Ch 1

5. Show that a projective transformation $\mathbb{P}^n \to \mathbb{P}^m$ uniquely determines the underlying linear map up to a scalar multiple,

i.e. for 2 linear map $\varphi, \psi : \mathbb{k}^{n+1} \to \mathbb{k}^{m+1}$

$$[\varphi] = [\psi] \iff \exists\, k \in \mathbb{k}^\times : \quad \psi = k \cdot \varphi$$

Def. A projective transformation between proj spaces $\mathbb{P}^n$ and $\mathbb{P}^m$ is a colinear map $\Phi : \mathbb{P}^n \to \mathbb{P}^m$

$$[x_0 : x_1 : \cdots : x_n] \mapsto [y_0 : \cdots : y_m]$$

i.e., Φ is a linear map $\mathbb{k}^{n+1} \to \mathbb{k}^{m+1}$ and $\Phi(x_0 : x_1 : \cdots : x_n) \sim \Phi(kx_0 : \cdots : kx_n)$.

Ans: Let $\varphi, \psi : \mathbb{k}^{n+1} \to \mathbb{k}^{m+1}$ be linear maps.

Tip: Let $\Phi : \mathbb{P}^n \to \mathbb{P}^m$ be a proj tran.

Assume that $[\psi] = [\varphi] = \Phi$.

Then for any $v \in \mathbb{k}^{n+1}$, $[\psi(v)] = [\varphi(v)]$

$\Rightarrow \psi(v) = k_v \cdot \varphi(v)$, $k_v \in \mathbb{k}^\times$. it may depend on v

In particular $[\psi(e_i)] = [\varphi(e_i)]$ where $\{e_i\}$'s are the basis

$\Rightarrow \psi(e_i) = k_i \cdot \varphi(e_i)$ where $k_i \in \mathbb{k}^\times$.

So we have $\psi(e_0) + \psi(e_1) + \cdots + \psi(e_n) = k_0 \varphi(e_0) + \cdots + k_n \varphi(e_n)$

but then $\psi(e_0) + \cdots + \psi(e_n)$

$= \psi(e_0 + \cdots + e_n)$ by linearity

$= k \cdot \varphi(e_0 + \cdots e_n)$, by putting $v = \Sigma e_i$

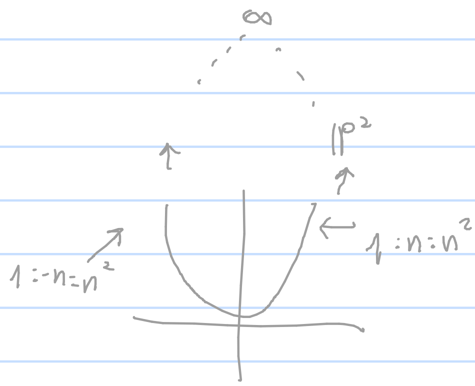
$= k\,\varphi(e_0) + \cdots + k\,\varphi(e_n)$

By comparing, $k_0 = k_1 = \cdots = k$.

So we have a fixed k s.t. $\psi = k \cdot \varphi$

The converse is obvious.

7. Consider the proj space $\mathbb{P}^2$.
Consider a plane $[1:x:y]$ and a parabola on it.
Show that the two branches of the parabola,
i.e. $[1:-n:n^2]$ and $[1:n:n^2]$,
intersect at $[0:0:1]$ when $n\to\infty$.



Ans: Note that $[1:n:n^2]$

$$= \left[\frac{1}{n^2} : \frac{1}{n} : 1\right] \text{ by def of proj sp}$$

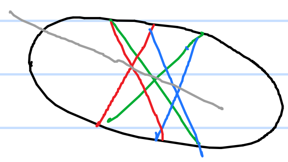
$$\to [0:0:1] \quad \text{as } n\to\infty$$

Similarly, $[1:-n:n^2]$

$$= \left[\frac{1}{n^2} : \frac{-1}{n} : 1\right]$$

$$\to [0:0:1]$$

Explanation: In proj space, there is a pt of infinity.
Why should we care about ∞?
Including the point ∞ completes our theory.
For example, there is Pascal's Thm



is true naturally if we consider projective case and count into ∞.

