

Exercises—Global Analysis

1. Consider the **cylinder** in \mathbb{R}^3 given by the equation

$$M := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = R^2\},$$

where $R > 0$. Show that M is a 2-dimensional submanifold in \mathbb{R}^3 . Moreover, give formula for local parametrizations and local trivializations, and a description of M as a local graph.

2. Consider a **double cone** given by rotating a line through 0 of slope α around the z -axis in \mathbb{R}^3 . It is given by the equation

$$z^2 = (\tan \alpha)^2(x^2 + y^2).$$

At which points is the double cone a smooth submanifold of \mathbb{R}^3 ? Around the points where it is give a formula for local parametrizations and trivializations, and a description of it as a local graph.

3. Denote by $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ the nm -dimensional vector space of linear maps from \mathbb{R}^n to \mathbb{R}^m . Consider the subset $\text{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ of linear maps in $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ of rank r . Show that $\text{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of dimension of $r(n + m - r)$ in $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

Hint: Let $T_0 \in \text{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ be a linear map of rank r and decompose \mathbb{R}^n and \mathbb{R}^m as follows

$$\mathbb{R}^n = E \oplus E^\perp \quad \text{and} \quad \mathbb{R}^m = F \oplus F^\perp, \quad (0.1)$$

where F equals the image of T_0 and E^\perp the kernel of T_0 , and $(\cdot)^\perp$ denotes the orthogonal complement. Note that $\dim E = \dim F = r$. With respect to (0.1) any $T \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ can be viewed as a matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \text{Hom}(E, F)$, $B \in \text{Hom}(E^\perp, F)$, $C \in \text{Hom}(E, F^\perp)$ and $D \in \text{Hom}(E^\perp, F^\perp)$. Show that the set of matrices T with A invertible defines an open neighbourhood of T_0 and characterize the elements in this neighbourhood that have rank r (equivalently, the ones that have an $(n - r)$ -dimensional kernel).

4. For $i = 1, \dots, n$ let (M_i, \mathcal{A}_i) be a smooth manifolds. Suppose $M := M_1 \times \dots \times M_n$ is endowed with the product topology. Then show that

$$\mathcal{A} := \{(U_1 \times \dots \times U_n, u_1 \times \dots \times u_n) : (U_i, u_i) \in \mathcal{A}_i\}$$

defines a smooth atlas on M and that the projections $\text{pr}_i : M \rightarrow M_i$ are smooth. Moreover show that, for any smooth manifold N , a map $f : N \rightarrow M$ is smooth if and only if $f_i := \text{pr}_i \circ f : N \rightarrow M_i$ is smooth for all i , and show that this property characterizes the smooth manifold structure on M uniquely.

5. Suppose (M_i, \mathcal{A}_i) are smooth manifolds for $i \in I$, where I is countable. Consider the disjoint union

$$M := \sqcup_{i \in I} M_i = \cup_{i \in I} \{(x, i) : x \in M_i\}$$

endowed with the disjoint union topology and denote by $\text{inj}_i : M_i \hookrightarrow M$ the canonical injections ($\text{inj}_i(x) = (x, i)$). Show that $\mathcal{A} := \cup_{i \in I} \mathcal{A}_i$ defines a smooth atlas on M and that the injections inj_i are smooth. Moreover, show that for any smooth manifold N , a map $f : M \rightarrow N$ is smooth if and only if $f_i := f \circ \text{inj}_i : M_i \rightarrow N$ is smooth for all i , and show that this property characterizes the smooth manifold structure on M uniquely.

6. Suppose $U \subset \mathbb{R}^m$ is open and $f : U \rightarrow \mathbb{R}^n$ a smooth map such that $D_x f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is of rank r for all $x \in U$.

Show that for any $x_0 \in U$ there exists a diffeomorphism ϕ between an open neighbourhood of x_0 and an open neighbourhood of $0 \in \mathbb{R}^m$ and a diffeomorphism ψ between an open neighbourhood of $y_0 = f(x_0)$ and an open neighbourhood of 0 in \mathbb{R}^n such that the locally defined map

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^r \times \mathbb{R}^{m-r} \rightarrow \mathbb{R}^r \times \mathbb{R}^{n-r}$$

has the form $(x_1, \dots, x_r, \dots, x_m) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$.

