Notation: vector - superscript, covector - subscript

4.4

Find a new basis $\alpha = \{e_1, e_2, e_3\}$ for $\mathbb{R}^3$ so that the resulting dual basis α^* for $(\mathbb{R}^3)^*$ is

$$\alpha^* = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}, \quad \left.\begin{aligned} f^1(x_1, x_2, x_3) &= 2x_1 - x_2 \\ f^2(x_1, x_2, x_3) &= x_2 - x_3 \\ f^3(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \end{aligned}\right\} \text{ linear functional}$$

unknown, variable

i.e. $f^1 = (2 \ \ -1 \ \ 0)$
$f^2 = (0 \ \ 1 \ \ -1)$
$f^3 = (1 \ \ 1 \ \ 1)$

basis s.t. f is dual basis

Ans: By the def of a dual basis, we must have

def:

$f^1(e_1) = 1, \quad f^2(e_1) = 0, \quad f^3(e_1) = 0$

That means to find e_1 is to solve

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore e_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \\ -\frac{1}{5} \end{pmatrix}$$

Similarly, for e_2, solve

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore e_2 = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ -\frac{3}{5} \end{pmatrix}$$

For e_3, solve

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore e_3 = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}$$

Rk. $$(e_1 \ \ e_2 \ \ e_3) = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}^{-1}$$

Check: $$\begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}(e_1 \ e_2 \ e_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4.5 Find a dual basis $\alpha^* = (f^1, f^2, f^3)$ to a basis α of $\mathbb{R}^3$:

$\alpha = (e_1, e_2, e_3)$

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Ans: Recall : $(e_1 \;\; e_2 \;\; e_3) = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}^{-1}$

$$\Leftrightarrow \quad (e_1 \;\; e_2 \;\; e_3)^{-1} = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}$$

So we find $\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}^{-1}$

$$= \begin{pmatrix} -\frac{1}{2} & -\frac{5}{4} & 1 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$$\therefore \quad \begin{aligned} f^1 &= \left(-\tfrac{1}{2} \quad -\tfrac{5}{4} \quad 1\right) \\ f^2 &= \left(\tfrac{1}{2} \quad -\tfrac{1}{4} \quad 0\right) \\ f^3 &= \left(0 \quad \tfrac{1}{2} \quad 0\right) \end{aligned}$$

Recall that a natural pairing of V and its dual space $[-,-]_V : V^* \times V \to \mathbb{k}$ is defined as follows:

Let $\varphi \in V^*$, $v \in V$,
then $[\varphi, v]_V := \varphi(v)$

Def. Let $f : V \to W$ be linear, the dual map $f^* : W^* \to V^*$ is defined by

$$f^*(\varphi) = \varphi \circ f \qquad \forall \varphi \in W^*$$

Rk. $[f^*(\varphi), v]_V = [\varphi, f(v)]_W$

7.1 Describe (in coordinates) the dual mapping to a linear map $A : \mathbb{k}^n \to \mathbb{k}^m$, i.e., $x \mapsto Ax$ for a matrix $A \in \mathrm{Mat}_{m \times n}\mathbb{k}$.

?

Ans: We want to find $A^* : (\mathbb{k}^m)^* \to (\mathbb{k}^n)^*$ s.t.

$$[A^* y, x]_{\mathbb{k}^n} = [y, Ax]_{\mathbb{k}^m} \qquad \forall x, y$$

$$\Leftrightarrow (A^* y) \cdot x = (y) \cdot (Ax) \qquad \forall x, y$$

$$\Leftrightarrow A^* y \cdot x = yA \cdot x \qquad \forall x, y$$

$$\Leftrightarrow A^* y = yA \qquad \forall y$$

So $A^* : (\mathbb{k}^m)^* \to (\mathbb{k}^n)^*$

$$y \mapsto y \cdot A$$

unit of vec space

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\Phi_V"'] & W \arrow[d, "\Phi_W"] \\ V^* & W^* \arrow[l, "f^*"] \end{tikzcd}$$

$f^* \cdot \Phi_W \cdot f(u) = \Phi_V(u)$

$f^*(\langle f(u), - \rangle) = \langle u, - \rangle$

$f^*(\langle f \qquad = \langle u, x \rangle$

7.2 Let $U := V \oplus W$. — $U = V \times W$ where $V \cap W = \{0\}$

Show that the composition $f: W \hookrightarrow U \to U/V$ is an iso.

Ans: We show that
$$\begin{aligned} f: W &\hookrightarrow U \to U/V \\ w &\mapsto w \mapsto w+V \end{aligned}$$
is injective ($\ker f = 0$) and surjective. hint

$\ker f = 0$:

Suppose $f(w) = 0 + V \in U/V$
$\Rightarrow w + V = 0 + V$
$\Rightarrow w \in V$
but since $w \in W$, this forces $w = 0$.

Surjective:

Suppose $u + V \in U/V$.
Since $U = V \oplus W$, write $u = v + w$ for some $v \in V, w \in W$.
Now since $v \in V$
we have $v + w - w \in V$
$\Rightarrow v + w + V = w + V$
$\Rightarrow u + V = w + V$
$\therefore \forall\ u + V,\ \exists\ w \in W : f(w) = w + V = u + V$.

7.3 Describe by a system of linear eqts, the subspace
$V = [(1,-1,0,0)^T, (1,0,-1,0)^T, (0,1,0,-1)^T] \subseteq \mathbb{k}^4$

Ans: Note that $\forall\ v \in V,\ w \in V^\perp$, $\langle v, w \rangle = 0$ } hint
and $\dim V^\perp = \dim \mathbb{k}^4 - \dim V = 1$
We find a basis $\{w\}$ for $V^\perp$:

Solve
$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = 0$$

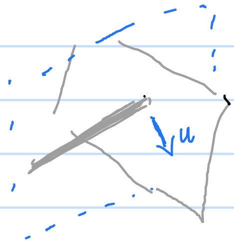
$\Rightarrow w = (1, 1, 1, 1)^T$

Now for an arbitrary $v \in V$,
$\langle v, (1,1,1,1)^T \rangle = 0$
$\Rightarrow v_1 + v_2 + v_3 + v_4 = 0$
which is our desired system of linear eqts

7.4 Describe all planes passing through the line

$$p: \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{cases}$$

(intersection of 2 planes) in $\mathbb{k}^3$.



?

Ans: The $\perp$ complement of a line is a plane, and vice versa. hint

We want to find $p^\perp$ and pick a vector u on it, then $u^\perp$ is a plane passing through p.

① Find the solution for p, hence a basis for p:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x = k(0, 1, 1) \qquad \text{for some } k \in \mathbb{k}$$

$$\Rightarrow \quad p = [(0, 1, 1)^T]$$

② Find $p^\perp$: $(0 \quad 1 \quad 1) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$

$$y = k(1, 0, 0), \quad y = k(0 \quad -1 \quad 1) \quad \text{for some } k \in \mathbb{k}$$

$$\Rightarrow \quad p^\perp = \{(0, -1, 1)^T, (1, 0, 0)^T\}$$

③ Pick an arbitrary $u \in p^\perp$, find $u^\perp$:

$$u = a \cdot (0, -1, 1)^T + b \cdot (1, 0, 0)^T.$$

Let $v \in u^\perp$,

$$\langle a \cdot (0, -1, 1) + b \cdot (1, 0, 0), v \rangle = 0$$

$$(b \quad -a \quad a) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

$$b v_1 - a v_2 + a v_3 = 0$$

So $u^\perp$, the required plane passing through p, is in the form

$$b v_1 - a v_2 + a v_3 = 0, \quad \text{for any } a, b \in \mathbb{k}.$$

7.5. Using the Motzkin elimination, decide the solvability of the system of inequalities:

$$\begin{cases} x, y, z \geq 0 \\ 4 \geq x+y+z \geq 2 \\ 3 \geq x+y \geq 1 \\ 2 \geq z \end{cases} \quad \text{--- } (*)$$

if it is solvable, find one solution.

Ans: Step I: Eliminate x, write x as the subject:

$x \geq 0$ — ③

$4 \geq x+y+z \iff 4-y-z \geq x$ — ①

$x+y+z \geq 2 \iff x \geq 2-y-z$ — ④

$3 \geq x+y \iff 3-y \geq x$ — ②

$x+y \geq 1 \iff x \geq 1-y$ — ⑤

$y \geq 0$, $z \geq 0$, $2 \geq z$

upper bound of x | lower bound of x | x-free

Solvable $\iff$ all up bds $\geq$ low bds.

① ≥ ③, $4-y-z \geq 0$

① ≥ ④, $4 \geq 2$ always true

① ≥ ⑤, $4-y-z \geq 1-y$

$3-z \geq 0$

② ≥ ③, $3-y \geq 0$

② ≥ ④, $3-y \geq 2-y-z$

$1+z \geq 0$

② ≥ ⑤, $3 \geq 1$, always true

Now we have

$$\begin{cases} 4-y-z \geq 0, & y \geq 0 \\ 3-z \geq 0, & z \geq 0 \\ 3-y \geq 0, & 2 \geq z \\ 1+z \geq 0 \end{cases} \quad \text{--- } (**)$$

Step II: Eliminate y:

$4-z \geq y$ — ①'

$3 \geq y$ — ②'

$y \geq 0$ — ③'

$z \geq 0$, $2 \geq z$, $3-z \geq 0$, $1+z \geq 0$

①' ≥ ③', $4-z \geq 0$

②' ≥ ③', $3 \geq 0$ always

Step III: Solve for z:

$\therefore 2 \geq z \geq 0$

Step IV: Find 1 particular solution:

Pick $z=1$, substitute to ineq in $(**)$,

$$3-y\geq 0 \qquad\qquad y\geq 0$$

$$\therefore\ 3\geq y\geq 0$$

Pick $y=1$, substitute to ineq $(*)$,

$$2\geq x \qquad\qquad x\geq 0$$

$$\therefore\ 2\geq x\geq 0$$