Ch 2

1. Def. A collineation is a bijective projective transformation

i.e. let $V \subseteq \mathbb{k}^{n+1}$, $W \subseteq \mathbb{k}^{m+1}$ be vector subspaces,

and let $\mathbb{P}(V) := V/\sim$, $\mathbb{P}(W) := W/\sim$.

Then a collineation Φ is a bijective colinear map

$$\Phi : \mathbb{P}(V) \to \mathbb{P}(W),$$

i.e. $\Phi : V \to W$ is a bijective linear map and

$$\Phi(x_0 : x_1 : \cdots : x_n) \sim \Phi(kx_0 : \cdots : kx_n).$$

Find a collineation mapping the parabola $(y_1)^2 + 4y_2 = 0$ onto the circle $x_1^2 + x_2^2 = 1$ in $\mathbb{P}^2$.

Ans:

Tipes

Define $\Phi : \mathbb{P}^2 \to \mathbb{P}^2$

$$[y_0 : y_1 : y_2] \mapsto [y_0 - y_2 : y_1 : y_0 + y_2]$$

Φ is linear as a map between $\mathbb{k}^3 \to \mathbb{k}^3$:

Φ can be expressed as a matrix

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

i.e. $M \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = (y_0 - y_2,\ y_1,\ y_0 + y_2)$ $\quad 4y_2 = -y_1^2$

Φ is bijection:

$$|M| = 1 - (-1) = 2 \neq 0, \text{ so } M^{-1} \text{ exists.}$$

$$\Rightarrow \Phi \text{ is bijection.}$$

Φ is well-defined on projective spaces:

$$\Phi([ky_0 : ky_1 : ky_2]) = [k(y_0 - y_2) : ky_1 : k(y_0 + y_2)]$$
$$= [y_0 - y_2 : y_1 : y_0 + y_2]$$
$$= \Phi([y_0 : y_1 : y_2])$$

Φ restricts to a parabola, with brings a circle:

Now consider $\Phi|_V : [1 : y_1 : -\frac{1}{4}y_1^2] \mapsto [1 + \frac{1}{4}y_1^2 : y_1 : 1 - \frac{1}{4}y_1^2]$

$$= [1 : \frac{y_1}{1+\frac{1}{4}y_1^2} : \frac{1-\frac{1}{4}y_1^2}{1+\frac{1}{4}y_1^2}]$$

$$=: [1 : x_1 : x_2]$$

Now compute $x_1^2 + x_2^2$

$$\left(\frac{y_1}{1+\frac{1}{4}y_1^2}\right)^2 + \left(\frac{1-\frac{1}{4}y_1^2}{1+\frac{1}{4}y_1^2}\right)^2$$

$$= \frac{y_1^2 + (1 - \frac{1}{4}y_1^2)^2}{(1 + \frac{1}{4}y_1^2)^2}$$

$$= \frac{y_1^2 + 1 - \frac{1}{2}y_1^2 + \frac{1}{16}y_1^2}{1 + \frac{1}{2}y_1^2 + \frac{1}{16}y_1^2} = 1$$

Recall that

if Q is a conic - quadratic equation:

$$\sum_{0\le i\le j\le n} a_{ij}\, x_i\, x_j = 0$$

Mult by x_0 (homo) and get symmetric $A = (a_{ij})$, $i,j = 0, \dots, n$

We obtain a bilinear form $f(x,y) := x^T A y$

Now define for (proj) points X, Y,

$$X \pitchfork Y \iff f(X,Y) = 0$$

Define $X^{\pitchfork} := \{ Y : X \pitchfork Y \}$

Show that if $X_0 \in Q$, then $X_0^{\pitchfork}$ is the tangent space of Q at X_0.

Pf. Consider a curve $x(t)$ on Q such that $x(0) = X_0 \in Q$.