Hint: The idea is that f locally around x_0 looks like $D_{x_0} f$, which is a linear map $\mathbb{R}^m \rightarrow \mathbb{R}^n$ of rank r , which up to a basis change has the form $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$.

- (a) Set $E_2 := \ker(D_{x_0} f) \subset \mathbb{R}^m$ and $E_1 := E_2^\perp$, and $F_1 := \text{Im}(D_{x_0} f) \subset \mathbb{R}^n$ and $F_2 := F_1^\perp$. Decompose

$$\mathbb{R}^m = E_1 \oplus E_2 \quad \text{and} \quad \mathbb{R}^n = F_1 \oplus F_2,$$

and consider f as a map $f = (f_1, f_2) : E_1 \oplus E_2 \rightarrow F_1 \oplus F_2$ defined on $U \subset E_1 \oplus E_2 = \mathbb{R}^m$.

- (b) Show that $\phi : E_1 \oplus E_2 \rightarrow F_1 \oplus F_2$ given by

$$\phi(x^1, x^2) = (f_1(x^1, x^2) - f_1(x_0^1, x_0^2), x^2 - x_0^2)$$

is a local diffeomorphism around $x_0 = (x_0^1, x_0^2)$ whose local inverse will be the required map.

(c) Show that $g := f \circ \phi^{-1} : F_1 \oplus E_2 \rightarrow F_1 \oplus F_2$ has the form

$$g(y^1, y^2) = (g_1((y^1, y^2)), g_2((y^1, y^2))) = (y^1 + y_0^1, g_2(y^1, 0)).$$

Now ψ is easily seen to be...?

7. Suppose M and N are manifolds of dimension m respectively n and let $f : M \rightarrow N$ be a smooth map of constant rank r . Deduce from (1) that for any fixed $y \in f(M)$ the preimage $f^{-1}(y) \subset M$ is a submanifold of dimension $m - r$ in M .
8. Consider the Grassmannian of r -planes in \mathbb{R}^n :

$$\text{Gr}(r, n) := \{E \subset \mathbb{R}^n : E \text{ is a } r\text{-dimensional subspace of } \mathbb{R}^n\}.$$

Denote by $\text{St}_r(\mathbb{R}^n)$ the set of r -tuples of linearly independent vectors in \mathbb{R}^n . Identifying an element $X \in \text{St}_r(\mathbb{R}^n)$ with a $n \times r$ matrix

$$X = (x^1, \dots, x^r) \quad x^i \in \mathbb{R}^n,$$

shows that $\text{St}_r(\mathbb{R}^n)$ equals the subset of rank r matrices in the vector space $\mathbf{M}_{n \times r}(\mathbb{R})$, which we know from Tutorial 1 is an open subset. Write

$$\pi : \text{St}_r(\mathbb{R}^n) \rightarrow \text{Gr}(r, n)$$

for the natural projection given by $\pi(X) = \text{span}(x^1, \dots, x^r)$ and equip $\text{Gr}(r, n)$ with the quotient topology with respect to π .

- (a) Fix $E \in \text{Gr}(r, n)$ and let $F \subset \mathbb{R}^n$ be a subspace of dimension $n - r$ such that $\mathbb{R}^n = E \oplus F$. Show that

$$U_{(E,F)} = \{W \in \text{Gr}(r, n) : W \cap F = \{0\}\} \subset \text{Gr}(r, n)$$

is an open neighbourhood of E .

- (b) Show that any element $W \in U_{(E,F)}$ determines a unique linear map

$$\widetilde{W} : E \rightarrow F$$

such that its graph equals W , i.e. $W = \{(x, \widetilde{W}x) : x \in E\}$.

- (c) Show that the map $u_{E,F} : U_{(E,F)} \rightarrow \text{Hom}(E, F)$ given by $u_{E,F}(W) = \widetilde{W}$ is a homeomorphism.

