in LTE assuming external source of heat this simplifies to (Field 1965, Lepp et al. 1985)

$$\mathcal{L}(\rho, T) = 0.$$

Equilibrium between heating and cooling

$$\mathcal{L}(\rho, T) = 0$$



- excitation of rotational levels of molecules and fine structure levels of atoms, strong dependence of $\mathcal{L}$ on T

Equilibrium between heating and cooling

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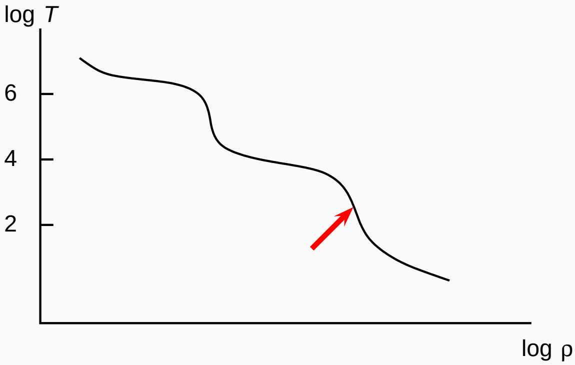

- rotational levels of molecules and fine structure levels of atoms exctited, corresponding Boltzmann factors $e^{-\epsilon/kT} \sim 1$, weak dependence of $\mathcal{L}$ on T

Equilibrium between heating and cooling

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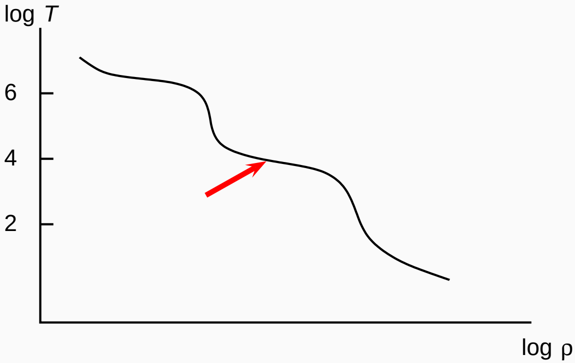

- excitation of levels of atoms and ions, strong dependence of $\mathcal{L}$ on T

Equilibrium between heating and cooling

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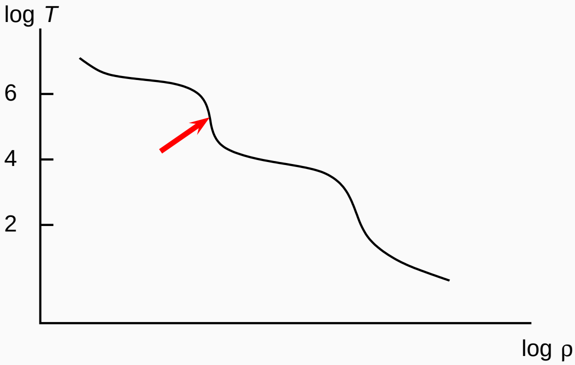

- levels of atoms and ions excited, corresponding Boltzmann factors $e^{-\epsilon/kT} \sim 1$, weak dependence of $\mathcal{L}$ on T

Equilibrium between heating and cooling

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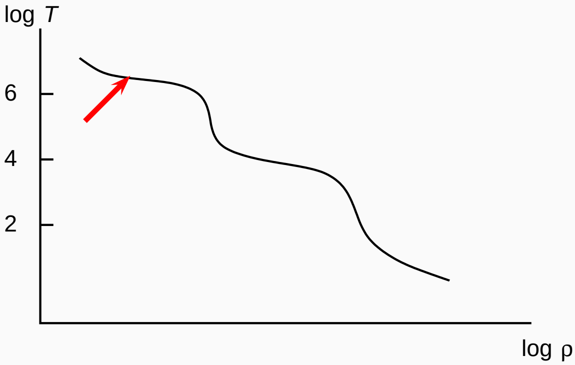
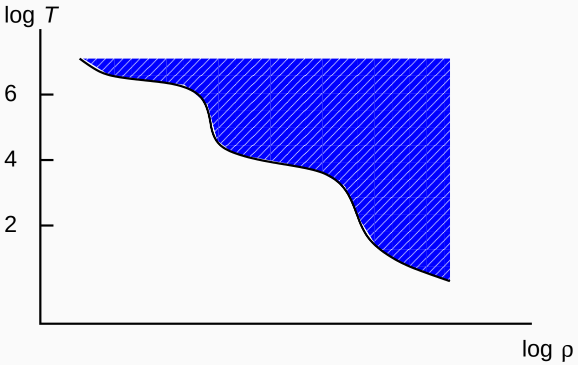


