CHAPTER 3

Notations and Numbers

Where our number symbols come from

We are so accustomed to today's number system, with its use of the ten decimal digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 (in Western countries), that it can come as a shock to realize that there are entirely different ways to write numbers. Even today, many cultures – Arabic, Chinese, Korean – use different symbols for the ten digits, although they all combine these symbols to form larger numbers using the same 'positional' method (hundreds, tens, units). But differences in notation can be more radical than that. There is nothing special about the number 10. It happens to be the number of human fingers and thumbs, which are ideal for counting, but if we had evolved seven fingers, or twelve, very similar systems would have worked equally well, perhaps better in some cases.

Roman numerals

Most Westerners know of at least one alternative system, Roman numerals, in which – for example – the year 2012 is written MMXII. Most of us are also aware, at least if reminded, that we employ two distinct methods for writing numbers that are not

whole numbers – fractions like $^3/_4$ and decimals such as 0.75. Yet another number notation, found on calculators, is the scientific notation for very large or very small numbers – such as 5×10^9 for five billion (often seen as 5E9 on calculator displays) or 5×10^{-6} for five millionths.

These symbolic systems developed over thousands of years, and many alternatives flourished in various cultures. We have already encountered the Babylonian sexagesimal system (which would come naturally to any creature that had 60 fingers), and the simpler and more limited Egyptian number symbols, with their strange treatment of fractions. Later, base-20 numbers were used in Central America by the Mayan civilization. Only recently did humanity settle on the current methods for writing numbers, and their use became established through a mixture of tradition and convenience. Mathematics is about concepts, not symbols – but a good choice of symbol can be very helpful.

Greek numerals

We pick up the story of number symbols with the Greeks. Greek geometry was a big improvement over Babylonian geometry, but Greek arithmetic – as far as we can tell from surviving sources – was not. The Greeks took a big step backwards; they did not use positional notation. Instead, they used specific symbols for multiples of 10 or 100, so that, for instance, the symbol for 50 bore no particular relationship to that for 5 or 500.

The earliest evidence of Greek numerals is from about 1100 BC. By 600 BC the symbols had changed, and by 450 BC they had changed again, with the adoption of the Attic system, which resembles Roman numerals. The Attic system used I, II, III and IIII for the numbers 1, 2, 3 and 4. For 5 the Greek capital letter pi (Π) was employed, probably because it is the first letter of penta. Similarly, 10 was written Δ, the first letter of deka; 100 was written H, the first letter of hekaton; 1000 was written Ξ, the first letter of chilioi; and

10,000 was written M, the first letter of myrioi. Later Π was changed to Γ. So the number 2178, for example, was written as

ΞΞΗΔΔΔΔΔΔΔΓ|||

Although the Pythagoreans made numbers the basis of their philosophy, it is not known how they wrote them. Their interest in square and triangular numbers suggests that they may have represented numbers by patterns of dots. By the classical period, 600–300 BC, the Greek system had changed again, and the 27 different letters of their alphabet were used to denote numbers from 1 to 900, like this:

1	2	3	4	5	6	7	8	9
α	β	γ	δ	ε	ϛ	ζ	η	θ
10	20	30	40	50	60	70	80	90
ι	κ	λ	μ	ν	ξ	ο	π	ϙ
100	200	300	400	500	600	700	800	900
ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ

These are the lower-case Greek letters, augmented by three extra letters derived from the Phoenician alphabet: ϛ (stigma), ϙ (koppa), and ϡ (sampi).

Using letters to stand for numbers might have caused ambiguity, so a horizontal line was placed over the top of the number symbols. For numbers bigger than 999, the value of a symbol could be multiplied by 1000 by placing a stroke in front of it.

The various Greek systems were reasonable as a method for recording the results of calculations, but not for performing the

calculations themselves. (Imagine trying to multiply σμγ by ωλδ, for instance.) The calculations themselves were probably carried out using an abacus, perhaps represented by pebbles in the sand, especially early on.