- (d) Show that

$$\mathcal{A} := \{(U_{(E,F)}, u_{(E,F)}) : E, F \subset \mathbb{R}^n \text{ complimentary subspaces of dimension } r \text{ resp. } n-r\}$$

is a smooth atlas for $\text{Gr}(r, n)$.

9. For a topological space M denote by $C^0(M)$ the vector space of continuous real-valued functions $f : M \rightarrow \mathbb{R}$. Any continuous map $F : M \rightarrow N$ between topological spaces M and N induces a map $F^* : C^0(N) \rightarrow C^0(M)$ given by $F^*(f) := f \circ F : M \rightarrow \mathbb{R}$.

- (a) Show that F^* is linear.
- (b) If M and N are (smooth) manifolds, show that $F : M \rightarrow N$ is smooth $\iff F^*(C^\infty(N)) \subset C^\infty(M)$.
- (c) If F is a homeomorphism between (smooth) manifolds, show that F is a diffeomorphism $\iff F^*$ is an isomorphism.

10. We have seen in the first tutorial that $\text{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ of dimension $r(n + m - r)$ in. For $X \in \text{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ compute the tangent space

$$T_X \text{Hom}_r(\mathbb{R}^n, \mathbb{R}^m) \subset T_X \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \cong \text{Hom}(\mathbb{R}^n, \mathbb{R}^m).$$

11. We have seen in the first tutorial that the Grassmannian manifold $\text{Gr}(r, n)$ can be realized as a submanifold of $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of dimension $r(n - r)$. For $E \in \text{Gr}(r, n)$ compute the tangent space

$$T_E \text{Gr}(r, n) \subset T_E \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cong \text{Hom}(\mathbb{R}^n, \mathbb{R}^n).$$

12. Consider the general linear group $\text{GL}(n, \mathbb{R})$ and the special linear group $\text{SL}(n, \mathbb{R})$. We have seen that they are submanifolds of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ (even so called Lie groups) and that $T_{\text{Id}} \text{GL}(n, \mathbb{R}) \cong M_n(\mathbb{R}) = \mathbb{R}^{n^2}$.

- (a) Compute the tangent space $T_{\text{Id}} \text{SL}(n, \mathbb{R})$ of $\text{SL}(n, \mathbb{R})$ at the identity Id .
- (b) Fix $A \in \text{SL}(n, \mathbb{R})$ and consider the conjugation $\text{conj}_A : \text{SL}(n, \mathbb{R}) \rightarrow \text{SL}(n, \mathbb{R})$ by A given by $\text{conj}_A(B) = ABA^{-1}$. Show that conj_A is smooth and compute the derivative $T_{\text{Id}} \text{conj}_A : T_{\text{Id}} \text{SL}(n, \mathbb{R}) \rightarrow T_{\text{Id}} \text{SL}(n, \mathbb{R})$.
- (c) Consider the map $\text{Ad} : \text{SL}(n, \mathbb{R}) \rightarrow \text{Hom}(T_{\text{Id}} \text{SL}(n, \mathbb{R}), T_{\text{Id}} \text{SL}(n, \mathbb{R}))$ given by $\text{Ad}(A) := T_{\text{Id}} \text{conj}_A$. Show that Ad is smooth and compute $T_{\text{Id}} \text{Ad}$.

13. Consider \mathbb{R}^n equipped with the standard inner product of signature (p, q) (where $p + q = n$) given by

$$\langle x, y \rangle := \sum_{i=1}^p x_i y_i - \sum_{i=p+1}^n x_i y_i$$

and the group of linear orthogonal transformation of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ given by

$$\text{O}(p, q) := \{A \in \text{GL}(n, \mathbb{R}) : \langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^n\}.$$

- (a) Show that

$$\text{O}(p, q) = \{A \in \text{GL}(n, \mathbb{R}) : A^{-1} = I_{p,q} A^t I_{p,q}\},$$

where $I_{p,q} = \begin{pmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{pmatrix}$, and that $\text{O}(p, q)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?