- matter is strongly ionized, excitation of inner shells of atoms, strong dependence of $\mathcal{L}$ on T

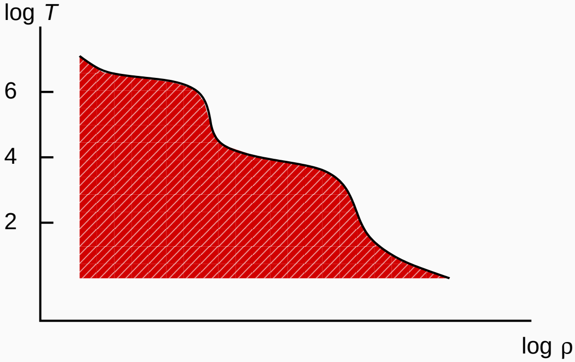
Cooling

$$\mathcal{L}(\rho, T) < 0$$



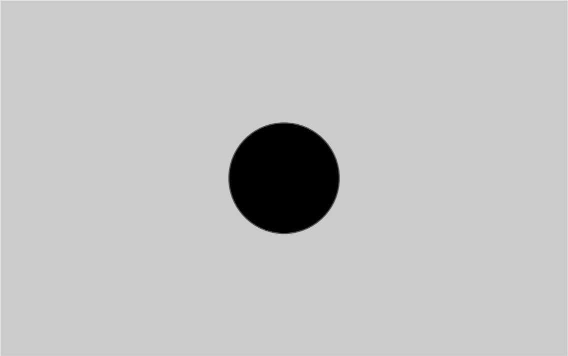
Heating

$$\mathcal{L}(\rho, T) > 0$$



Bubble with perturbed temperature

- small perturbation of temperature and density in a bubble
- bubble in a mechanical equilibrium with external environment: $p \sim \rho T = \text{const.}$



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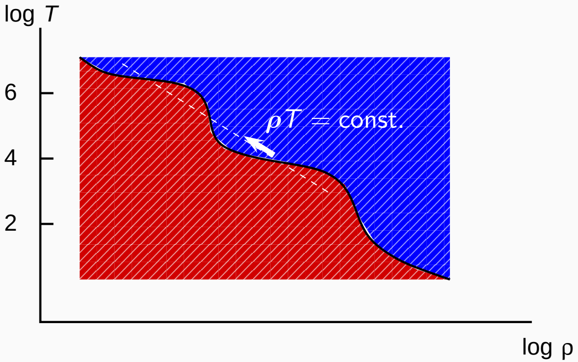

- increase of temperature and decrease of density $\Rightarrow$ more cooling $\Rightarrow$ *stability*

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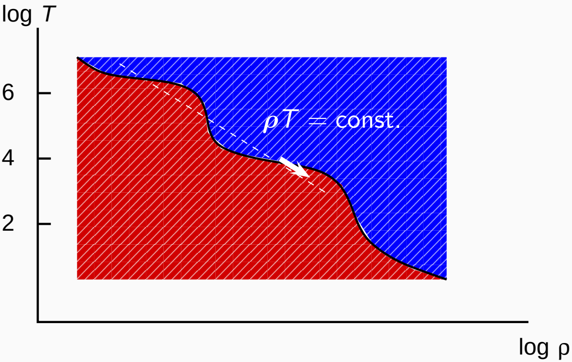