The Greeks wrote fractions in several ways. One was to write the numerator, followed by a prime (′), and then the denominator, followed by a double prime (″). Often the denominator was written twice. So $^{21}/_{47}$ would be written as

κα′ μζ″ μζ″,

where κα is 21 and μζ is 47. They also used Egyptian-style fractions, and there was a special symbol for $^{1}/_{2}$. Some Greek astronomers, notably Ptolemy, employed the Babylonian sexagesimal system for precision, but using Greek symbols for the component digits. It was all very different from what we use today. In fact, it was a mess.

Indian number symbols

The ten symbols currently used to denote decimal digits are often referred to as *Hindu–Arabic* numerals, because they originated in India and were taken up and developed by the Arabs.

The earliest Indian numerals were more like the Egyptian system. For example, Khasrosthi numerals, used from 400 BC to AD 100, represented the numbers from 1 to 8 as

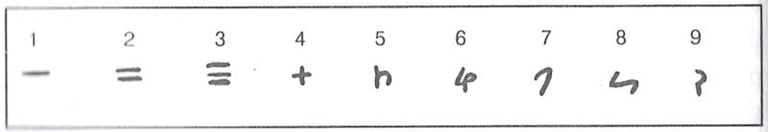
I II III X IX IIX IIIX XX

with a special symbol for 10. The first traces of what eventually became the modern symbolic system appeared around 300 BC in the Brahmi numerals. Buddhist inscriptions from the time include precursors of the later Hindu symbols for 1, 4 and 6. However, the Brahmi system used different symbols for multiples of ten or multiples of 100, so it was similar to the Greek number symbolism, except that it used special symbols rather than letters of the alphabet. The Brahmi system was not a positional system. By AD 100 there are records of the full Brahmi system. Inscriptions in caves and on coins show that it continued in use until the fourth century.

Between the fourth and sixth centuries, the Gupta Empire gained control of a large part of India, and the Brahmi numerals developed into Gupta numerals. From there they developed into Nagari numerals. The idea was the same, but the symbols differed.

The Indians may have developed positional notation by the first century, but the earliest datable documentary evidence for positional notation places it in 594. The evidence is a legal document which bears the date 346 in the Chedii calendar, but some scholars believe this date may be a forgery. Nevertheless, it is generally agreed that positional notation was in use in India from about 400 onwards.

There is a problem with the use of only the symbols 1–9: the notation is ambiguous. What does 25 mean, for instance? It might (in our notation) mean 25, or 205, or 2005 or 250, etc. In positional notation, where the meaning of a symbol depends on its location, it is important to specify that location without ambiguity. Today we do that by using a tenth symbol, zero (0). But it took early civilizations a long time to recognize the problem and solve it in that manner. One reason was philosophical: how can 0 be a number when a number is a quantity of things? Is nothing a quantity? Another was practical: usually it was clear from the context whether 25 meant 25 or 250 or whatever.



Brahmi numerals 1–9

Some time before 400 BC – the exact date is unknown – the Babylonians introduced a special symbol to show a missing position in their number notation. This saved the scribes the effort of leaving a carefully judged space, and made it possible to work out what a number meant even if it was written sloppily. This invention was forgotten, or not transmitted to other cultures, and eventually rediscovered by the Hindus. The Bakhshali manuscript, the date of which is disputed but lies somewhere between AD 200 and 1100, uses a heavy dot •. The Jain text *Lokavibhaaga* of AD 458 uses the concept of zero, but not a symbol. A positional system that lacked the numeral zero was introduced by Aryabhata around AD 500. Later Indian mathematicians had names for zero, but did not use a symbol. The first undisputed use of zero in positional notation occurs on a stone tablet in Gwalior dated to AD 876.

Brahmagupta, Mahavira and Bhaskara

The key Indian mathematicians were Aryabhata (born AD 476), Brahmagupta (born AD 598), Mahavira (9th century) and Bhaskara (born 1114). Actually they should be described as astronomers, because mathematics was then considered to be an astronomical technique. What mathematics existed was written down as chapters in astronomy texts; it was not viewed as a subject in its own right.

Aryabhata tells us that his book *Aryabhatiya* was written when he was 23 years old. Brief though the mathematical section of his book is, it contains a wealth of material: an alphabetic system of numerals, arithmetical rules, solution methods for linear and quadratic equations, trigonometry (including the sine function and the 'versed sine' $1 - \cos\theta$). There is also an excellent approximation, 3.1416, to π.