Let $x(t) = (1, x_1(t), \dots, x_n(t))$. $\quad x_0(t) = 1$, $\mathbb{P}^n$

Then we may write

Since $x(t)$ is on Q, $\sum_{i,j=0}^{n} a_{ij} \cdot x_i(t) \cdot x_j(t) = 0$

Differentiate w.r.t. t,

$$\sum_{i,j=0}^{n} a_{ij} \left(x_i'(t)\, x_j(t) + x_i(t)\, x_j'(t) \right) = 0$$

Sub $t = 0$, which can be written as

$$2 \cdot \left(x_0'(t) \ \cdots \ x_n'(t) \right) A \cdot \begin{pmatrix} x_0(t) \\ \vdots \\ x_n(t) \end{pmatrix} = 0$$

$\therefore [x'(0)] \pitchfork [x(0)] = X_0$

Now since $X_0 \in Q$, $X_0 \pitchfork X_0$

Hence $X_0 + [x'(0)] \pitchfork X_0 \Rightarrow X_0 + t \cdot x'(0) \subseteq X_0^{\pitchfork}$

Tangent line of $x(t)$ at X_0

contain $y(t), z(t)$'s..

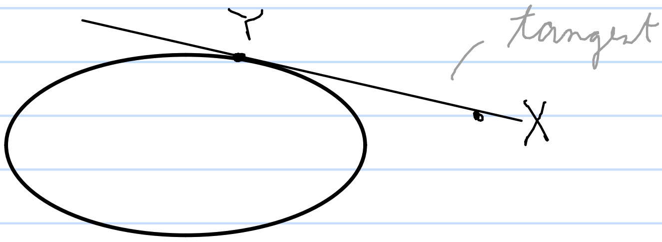
$\mathbb{k}[X]$

$\mathfrak{m}/\mathfrak{m}^2$ - linear terms

Reasoning — how to understand $X^{\pitchfork}$?

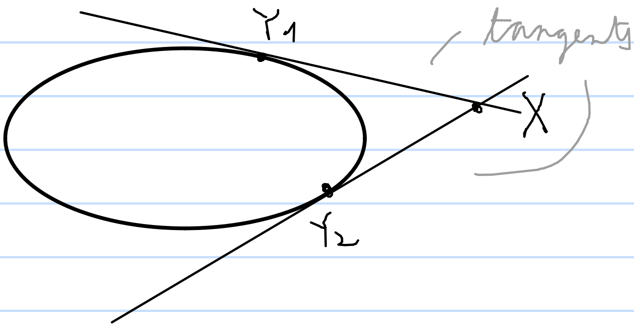
Case I: if $X \in Q$, i.e., X is on the conic section, then $X^{\pitchfork}$ is the tangent space of Q at X. (We will prove this in the tutorial).

Case II: if $X \notin Q$ & $X^TAX \geq 0$, i.e. X is 'outside' the conic section, then $X^{\pitchfork}$ contains at least one point $Y \in Q$, because



there is always a tangent of Q at Y that passes through X.

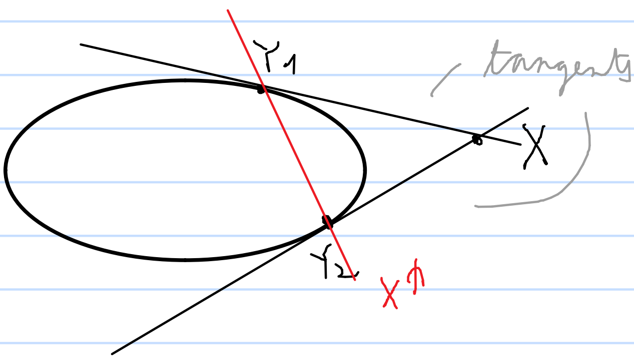
Indeed, conic sections always have such two $Y_1, Y_2 \in Q$



Then since $Y_1, Y_2 \in Q$, by Case I, $X \in Y_1^{\pitchfork}$ & $X \in Y_2^{\pitchfork}$ as X lies on the tangents $\overrightarrow{Y_1X}$, $\overrightarrow{Y_2X}$.

Note that $X \in Y^{\pitchfork} \Leftrightarrow Y \in X^{\pitchfork}$, so both $Y_1, Y_2 \in X^{\pitchfork}$.

Thus, $X^{\pitchfork}$ should be a line passing through Y_1 & Y_2



3. Find the tangents of the conic section

$$Q: \ 2x_1^2 - 4x_1x_2 + x_2^2 - 2x_1 + 6x_2 - 3 = 0$$

passing through the point $(3,4) \in k^2$

Ans: 0. We introduce x_0 so that we can work in proj space

new quest: point $[1:3:4] \in \mathbb{P}^2$.

1. Write down the symmetric matrix A for the bilinear form:

Homogenise and obtain

$$2x_1^2 - 4x_1x_2 + x_2^2 - 2x_0x_1 + 6x_0x_2 - 3x_0^2$$

tips

Now, $\mathring{a}_{11} = 2$, $\mathring{a}_{12} = -4$, $\mathring{a}_{22} = 1$

$\mathring{a}_{01} = -2$, $\mathring{a}_{02} = 6$, $\mathring{a}_{00} = 3$

so

$$\mathring{A} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & -3 & -2 & 6 \\ 1 & 0 & 2 & -4 \\ 2 & 0 & 0 & 1 \end{array}$$

Because we want to use tools from bilinear forms, easier for computations.

Symmetrise:

$$A = \begin{pmatrix} -3 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & 1 \end{pmatrix}$$

2. Find $[1:3:4]^{\perp}$:

Compute $f((1,3,4), y) = (1,3,4)\, A \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$

$$= (6, -3, 1) \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

$\therefore$ $X^{\perp}$ is given by

$$6y_0 - 3y_1 + y_2 = 0$$

Or in projective notation, put $y_0 = 1$,

$$6 - 3y_1 + y_2 = 0$$

3. find the intersection pts of $X^{\perp}$ & Q.

$$\begin{cases} 2x_1^2 - 4x_1x_2 + x_2^2 - 2x_1 + 6x_2 - 3 = 0 \\ 6 - 3y_1 + y_2 = 0 \end{cases}$$

$\Rightarrow$ $(1,1,-3)$ & $(1,3,3)$

$(1,-3) + \lambda(2,7)$ & $(3,3) + \lambda(0,1)$

4. Find the tangents of the conic section

$Q: \quad 4x_1 + 2x_2 - 4x_1 x_2 - 4 = 0$

parallel to the direction of $(1,2) \in \mathbb{k}^2$ from the origin

Ans: 0. We introduce x_0 so that we can work in proj space

now quest: $[0:1:2]$ is in the tangent space
(the end of the tangent)

1. Write down the symmetric matrix A for the bilinear form:

Homogenise and obtain

$4x_0x_1 + 2x_0x_2 - 4x_1x_2 - 4x_0^2$

Now $\mathring{a}_{01} = 4$, $\mathring{a}_{02} = 2$, $\mathring{a}_{12} = -4$, $\mathring{a}_{00} = -4$

so $\mathring{A}$

$$= \begin{pmatrix} -4 & 4 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

symmetrise: A

$$= \begin{pmatrix} -4 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 0 \end{pmatrix}$$

assuming $X \in Q$

2. Find X s.t. $[0:1:2] \in X^{\pitchfork}$

$$f(x, (0,1,2)) = (x_0 \ x_1 \ x_2) A \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$ where $X = x = [x_0 : x_1 : x_2]$

$$= (x_0 \ x_1 \ x_2) \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$$

So our desired X satisfies

$$4x_0 - 4x_1 - 2x_2 = 0$$

and X should lie on the conic, so

$$4x_0x_1 + 2x_0x_2 - 4x_1x_2 - 4x_0^2 = 0$$

Assume $x_0 \neq 0$, we can put $x_0 = 1$ in our proj setting,

$$\begin{cases} 4 - 4x_1 - 2x_2 = 0 & \text{— (1)} \\ 4x_1 + 2x_2 - 4x_1x_2 - 4 = 0 & \text{(2)} \end{cases}$$

(1) $\Rightarrow$ $x_2 = 2 - 2x_1$ — (3)

Sub (3) into (2), $4x_1 + 2(2-2x_1) - 4x_1(2-2x_1) - 4 = 0$

$x_1(x_1 - 1) = 0$

$\Rightarrow x_1 = 0$ or $x_1 = 1$

Sub into (3), we get $X = [1:0:2]$ or $X = [1:1:0]$

$\therefore$ The tangents are $X + \lambda(0,1,2)$, i.e. $X + [0:1:2]$