Linear programming & simplex method

E.g. Minimise $10x_1 + x_2$ — objective function (decision variable)

subject to $x_1 + x_2 \le 2$ — constraint

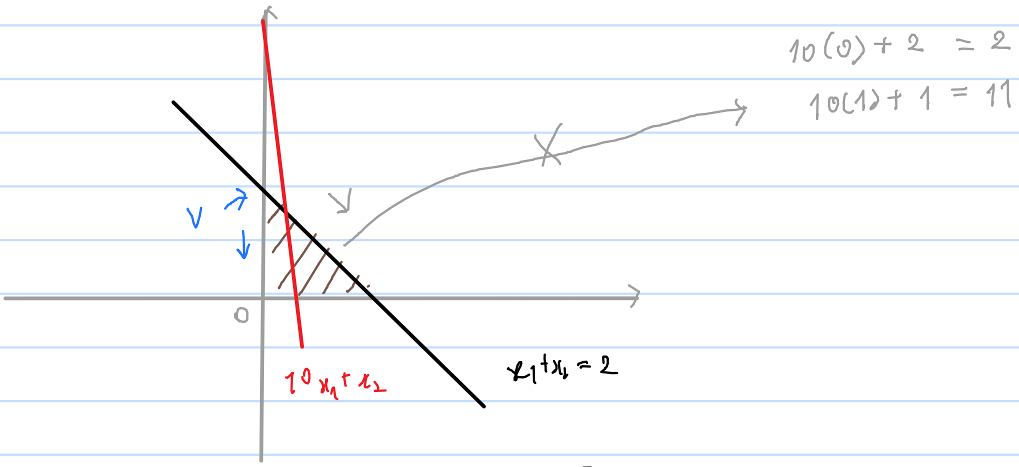
$x_1 \ge 0$

$x_2 \ge 0$

We can rewrite as : $\min\{(10 \;\; 1)\, x \mid \begin{pmatrix}1\\1\end{pmatrix} x \le 2,\; x \ge 0\}$

We can solve this by simplex method.

Idea : The constraints form a (convex) polyhedron.



1. the red line represents the objective function $10x_1 + x_2$
2. start at any feasible vertice v
3. find an edge along which the obj func is improving (in our case, decreasing)
4. arrive at another vertice
5. repeat 3 ~ 4 until we cannot find an edge along which the obj func is improving

In our E.g., $x = (0, 0)$ is the solution.

Canonical form

A linear program is in the canonical form $\Leftrightarrow$
$\min(c^T x \mid Ax = b, x \geq 0)$ where $b \geq 0$, and A contains an identity submatrix

Turning a general LP into the canonical form

for an inequality
$$\sum_j a_{ij} x_j \leq b_i$$
we can add a slack variable s_i,
i.e., $$\sum_j a_{ij} x_j + s_i = b_i$$
$$s_i \geq 0$$

for an inequality
$$\sum_j a_{ij} x_j \geq b_i$$
we can subtract a slack variable s_i,
i.e., $$\sum_j a_{ij} x_j - s_i = b_i$$
$$s_i \geq 0$$

Applying simplex method

$z = 10x_1 + x_2$
$\Rightarrow -z + 10x_1 + x_2 = 0$

Draw a table

x_1	x_2	s_1	$-z$	RHS	
1	1	1	0	2	— constraint
10	1	0	1	0	— obj fun

1. variables corresponding to the id submatrix = basic variables
(in our case, $s_1 = (1\ 0)^T$, $-z = (0\ 1)^T$)
2. obtain a basic feasible sol by setting non-basic var $= 0$,
(in our case, $x_1 = 0$, $x_2 = 0$, so $s_1 = 2$, $z = 0$)
3. if all the coefficient c_j of the obj fun row is ≥ 0, optimal
otherwise,
improve a basic feasible sol:
i) choose an entering variable x_j with coeff $c_j < 0$,
ii) pick $i = \arg\min_k \left(\frac{b_k}{a_{kj}} \mid a_{kj} > 0\right)$ and pivot on (i, j)
i.e., turn the column j into e_i (k pivot + other row)
maximise x_j
set non-basic $= 0$, set $x_j = \Delta$
express others, find the one getting to 0 first
iii) repeat until we get back to 3.