Brahmagupta was the author of two books: *Brahma Sputa Siddhanta* and *Khanda Khadyaka*. The first is the most important; it is an astronomy text with several sections on mathematics, with arithmetic and the

What arithmetic did for them

The oldest surviving Chinese mathematics text is the *Chiu Chang*, which dates from about AD100. A typical problem is: Two and a half piculs of rice are bought for $\frac{3}{7}$ of a tael of silver. How many piculs can be bought for 9 taels? The proposed solution uses what medieval mathematicians called the 'rule of three'. In modern notation, let x be the required quantity. Then

$$\frac{x}{9} = \frac{5/2}{3/7}$$

so $x = 52\frac{1}{2}$ piculs. A picul is about 65 kilograms.

verbal equivalent of simple algebra. The second book includes a remarkable method for interpolating sine tables – that is, finding the sine of an angle from the sines of a larger angle and a smaller one.

Mahavira was a Jain, and he included a lot of Jain mathematics in his *Ganita Sara Samgraha*. This book included most of the contents of those of Aryabhata and Brahmagupta, but went a great deal further and was generally more sophisticated. It included fractions, permutations and combinations, the solution of quadratic equations, Pythagorean triangles and an attempt to find the area and perimeter of an ellipse.

Bhaskara (known as 'the teacher') wrote three important works: *Lilavati*, *Bijaganita* and *Siddhanta Siromani*. According to Fyzi, court poet of the Mogul emperor Akbar, Lilavati was the name of Bhaskara's daughter. Her father cast his daughter's horoscope, and determined the most auspicious time for her wedding. To dramatize his forecast, he put a cup with a hole in it inside a bowl of water, constructed so that it would sink when the propitious moment arrived. But Lilavati leaned over the bowl and a pearl from her clothing fell into the cup and blocked the hole. The cup did not sink, which meant that Lilavati could never get married. To cheer her up, Bhaskara

wrote a mathematics textbook for her. The legend does not record what she thought of this.

Lilavati contains sophisticated ideas in arithmetic, including the method of casting out the nines, in which numbers are replaced by the sum of their digits to check calculations. It contains similar rules for divisibility by 3, 5, 7 and 11. The role of zero as a number in its own right is made clear. *Bijaganita* is about the solution of equations. *Siddhanta Siromani* deals with trigonometry: sine tables and various trigonometric relations. So great was Bhaskara's reputation that his works were still being copied around 1800.

The Hindu System

The Hindu system started to spread into the Arabic world, before it was fully developed in its country of origin. The scholar Severus Sebokht writes of its use in Syria in 662: 'I will omit all discussion of the science of the Indians ... of their subtle discoveries in astronomy ... and of their valuable methods of calculation ... I wish only to say that this computation is done by means of nine signs.'

In 776 a traveller from India appeared at the court of the Caliph and demonstrated his prowess in the 'siddhanta' method of calculation, along with trigonometry and astronomy. The basis for the computational methods seems to have been the *Brahmasphutasiddhanta* of Brahmagupta, written in 628, but whichever book it was, it was promptly translated into Arabic.

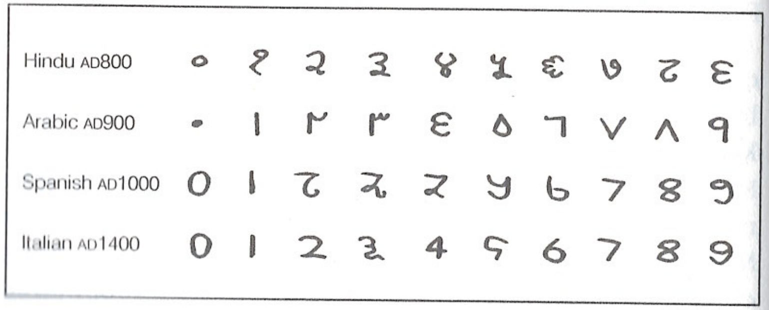
Initially the Hindu numerals were mainly used by scholars; older methods remained in widespread use among the Arabic business community and in daily life, until about 1000. But Al-Khwarizmi's *On Calculation with Hindu Numerals* of 825 made the Hindu system widely known in the Arab world. The mathematician Al-Kindi's four-volume treatise *On the Use of the Indian Numerals* (*Ketab fi Isti'mal al-'Adad al-Hindi*) of 830 increased awareness of the possibility of performing all numerical calculations using only the ten digits.