- decrease of temperature and increase of density ⇒ more heating ⇒ *stability*

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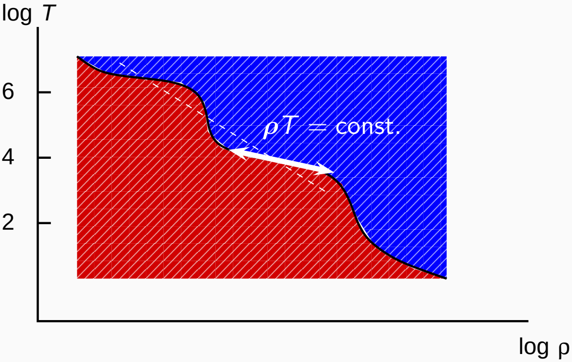

- region of stability

Bubble with perturbed temperature

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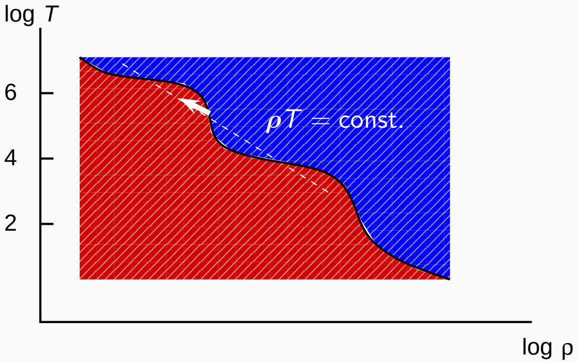

- increase of temperature and decrease of density ⇒ more heating ⇒ *instability*

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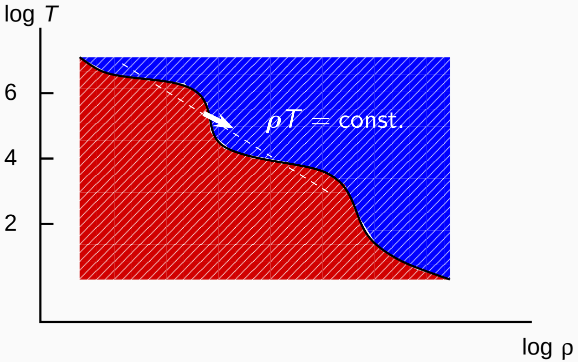

- decrease of temperature and increase of density ⇒ more cooling ⇒ *instability*

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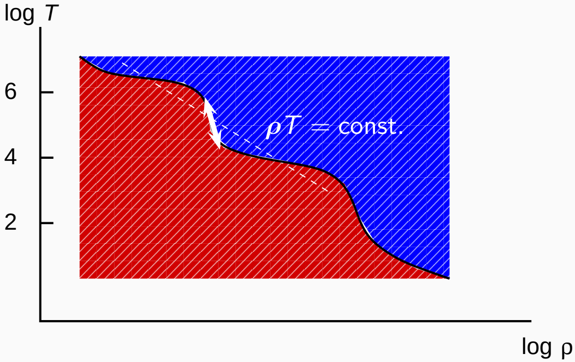

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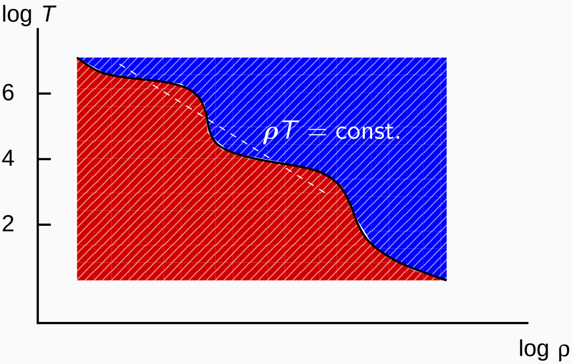
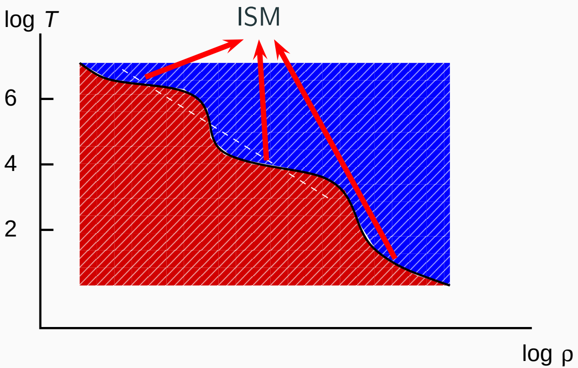