Summary

$\exists j: c_j < 0 \xrightarrow{\text{no}}$ basic sol is optimal
↓ yes, pick this j
$\exists i: a_{ij} > 0 \xrightarrow{\text{no}}$ unbounded, no solution
↓ yes
choose $i = \arg\min_k \left(\frac{b_k}{a_{kj}} \mid a_{kj} > 0\right)$, pivot on (i,j), find basic sol

5.6 $\min\{(-2 \quad 3 \quad -1)x \mid \begin{pmatrix} 1 & 1 & 1 \\ 4 & -3 & 1 \\ 2 & 1 & -1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 3 \\ 10 \end{pmatrix}, x \geq 0\}$

Ans: Turn this into the canonical form:

$\min\{(-2 \quad 3 \quad -1 \quad 0 \quad 0 \quad 0)x \mid \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & -3 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 3 \\ 10 \end{pmatrix}, x \geq 0, s_1, s_2, s_3 \geq 0\}$

① Table

x_1	x_2	x_3	s_1	s_2	s_3	$-Z$	RHS
1	1	1	1	0	0	0	10
4	-3	1	0	1	0	0	3
2	1	-1	0	0	1	0	10
-2	3	-1	0	0	0	1	0

basic: $s_1, s_2, s_3, -Z$

$\Rightarrow$ set $x_1 = x_2 = x_3 = 0$, obtain $s_1 = 10$, $s_2 = 3$, $s_3 = 10$

② $c_1 < 0$, $\exists\, a_{i1} > 0$.

min $= \frac{3}{4}$

$\therefore$ pivot on $(2, 1)$:

x_1	x_2	x_3	s_1	s_2	s_3	$-Z$	RHS	
0	$\frac{7}{4}$	$\frac{3}{4}$	1	$-\frac{1}{4}$	0	0	$\frac{37}{4}$	$+(-\frac{1}{4})$ pivot
4	-3	1	0	1	0	0	3	
0	$\frac{5}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	1	0	$\frac{17}{2}$	$+(-\frac{1}{2})$ pivot
0	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	1	$\frac{3}{2}$	$+\frac{1}{2}$ pivot

x_1	x_2	x_3	s_1	s_2	s_3	$-Z$	RHS	
0	$\frac{7}{4}$	$\frac{3}{4}$	1	$-\frac{1}{4}$	0	0	$\frac{37}{4}$	
1	$-\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{3}{4}$	$\times\frac{1}{4}$
0	$\frac{5}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	1	0	$\frac{17}{2}$	
0	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	1	$\frac{3}{2}$	

basic: s_1, x_1, s_3

$\Rightarrow$ set $x_2 = x_3 = s_2 = 0$, obtain $s_1 = \frac{37}{4}$, $x_1 = \frac{3}{4}$, $s_3 = \frac{17}{2}$

$-\frac{1}{2} + 6\times\frac{1}{4}$

③ $c_3 < 0$, $\exists\, a_{i3} > 0$, min $= 3$.

$\therefore$ pivot on $(2, 3)$:

x_1	x_2	x_3	s_1	s_2	s_3	$-Z$	RHS	
-3	4	0	1	-1	0	0	7	$+(-3)$ pivot
1	$-\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{3}{4}$	
-6	-2	0	0	1	1	0	13	$+6$ pivot
2	0	0	0	1	0	1	3	$+2$ pivot

x_1	x_2	x_3	s_1	s_2	s_3	$-z$	RHS	
-3	4	0	1	-1	0	0	7	
4	-3	1	0	1	0	0	3	$\times 4$
-6	-2	0	0	1	1	0	13	
2	0	0	0	1	0	1	3	

basic : s_1, x_3, s_3, $-z$

$\Rightarrow$ set $x_1 = x_2 = s_2 = 0$, obtain $s_1 = 7$, $x_3 = 3$, $s_3 = 13$, $-z = 3$

(4) No $c_j < 0$, so we arrive optimal sol.

i.e. $z = -3$, $x_1 = 0$, $x_2 = 0$, $x_3 = 3$

I = E =

Artificial problem

If we have a LP

$\min(c^T x \mid Ax = b, x \geq 0)$ where $b \geq 0$,

but A does not contain identity submatrix,

we introduce artificial variables t_i, the new LP is

$\min((1\ 1 \cdots 1)^T t \mid Ax + Et = b, x \geq 0, t \geq 0)$ where $b \geq 0$

However, this is not canonical now as $A + E$ does not contain an identity submatrix.

We can subtract all the rows of constraints

i.e., new obj fun = row of obj fun $- \Sigma$ rows of constraints,

⇒ this gives the canonical form.

5.7 $\min\left\{ (2\ 4\ 7\ 2\ 5)x \;\middle|\; \begin{pmatrix} 1 & 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix},\ x \geq 0 \right\}$

Ans: ① Pre-table

x_1	x_2	x_3	x_4	x_5	t_1	t_2	t_3	$-z$	RHS
1	1	2	1	2	1	0	0	0	7
1	2	3	1	1	0	1	0	0	6
1	1	1	2	1	0	0	1	0	4
0	0	0	0	0	1	1	1	1	0

⇒ Table

x_1	x_2	x_3	x_4	x_5	t_1	t_2	t_3	$-z$	RHS
1	1	2	1	2	1	0	0	0	7
1	2	3	1	1	0	1	0	0	6
1	1	1	2	1	0	0	1	0	4
-3	-4	-6	-4	-4	0	0	0	1	-17

basic: t_1, t_2, t_3, $-z$

⇒ set $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, obtain $t_1 = 7$, $t_2 = 6$, $t_3 = 4$

② $c_3 < 0$, $a_{i3} \geq 0$, min $= 2$ (3,4) is also ok

∴ pivot on (2,3):

x_1	x_2	x_3	x_4	x_5	t_1	t_2	t_3	$-z$	RHS	
$\frac{1}{3}$	$\frac{-1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	1	$\frac{-2}{3}$	0	0	3	$+ (\frac{-2}{3})$ pivot
1	2	3	1	1	0	1	0	0	6	
$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{5}{3}$	$\frac{2}{3}$	0	$\frac{-1}{3}$	1	0	2	$+ (\frac{-1}{3})$ pivot
-1	0	0	-2	-2	0	2	0	1	-5	$+ 2$ pivot

x_1	x_2	x_3	x_4	x_5	t_1	t_2	t_3	$-z$	RHS	
$\frac{1}{3}$	$\frac{-1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	1	$\frac{-2}{3}$	0	0	3	
$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	2	$\times \frac{1}{3}$
$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{5}{3}$	$\frac{2}{3}$	0	$\frac{-1}{3}$	1	0	2	
-1	0	0	-2	-2	0	2	0	1	-5	

basic: t_1, x_3, t_3, $-z$

⇒ set $x_1 = x_2 = x_4 = x_5 = t_2 = 0$, obtain $t_1 = 3$, $x_3 = 2$, $t_3 = 2$

③ $c_1 < 0$, $a_{i1} > 0$, $\min = 3$

$\therefore$ pivot on $(3, 1)$:

x_1	x_2	x_3	x_4	x_5	t_1	t_2	t_3	$-z$	RHS	
0	$\frac{-1}{2}$	0	$\frac{-1}{2}$	1	1	$\frac{-1}{2}$	$\frac{-1}{2}$	0	2	$+ (\frac{-1}{2})$ pivot
0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{-1}{2}$	0	1	$+ (\frac{-1}{2})$ pivot
$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{5}{3}$	$\frac{2}{3}$	0	$\frac{-1}{3}$	1	0	2	
0	$\frac{1}{2}$	0	$\frac{1}{2}$	-1	0	$\frac{3}{2}$	$\frac{3}{2}$	1	-2	$+ \frac{3}{2}$ pivot

x_1	x_2	x_3	x_4	x_5	t_1	t_2	t_3	$-z$	RHS	
0	$\frac{-1}{2}$	0	$\frac{-1}{2}$	1	1	$\frac{-1}{2}$	$\frac{-1}{2}$	0	2	
0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{-1}{2}$	0	1	
1	$\frac{1}{2}$	0	$\frac{5}{2}$	1	0	$\frac{-1}{2}$	3	0	3	$\times \frac{3}{2}$
0	$\frac{1}{2}$	0	$\frac{1}{2}$	-1	0	$\frac{3}{2}$	$\frac{3}{2}$	1	-2	

basic: t_1, x_3, x_1, $-z$

$\Rightarrow$ set $x_2 = x_4 = x_5 = t_2 = 0$, obtain

④ $c_5 = -1 < 0$, $a_{15}, a_{35} > 0$, $\min = 2$

$\therefore$ pivot on $(1, 5)$:

x_1	x_2	x_3	x_4	x_5	t_1	t_2	t_3	$-z$	RHS	
0	$\frac{-1}{2}$	0	$\frac{-1}{2}$	1	1	$\frac{-1}{2}$	$\frac{-1}{2}$	0	2	
0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{-1}{2}$	0	1	
1	1	0	3	0	-1	0	$\frac{7}{2}$	0	1	$+ (-1)$ pivot
0	0	0	0	0	1	1	1	1	0	$+$ pivot

basic: x_5, x_3, x_1, $-z$

$\Rightarrow$ set $x_2 = x_4 = t_1 = t_2 = t_3 = 0$,

obtain $x_1 = 1$, $x_3 = 1$, $x_5 = 2$, $z = -2$

⑤ Since all $c_j \geq 0$, optimal solution is arrived.

$x = (1 \;\; 0 \;\; 1 \;\; 0 \;\; 2)$, $z = -2$