The Dark Ages?

While Arabia and India were making significant advances in mathematics and science, Europe was comparatively stagnant, although the medieval period was not quite the 'Dark Ages' of popular conception. Some advances were made, but these were slow and not particularly radical. The pace of change began to accelerate when word of the Eastern discoveries came to Europe.

Italy lies closer to the Arabian world than most parts of Europe, so it was probably inevitable that Arab advances in mathematics made their way to Europe through Italy. Venice, Genoa and Pisa were significant trading centres, and merchants sailed from these ports to North Africa and the eastern end of the Mediterranean. They exchanged wool and European wood for silks and spices.

There was metaphorical trade in ideas as well as literal trade in goods. Arabian discoveries in science and mathematics made their way along the trade routes, often by word of mouth. As trade made Europe more prosperous, barter gave way to money, and keeping accounts and paying taxes became more complex. The period's equivalent of a pocket calculator was the abacus, a device in which beads moving on wires represented numbers. However, those numbers also had to be written down on paper, for legal purposes and for general record-keeping. So the merchants needed



Evolution of western number symbols

a good number notation as well as methods for doing calculations quickly and accurately.

An influential figure was Leonardo of Pisa, also known as Fibonacci, whose book *Liber Abbaci* was published in 1202. (The Italian word 'abbaco' usually means 'calculation', and need not imply the use of the abacus, a Latin term.) In this book, Leonardo introduced Hindu–Arabic number symbols to Europe.

The *Liber Abbaci* includes, and promoted, one further notational device that remains in use today: the horizontal bar in a fraction, such as $\frac{3}{4}$ for 'three-quarters'. The Hindus employed a similar notation, but without the bar; the bar seems to have been introduced by the Arabs. Fibonacci employed it widely, but his usage differed from what we do today in some respects. For instance, he would use the same bar as part of several different fractions.

Because fractions are very important in our story, it may be worth adding a few comments on the notation. In a fraction like $\frac{3}{4}$, the 4 on the bottom tells us to divide the unit into four equal parts, and the 3 on top then tells us to select three of those pieces. More formally, 4 is the *denominator* and 3 is the *numerator*. For typographical convenience, fractions are often written on a single line in the form 3/4, or sometimes in the compromise form ³⁄₄. The horizontal bar then mutates into a diagonal slash.

On the whole, however, we seldom use fractional notation in practical work. Mostly we use decimals – writing π as 3.14159, say, which is not exact, but close enough for most calculations. Historically, we have to make a bit of a leap to get to decimals, but we are following chains of ideas, not chronology, and it will be much simpler to make the leap anyway. We therefore jump forward to 1585, when William the Silent chose the Dutchman Simon Stevin as private tutor to his son Maurice of Nassau.

Building on this recognition, Stevin made quite a career for himself, becoming Inspector of Dykes, Quartermaster-General of the Army and eventually the Minister of Finance. He quickly realized

Leonardo of Pisa (Fibonacci)

1170–1250

Leonardo, Italian born, grew up in North Africa, where his father Guilielmo was working as a diplomat on behalf of merchants trading at Bugia (in modern Algeria). He accompanied his father on his numerous travels, encountered the Arabic system for writing numbers and understood its importance. In his *Liber Abbaci* of 1202 he writes: 'When my father, who had been appointed by his country as public notary in the customs at Bugia acting for the Pisan merchants going there, was in charge, he summoned me to him while I was still a child, and having an eye to usefulness and future convenience, desired me to stay there and receive instruction in the school of accounting. There, when I had been introduced to the art of the Indians' nine symbols through remarkable teaching, knowledge of the art very soon pleased me above all else.'

The book introduced the Hindu–Arabic notation to Europe, and formed a comprehensive arithmetic text, containing a wealth of material related to trade and currency conversion. Although it took several centuries for Hindu–Arabic notation to displace the traditional abacus, the advantages of a purely written system of calculation soon became apparent.