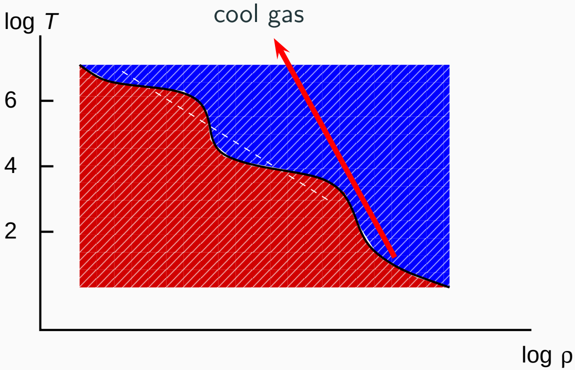
Field criterion of thermal stability: $\left(\frac{\partial \mathcal{L}}{\partial T}\right)_p < 0.$

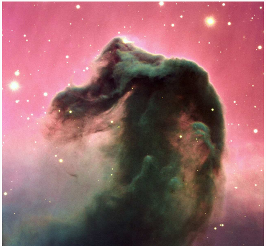
Three phases of interstellar medium

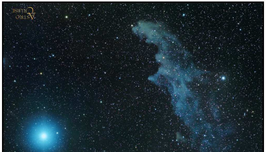


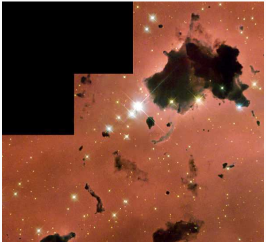
- three regions of stability ⇒ three phases of interstellar medium (cool gas, warm ionized gas, and hot coronal gas)

Three phases of interstellar medium









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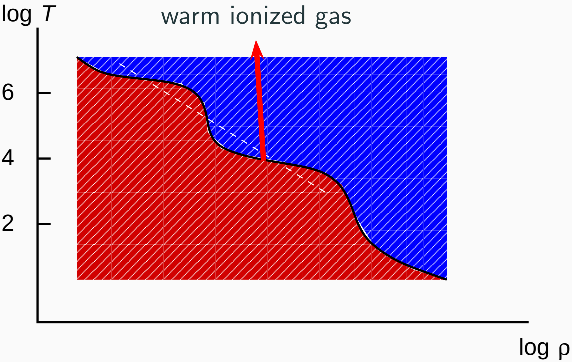

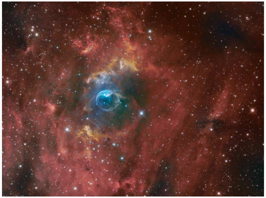

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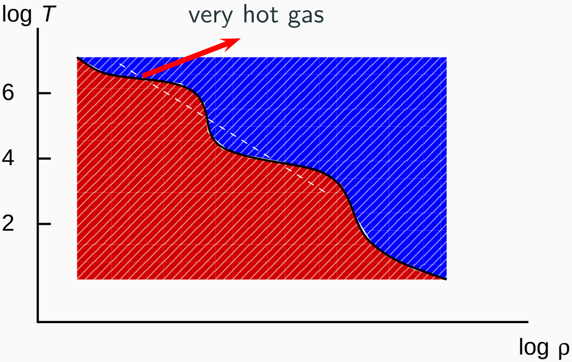
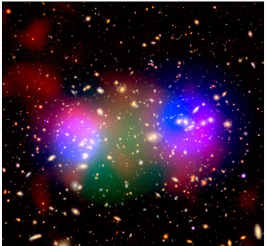
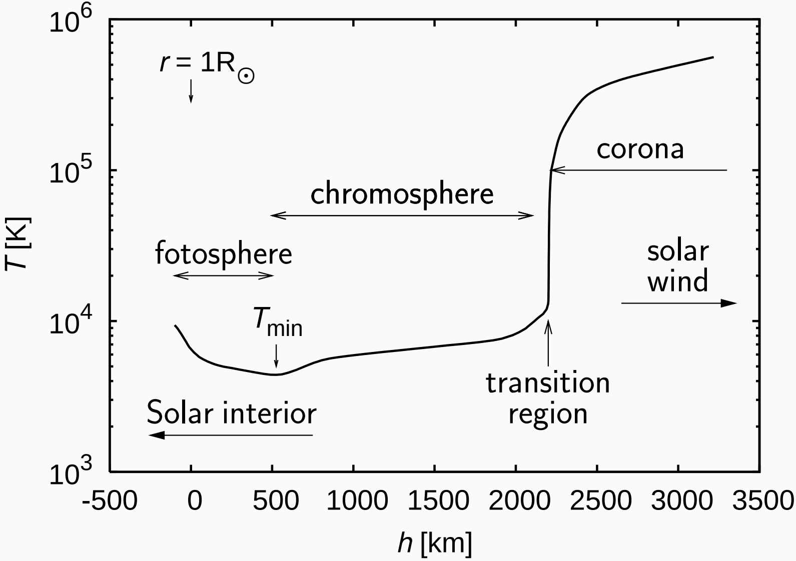