- (b) Show that $\text{O}(p, q)$ is a subgroup of $\text{GL}(n, \mathbb{R})$ with respect to matrix multiplication μ and that $\mu : \text{O}(p, q) \times \text{O}(p, q) \rightarrow \text{O}(p, q)$ is smooth (i.e. that $\text{O}(p, q)$ is a Lie group.)

(c) Compute the tangent space $T_{\text{Id}}\mathbf{O}(p, q)$ of $\mathbf{O}(p, q)$ at the identity Id .

14. Suppose $M = \mathbb{R}^3$ with standard coordinates (x, y, z) . Consider the vector field

$$\xi(x, y, z) = 2\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}.$$

How does this vector field look like in terms of the coordinate vector fields associated to the cylindrical coordinates (r, ϕ, z) , where $x = r \cos \phi$, $y = r \sin \phi$ and $z = z$? Or with respect to the spherical coordinates (r, ϕ, θ) , where $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$?

15. Consider \mathbb{R}^3 with coordinates (x, y, z) and the vector fields

$$\xi(x, y, z) = (x^2 - 1)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}$$

$$\eta(x, y, z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2xz^2\frac{\partial}{\partial z}.$$

Are they tangent to the cylinder $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$ with radius 1 (i.e. do they restrict to vector fields on M)?

16. Suppose $M = \mathbb{R}^2$ with coordinates (x, y) . Consider the vector fields $\xi(x, y) = y\frac{\partial}{\partial x}$ and $\eta(x, y) = \frac{x^2}{2}\frac{\partial}{\partial y}$ on M . We computed in class their flows and saw that they are complete. Compute $[\xi, \eta]$ and its flow? Is $[\xi, \eta]$ complete?

17. Let M be a (smooth) manifold and $\xi, \eta \in \mathfrak{X}(M)$ two vector fields on M . Show that

(a) $[\xi, \eta] = 0 \iff (\text{Fl}_t^\xi)^*\eta = \eta$, whenever defined $\iff \text{Fl}_t^\xi \circ \text{Fl}_s^\eta = \text{Fl}_s^\eta \circ \text{Fl}_t^\xi$, whenever defined.

(b) If N is another manifold, $f : M \rightarrow N$ a smooth map, and ξ and η are f -related to vector fields $\tilde{\xi}$ resp. $\tilde{\eta}$ on N , then $[\xi, \eta]$ is f -related to $[\tilde{\xi}, \tilde{\eta}]$.

18. Consider the general linear group $\text{GL}(n, \mathbb{R})$. For $A \in \text{GL}(n, \mathbb{R})$ denote by

$$\lambda_A : \text{GL}(n, \mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R}) \quad \lambda_A(B) = AB$$

$$\rho_A : \text{GL}(n, \mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R}) \quad \rho_A(B) = BA$$

left respectively right multiplication by A , and by $\mu : \text{GL}(n, \mathbb{R}) \times \text{GL}(n, \mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R})$ the multiplication map.

(a) Show that λ_A and ρ_A are diffeomorphisms for any $A \in \text{GL}(n, \mathbb{R})$ and that

$$T_B\lambda_A(B, X) = (AB, AX) \quad T_B\rho_A(B, X) = (BA, XA),$$

where $(B, X) \in T_B\text{GL}(n, \mathbb{R}) = \{(B, X) : X \in M_n(\mathbb{R})\}$.

(b) Show that

$$T_{(A,B)}\mu((A, B), (X, Y)) = T_B\lambda_A Y + T_A\rho^B X = (AB, AY + XB)$$

where $(A, B) \in \mathbf{GL}(n, \mathbb{R}) \times \mathbf{GL}(n, \mathbb{R})$ and $(X, Y) \in M_n(\mathbb{R}) \times M_n(\mathbb{R})$.

(c) For any $X \in M_n(\mathbb{R}) \cong T_{Id}\mathbf{GL}(n, \mathbb{R})$ consider the maps

$$L_X : \mathbf{GL}(n, \mathbb{R}) \rightarrow T\mathbf{GL}(n, \mathbb{R}) \quad L_X(B) = T_{Id}\lambda_B(Id, X) = (B, BX).$$

$$R_X : \mathbf{GL}(n, \mathbb{R}) \rightarrow T\mathbf{GL}(n, \mathbb{R}) \quad R_X(B) = T_{Id}\rho_B(Id, X) = (B, XB).$$

Show that L_X and R_X are smooth vector field and that $\lambda_A^*L_X = L_X$ and $\rho_A^*R_X = R_X$ for any $A \in \mathbf{GL}(n, \mathbb{R})$. What are their flows? Are these vector fields complete?

(d) Show that $[L_X, R_Y] = 0$ for any $X, Y \in M_n(\mathbb{R})$.

19. Suppose α_j^i for $i = 1, \dots, k$ and $j = 1, \dots, n$ are smooth real-valued functions defined on some open set $U \subset \mathbb{R}^{n+k}$ satisfying

$$\frac{\partial \alpha_j^i}{\partial x^k} + \sum_{\ell=1}^k \alpha_k^\ell \frac{\partial \alpha_j^i}{\partial z^\ell} = \frac{\partial \alpha_k^i}{\partial x^j} + \sum_{\ell=1}^k \alpha_j^\ell \frac{\partial \alpha_k^i}{\partial z^\ell},$$

where we write $(x, z) = (x^1, \dots, x^n, z^1, \dots, z^k)$ for a point in \mathbb{R}^{n+k} . Show that for any point $(x_0, z_0) \in U$ there exists an open neighbourhood V of x_0 in \mathbb{R}^n and a unique C^∞ -map $f : V \rightarrow \mathbb{R}^k$ such that

$$\frac{\partial f^i}{\partial x^j}(x^1, \dots, x^n) = \alpha_j^i(x^1, \dots, x^n, f^1(x), \dots, f^k(x)) \quad \text{and} \quad f(x_0) = z_0.$$

In the class/tutorial we proved this for $k = 1$ and $j = 2$.

20. Which of the following systems of PDEs have solutions $f(x, y)$ (resp. $f(x, y)$ and $g(x, y)$) in an open neighbourhood of the origin for positive values of $f(0, 0)$ (resp. $f(0, 0)$ and $g(0, 0)$)?

(a) $\frac{\partial f}{\partial x} = f \cos y$ and $\frac{\partial f}{\partial y} = -f \log f \tan y$.

(b) $\frac{\partial f}{\partial x} = e^{xf}$ and $\frac{\partial f}{\partial y} = xe^{yf}$.

(c) $\frac{\partial f}{\partial x} = f$ and $\frac{\partial f}{\partial y} = g$; $\frac{\partial g}{\partial x} = g$ and $\frac{\partial g}{\partial y} = f$.

21. Suppose $E \rightarrow M$ is a (smooth) vector bundle of rank k over a manifold M . Then E is called *trivializable*, if it is isomorphic to the trivial vector bundle $M \times \mathbb{R}^k \rightarrow M$.

(a) Show that $E \rightarrow M$ is trivializable $\iff E \rightarrow M$ admits a global frame, i.e. there exist (smooth) sections s_1, \dots, s_k of E such that $s_1(x), \dots, s_k(x)$ span E_x for any $x \in M$.

(b) Show that the tangent bundle of any Lie group G is trivializable.

(c) Recall that \mathbb{R}^n has the structure of a (not necessarily associative) normed division algebra over \mathbb{R} for $n = 1, 2, 4, 8$. Use this to show that the tangent bundle of the spheres $S^1 \subset \mathbb{R}^2$, $S^3 \subset \mathbb{R}^4$ and $S^7 \subset \mathbb{R}^8$ is trivializable.

22. Let V be a finite dimensional real vector space and consider the subspace of r -linear alternating maps $\Lambda^r V^* = L_{\text{alt}}^r(V, \mathbb{R})$ of the vector space of r -linear maps $L^r(V, \mathbb{R}) = (V^*)^{\otimes r}$. Show that for $\omega \in L^r(V, \mathbb{R})$ the following are equivalent:

(a) $\omega \in \Lambda^r V^*$

(b) For any vectors $v_1, \dots, v_r \in V$ one has

$$\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

(c) ω is zero whenever one inserts a vector $v \in V$ twice.