Leonardo is often known by his nickname 'Fibonacci', which means 'son of Bonaccio', but this name is not recorded before the 18th century and was probably invented then by Guillaume Libri.

the need for accurate accounting procedures, and he looked to the Italian arithmeticians of the Renaissance period, and the Hindu–Arabic notation transmitted to Europe by Leonardo of Pisa. He found fractional calculations cumbersome, and would have preferred the precision and tidiness of Babylonian sexagesimals, were it not for the use of base-60. He tried to find a system that

combined the best of both, and invented a base-10 analogue of the Babylonian system: decimals.

He published his new notational system, making it clear that it had been tried, tested and found to be entirely practical by entirely practical men. In addition, he pointed out its efficacy as a business tool: 'all computations that are met in business may be performed by integers alone without the aid of fractions'.

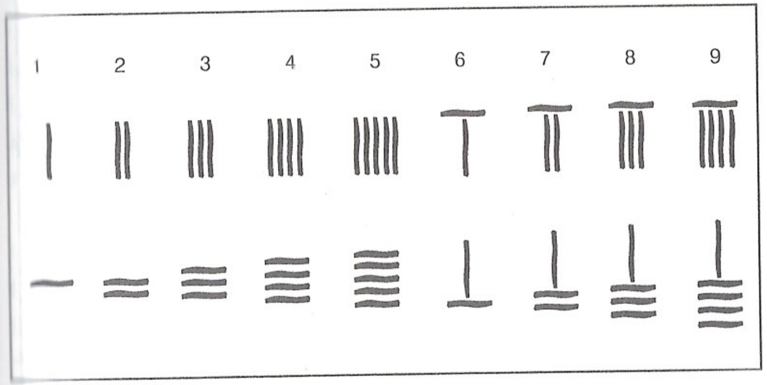
Negative numbers

Mathematicians call the system of whole numbers the **natural numbers**. Including negative numbers as well, we obtain the integers. The **rational numbers** (or merely 'rationals') are the positive and negative fractions, the **real numbers** (or merely 'reals') are the positive and negative decimals, going on forever if necessary.

How did negative numbers come into the story?

Early in the first millennium, the Chinese employed a system of 'counting rods' instead of an abacus. They laid the rods out in patterns to represent numbers.

The top row of the picture shows **heng** rods, which represented units, hundreds, tens of thousands and so on, according to their



Ancient Chinese counting rods

position in a row of such symbols. The bottom row shows *tsung* rods, which represented tens, thousands and so on. So the two types alternated. Calculations were performed by manipulating the rods in systematic ways.

When solving a system of linear equations, the Chinese calculators would arrange the rods in a table. They used red rods for terms that were supposed to be added and black rods for terms that were supposed to be subtracted. So to solve equations that we would write as

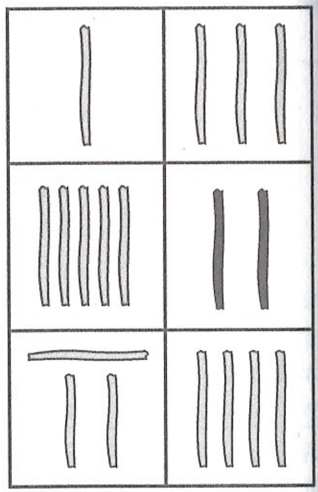
$$3x - 2y = 4$$

$$x + 5y = 7$$

they would set out the two equations as two columns of a table: one with the numbers 3 (red), 2 (black), 4 (red), and the other 1 (red), 5 (red), 7 (red).

The red/black notation was not really about negative numbers, but the operation of subtraction. However, it set the stage for a concept of negative numbers, *cheng fu shu*. Now a negative number was represented by using the same arrangement of rods as for the corresponding positive number, by placing another rod diagonally over the top.

To Diophantus, all numbers had to be positive, and he rejected negative solutions to equations. Hindu mathematicians found negative numbers useful to represent



Laying out equations, Chinese style. Shaded rods are red

debts in financial calculations – owing someone a sum of money was worse, financially, than having no money, so a debt clearly should be less than zero. If you have 3 pounds and pay out 2, then you are left with $3 - 2 = 1$. By the same token, if you owe a debt of 2 pounds and acquire 3, your net worth is $-2 + 3 = 1$. Bhaskara remarks that a particular problem had two solutions, 50 and -5, but he remained nervous about the second solution, saying that it was 'not to be taken; people do not approve of negative solutions'.