Existence of solar transition region



Cooling in prominence regions

After beeing elevated by coronal heating, the solar material quickly cools down reaching the temperatures at which it can emit, for instance, Hα line.

Radiative vs. adiabatic shocks

Post-shock temperature distribution

In the post-shock region, the gas can be typically cooled-down either adiabatically or radiatively. The importance of these processes can be determined from the energy equation

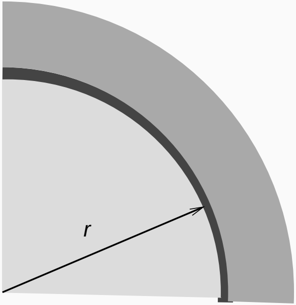
$$\frac{\partial\left(\rho\epsilon\right)}{\partial t} + \nabla\cdot\left(\rho\epsilon\boldsymbol{v}\right) = -\rho^2\Lambda(T) - p\nabla\cdot\boldsymbol{v},$$

where $\Lambda(T)$ is the cooling function.
Assuming stationary spherically symmetric post-shock flow with constant flow velocity

$$\frac{3}{2}\frac{k}{\mu}\frac{v}{r^2}\frac{\mathrm{d}\left(r^2\rho T\right)}{\mathrm{d}r} = -\rho^2\Lambda(T) - \frac{\rho k T}{\mu}\frac{2v}{r}.$$

From continuity equation at constant speed $r^2\rho =$const. the temperature gradient is

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{4T}{3r} - \frac{2}{3}\frac{\mu}{k}\frac{\rho}{v}\Lambda(T).$$



Radiative vs. adiabatic shocks

The first right-hand side term in energy equation

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{4T}{3r} - \frac{2}{3}\frac{\mu}{k}\frac{\rho}{v}\Lambda(T)$$

describes adiabatic cooling, while the second right-hand side term stands for radiative cooling. When adiabatic cooling dominates, from the energy equation follows that the post-shock region is large, comparable with r. On the other hand, the post-shock region is significantly thinner when radiative cooling dominates.

Isothermal shocks

When the matter is in the radiative equilibrium with diluted radiation from star, then in a limit of geometrically thin cooling layer the shock can be considered as isothermal. The corresponding jump conditions can be in such a case derived by putting $\gamma = 1$ in the Rankine-Hugoniot jump conditions

$$\frac{T_1}{T_0} = 1,$$

$$\frac{\rho_1}{\rho_0} = \frac{p_1}{p_0} = M_0^2,$$

$$M_1^2 = \frac{1}{M_0^2}.$$

The jump conditions now allow for arbitrary large jump density. For instance, This can describe shocks in nebulae around hot stars, which are in ionization equilibrium due to a stron ionizing source.

Suggested reading

J. Castor: Radiation hydrodynamics

D. Mihalas & B. W. Mihalas: Foundations of Radiation Hydrodynamics

F. H. Shu: The physics of astrophysics: II. Hydrodynamics

Y. B. Zeldovich, Y. P. Raizer: Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena