Recall that
if Q is a conic - quadratic equation:
$$\sum_{0 \le i \le j \le n} a_{ij} x_i x_j = 0$$

Mult by x_0 (homo) and get symmetric $A = (a_{ij})$, $i,j = 0, \dots, n$
We obtain a bilinear form $f(x,y) := x^T A y$

Now define for (proj) points X, Y,
$X \pitchfork Y \Leftrightarrow f(X,Y) = 0$

Define $X^{\pitchfork} := \{ Y : X \pitchfork Y \}$

Show that if $X_0 \in Q$, then $X_0^{\pitchfork}$ is the tangent space of Q at X_0.

Pf. Consider a curve $x(t)$ on Q such that $x(0) = X_0 \in Q$.
Set $x(t) = (1, x_1(t), \dots, x_n(t))$. $x_0(t) = 1$, $\mathbb{P}^n$
Then we may write
Since $x(t)$ is on Q, $\sum_{i,j=0}^{n} a_{ij} \cdot x_i(t) \cdot x_j(t) = 0$
Differentiate w.r.t. t,
$$\sum_{i,j=0}^{n} a_{ij} \left(x_i'(t) x_j(t) + x_i(t) x_j'(t) \right) = 0$$
Sub $t = 0$, which can be written as
$$2 \cdot (x_0'(t) \cdots x_n'(t)) \, A \cdot \begin{pmatrix} x_0(t) \\ \vdots \\ x_n(t) \end{pmatrix} = 0$$

$\therefore [x'(0)] \pitchfork [x(0)] = X_0$
Now since $X_0 \in Q$, $X_0 \pitchfork X_0$
Hence $X_0 + [x'(0)] \pitchfork X_0 \Rightarrow X_0 + \lambda \cdot x'(0) \subseteq X_0^{\pitchfork}$

Tangent line of $x(t)$ at X_0

contain $y(t), z(t)$'s..

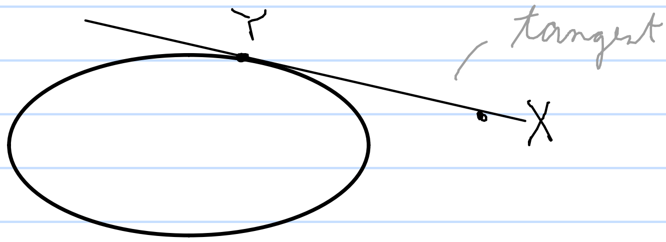
$\mathbb{k}[X]$

$\mathfrak{m}/\mathfrak{m}^2$ - linear terms

Reasoning — how to understand $X^{\pitchfork}$?

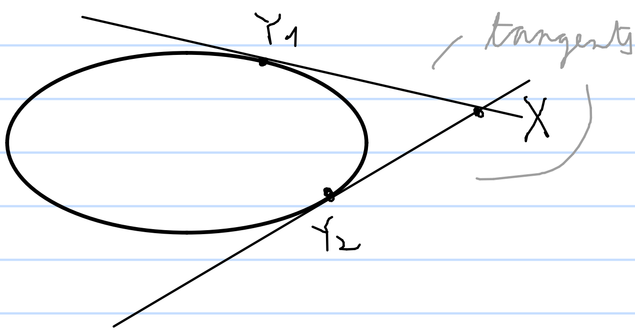
Case I: if $X \in Q$, i.e., X is on the conic section, then $X^{\pitchfork}$ is the tangent space of Q at X. (We will prove this in the tutorial).

Case II: if $X \notin Q$ & $X^TAX \geq 0$, i.e. X is 'outside' the conic section, then $X^{\pitchfork}$ contains at least one point $Y \in Q$, because



there is always a tangent of Q at Y that passes through X.

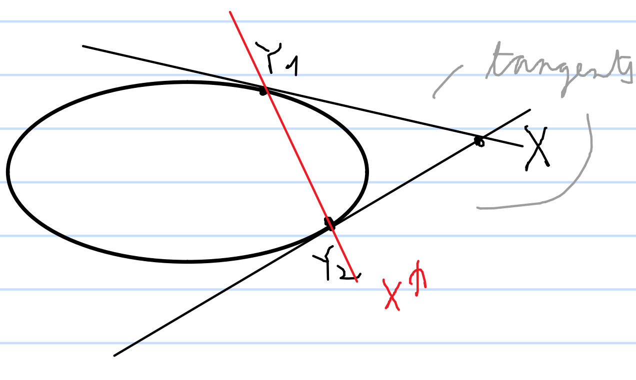
Indeed, conic sections always have such two $Y_1, Y_2 \in Q$



Then since $Y_1, Y_2 \in Q$, by Case I, $X \in Y_1^{\pitchfork}$ & $X \in Y_2^{\pitchfork}$ as X lies on the tangents $\overrightarrow{Y_1X}$, $\overrightarrow{Y_2X}$.

Note that $X \in Y^{\pitchfork} \Leftrightarrow Y \in X^{\pitchfork}$, so both $Y_1, Y_2 \in X^{\pitchfork}$.

Thus, $X^{\pitchfork}$ should be a line passing through Y_1 & Y_2



Ch 2

remind them of tests / quizzes

4. Find the tangents of the conic section

$Q: \quad 4x_1 + 2x_2 - 4x_1x_2 - 4 = 0$

parallel to the direction of $(1,2) \in \mathbb{K}^2$ from the origin

Ans: 0. We introduce x_0 so that we can work in proj space
new quest: $[0:1:2]$ is in the tangent space
(the end of the tangent)

1. Write down the symmetric matrix A for the bilinear form:
Homogenise and obtain

$$4x_0x_1 + 2x_0x_2 - 4x_1x_2 - 4x_0^2$$

Now $\mathring{a}_{01} = 4$, $\mathring{a}_{02} = 2$, $\mathring{a}_{12} = -4$, $\mathring{a}_{00} = -4$

so $\mathring{A}$

$$= \begin{pmatrix} -4 & 4 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

symmetrise: A

$$= \begin{pmatrix} -4 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 0 \end{pmatrix}$$

assuming $X \in Q$

2. Find X s.t. $[0:1:2] \in X^{\perp}$

$$f(x, (0,1,2)) = (x_0 \ x_1 \ x_2) \, A \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \qquad \text{where } X = x = [x_0 : x_1 : x_2]$$

$$= (x_0 \ x_1 \ x_2) \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$$

So our desired X satisfies

$$4x_0 - 4x_1 - 2x_2 = 0$$

and X should lie on the conic, so

$$4x_0x_1 + 2x_0x_2 - 4x_1x_2 - 4x_0^2 = 0$$

Assume $x_0 \neq 0$, we can put $x_0 = 1$ in our proj setting,

$$\begin{cases} 4 - 4x_1 - 2x_2 = 0 & — ① \\ 4x_1 + 2x_2 - 4x_1x_2 - 4 = 0 & ② \end{cases}$$

① $\Rightarrow$ $x_2 = 2 - 2x_1$ — ③

Sub ③ into ②, $\quad 4x_1 + 2(2-2x_1) - 4x_1(2-2x_1) - 4 = 0$

$$x_1(x_1 - 1) = 0$$

$$\Rightarrow x_1 = 0 \quad \text{or} \quad x_1 = 1$$

Sub into ③, we get $X = [1:0:2]$ or $X = [1:1:0]$

$\therefore$ The tangents are $X + \lambda(0,1,2)$, i.e. $X + [0:1:2]$

5. In $\mathbb{P}^2$:

Find the projective conic section $\bar{Q}$ passing through the points $A_1 = [0:1:1]$, $A_2 = [1:0:1]$, $A_3 = [1:1:0]$, $A_4 = [1:1:-1]$, $A_5 = [1:-1:1]$

Ans : Let $\bar{Q}: a x_1^2 + b \cdot x_2^2 + c \cdot x_1 x_2 + d x_0 x_1 + e x_0 x_2 + f x_0^2 = 0$

Sub A_2,

$$b + e + f = 0 \quad \text{— ①}$$

Sub A_3,

$$a + d + f = 0 \quad \text{— ②}$$

Sub A_4,

$$a + b - c + d - e + f = 0 \quad \text{— ③}$$

Sub A_5,

$$a + b - c - d + e + f = 0 \quad \text{— ④}$$

Sub A_1,

$$a + b + c = 0 \quad \text{— ⑤}$$

③ − ④,
$$2d - 2e = 0$$
$$d = e \quad \text{— ⑥}$$

Now sub ⑥ into ①, ②,
$$a = b \quad \text{— ⑦}$$

Sub ⑦ into ⑤,
$$2a + c = 0$$
$$c = -2a \quad \text{— ⑧}$$

Rewrite ③ and ④,
$$2a + 2a + f = 0$$
$$\Rightarrow f = -4a \quad \text{— ⑨}$$

Put ⑨ into ②,
$$a + d - 4a = 0$$
$$d = 3a$$

$\therefore \quad (a, a, -2a, 3a, 3a, -4a) = (a, b, c, d, e, f)$

$\bar{Q} = x_1^2 + x_2^2 - 2x_1x_2 + 3x_0x_1 + 3x_0x_2 - 4x_0^2 = 0$

Def. (affine) canonical form in 2D

A canonical ellipse is of the form $\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 = 1$

" hyperbola is " $\left(\frac{x_1}{a_1}\right)^2 - \left(\frac{x_2}{a_2}\right)^2 = 1$

" parabola is " $a_1 x_1^2 - a_2 x_2 = 0$

Def. Canonical form in 3D quadratic surfaces

ellipsoid : $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{x_3^2} = 1$

cone : $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2} = 0$

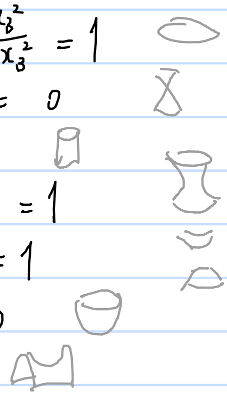
cylinder : $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1$

connected hyperboloid : $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2} = 1$

disconnected hyperboloid : $-\frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$

elliptic paraboloid : $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} - \frac{x_3}{a_3} = 0$

hyperbolic paraboloid : $\frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \frac{x_3}{a_3} = 0$



Affine approach:

We may sacrifice the distinction between translations

i.e., $n x_0 = x_0$ so $x_1 + 1 \doteq x_1 + x_0 \doteq x_1$

for simpler expressions, and do it in the affine setting

1. Find A, A_0 by ignoring all a_{0j}, a_{i0}.
   Check $\operatorname{rk} A$, $\operatorname{rk} A_0$.

2. Calculate the eigenvalues λ of A_0:

Case I – all $\lambda \neq 0$:

I.3. Find a_{00} s.t.

$$\begin{vmatrix} a_{00} & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_4 \end{vmatrix} = |A|$$

$\because$ obtain from transl & rotation

$\therefore$ det unchange

I.4 Then the canonical form is given by

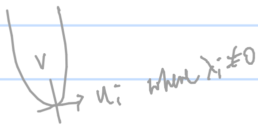
$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + a_{00} = 0$$

Case II – some $\lambda = 0$:

A

$v \in Q$

a) If $\operatorname{rk} A - \operatorname{rk} A_0 = 2$:



II.3. Find vertex v of Q by solving $\begin{cases} f(v,v) = 0 \\ f(u_i, v) = 0 \end{cases}$ for $\lambda_i \neq 0$

II.4. Find the linear term $c_{0i} y_i$ by solving $f(u_i, v) = \frac{c_{0i}}{2}$

II.5. Then the canonical form is given by

i) If there are two λ's $= 0$,

$$\lambda_1^2 y_1^2 + \Sigma c_{0i} y_i = 0$$

parabolic cylin

ii) If only one $\lambda = 0$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \Sigma c_{0i} y_i = 0$$

if all λ's are of the same sign $\rightsquigarrow$ elliptic paraboloid 

if not $\rightsquigarrow$ hyperbolic paraboloid 

b) If $\operatorname{rk} A - \operatorname{rk} A_0 = 1$:

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$$

if all λ's are of the same sign $\rightsquigarrow$ elliptic cylinder 

if not $\rightsquigarrow$ hyperbolic cylinder 

c) If $\operatorname{rk} A - \operatorname{rk} A_0 = 0$:

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 0$$

intersecting planes 

There are other situations, eg. double plane $x^2 = 0$

parallel planes $x^2 = a^2$

8. Determine the canonical form of the conic surface

$Q: \quad x_1^2 + x_2^2 + 5x_3^2 - 6x_1x_2 - 2x_1x_3 + 2x_2x_3 - 6x_1 + 6x_2 - 6x_3 + 9 = 0$

Ans: 0. Homogenise Q:

$$x_1^2 + x_2^2 + 5x_3^2 - 6x_1x_2 - 2x_1x_3 + 2x_2x_3 - 6x_0x_1 + 6x_0x_2 - 6x_0x_3 + 9x_0^2$$

1. $A = \begin{pmatrix} 9 & -3 & 3 & -3 \\ -3 & 1 & -3 & -1 \\ 3 & -3 & 1 & 1 \\ -3 & -1 & 1 & 5 \end{pmatrix}$ $\quad A_0 = \begin{pmatrix} 1 & -3 & -1 \\ -3 & 1 & 1 \\ -1 & 1 & 5 \end{pmatrix}$

2. Find eigenvalues of A_0:

$$A_0 v = \lambda v$$

$\lambda_1 = 6, \ v_1 = (-1, 1, 2), \quad \lambda_2 = 3, \ v_2 = (1, -1, 1), \quad \lambda_3 = -2, \quad v_3 = (1, 1, 0)$

Case I. all $\lambda \neq 0$

3. Find a_{00} s.t.

$$\begin{vmatrix} a_{00} & & & \\ & 6 & & \\ & & 3 & \\ & & & -2 \end{vmatrix} = |A|$$

$$-36 \ a_{00} = 0 \Rightarrow a_{00} = 0$$

4. The canonical form is given by

$$6y_1^2 + 3y_2^2 - 2y_3^2 = 0$$