Despite these misgivings, negative numbers gradually became accepted. Their interpretation, in a real calculation, needed care. Sometimes they made no sense, sometimes they might be debts, sometimes they might mean a downwards motion instead of an upwards one. But interpretation aside, their arithmetic worked perfectly well, and they were so useful as a computational aid that it would have been silly not to use them.

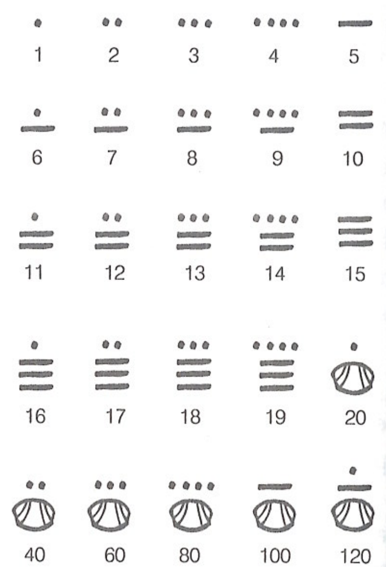
Arithmetic lives on

Our number system is so familiar that we tend to assume that it is the only possible one, or at least the only sensible one. In fact, it evolved, laboriously and with lots of dead ends, over thousands of years. There are many alternatives; some were used by earlier cultures, like the Mayans. Different notations for the numerals 0–9 are in use today in some countries. And our computers represent numbers internally in binary, not decimal: their programmers ensure that the numbers are turned back into decimal before they appear on the screen or in a print-out.

Since computers are now ubiquitous, is there any point in teaching arithmetic any more? Yes, for several reasons. Someone has to be able to design and build calculators and computers, and make them do the right job; this requires *understanding* arithmetic – how and why it works, not just how to do it. And if your only arithmetical ability is reading what's on a calculator, you probably won't notice if the supermarket gets your bill wrong. Without internalizing the basic operations of arithmetic, the whole of mathematics will be inaccessible to you. You might not worry about that, but modern civilization would quickly break down if we stopped teaching arithmetic, because you can't spot the future engineers and scientists at the age of five. Or even the future bank managers and accountants.

Of course, once you have a basic grasp of arithmetic by hand, using a calculator is a good way to save time and effort. But, just as you won't learn to walk by always using a crutch, you won't learn to think sensibly about numbers by relying solely on a calculator.

Mayan Numerals



A remarkable number system, which used base-20 notation instead of base-10, was developed by the Mayans, who lived in South America around 1000. In the base-20 system, the symbols equivalent to our 347 would mean

$3 \times 400 + 4 \times 20 + 7 \times 1$

(Since $20 \times 20 = 400$)

which is 1287 in our notation. The actual symbols are shown here.

Early civilizations that use base-10 probably did so because humans have ten fingers (including thumbs). It has been suggested that the Mayans counted on their toes as well, which is why they used base-20.

What arithmetic does for us

We use arithmetic throughout our daily lives, in commerce, and in science. Until the development of electronic calculators and computers, we either did the calculations by hand, with pen and paper, or we used aids such as the abacus or a ready reckoner (a printed book of tables of multiples of amounts of money). Today most arithmetic goes on electronically behind the scenes – supermarket checkout tills now tell the operator how much change to give back, for instance, and banks total up what is in your account automatically, rather than getting their accountant to do it. The quantity of arithmetic 'consumed' by a typical person during the course of a single day is substantial.

Computer arithmetic is not actually carried out in decimal format. Computers use base-2, or binary, rather than base-10. In place of units, tens, hundreds, thousands and so on, computers use 1, 2, 4, 8, 16, 32, 64, 128, 256, and so on – the powers of two, each twice its predecessor. (This is why the memory card for your digital camera comes in funny sizes like 256 megabytes.) In a computer, the number 100 would be broken down as 64+32+4 and stored in the form 1100100.