(d) $\omega(v_1, \dots, v_k) = 0$, whenever $v_1, \dots, v_k \in V$ are linearly dependent vectors.

23. Let V be a finite dimensional real vector space. Show that the vector space $\Lambda^* V^* := \bigoplus_{r \geq 0} \Lambda^r V^*$ is an associative, unital, graded-anticommutative algebra with respect to the wedge product \wedge , i.e. show that the following holds:

(a) $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$ for all $\omega, \eta, \zeta \in \Lambda^* V^*$.

(b) $1 \in \mathbb{R} = \Lambda^0 V^*$ satisfies $1 \wedge \omega = \omega \wedge 1 = \omega$ for all $\omega \in \Lambda^* V^*$.

(c) $\Lambda^r V^* \wedge \Lambda^s V^* \subset \Lambda^{r+s} V^*$.

(d) $\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega$ for $\omega \in \Lambda^r V^*$ and $\eta \in \Lambda^s V^*$.

Moreover, show that for any linear map $f : V \rightarrow W$ the linear map $f^* : \Lambda^* W^* \rightarrow \Lambda^* V^*$ is a morphism of graded unital algebras, i.e. $f^* 1 = 1$, $f^*(\Lambda^r W^*) \subset \Lambda^r V^*$ and $f^*(\omega \wedge \eta) = f^* \omega \wedge f^* \eta$.

24. Let V be a finite dimensional real vector space. Show that:

(a) If $\omega_1, \dots, \omega_r \in V^*$ and $v_1, \dots, v_r \in V$, then

$$\omega_1 \wedge \dots \wedge \omega_r(v_1, \dots, v_r) = \det((\omega_i(v_j))_{1 \leq i, j \leq r}).$$

In particular, $\omega_1, \dots, \omega_r$ are linearly independent $\iff \omega_1 \wedge \dots \wedge \omega_r \neq 0$.

(b) If $\{\lambda_1, \dots, \lambda_n\}$ is a basis of V^* , then

$$\{\lambda_{i_1} \wedge \dots \wedge \lambda_{i_r} : 1 \leq i_1 < \dots < i_r \leq n\}$$

is a basis of $\Lambda^r V^*$.

25. Let V be a finite dimensional real vector space. An element $\mu \in L^r(V, \mathbb{R})$ is called *symmetric*, if $\mu(v_1, \dots, v_r) = \mu(v_{\sigma(1)}, \dots, v_{\sigma(r)})$ for any vectors $v_1, \dots, v_r \in V$ and any permutation $\sigma \in S^r$. Denote by $S^r V^* \subset L^r(V, \mathbb{R})$ the subspace of symmetric elements in the vector space $L^r(V, \mathbb{R})$.

(a) For $\mu \in L^r(V, \mathbb{R})$ show that

$$\mu \in S^r V^* \iff \mu(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = \mu(v_1, \dots, v_j, \dots, v_i, \dots, v_k),$$

for any vectors $v_1, \dots, v_r \in V$.

(b) Consider the map $\text{Sym} : L^r(V, \mathbb{R}) \rightarrow L^r(V, \mathbb{R})$ given by

$$\text{Sym}(\mu)(v_1, \dots, v_r) = \frac{1}{r!} \sum_{\sigma \in S^r} \mu(v_{\sigma(1)}, \dots, v_{\sigma(r)}).$$

Show that $\text{Image}(\text{Sym}) = S^r V^*$ and that $\mu \in S^r V^* \iff \text{Sym}(\mu) = \mu$.

26. Let V be a finite dimensional real vector space and set $S(V^*) := \bigoplus_{r=0}^{\infty} S^r V^*$ with the convention $S^0 V^* = \mathbb{R}$ and $S^1 V^* = V^*$. For $\mu \in S^r V^*$ and $\nu \in S^t V^*$ define their symmetric product by

$$\mu \odot \nu := \text{Sym}(\mu \otimes \nu) \in S^{r+t} V^*.$$

By bilinearity, we extend this to a \mathbb{R} -bilinear map $\odot : S(V^*) \times S(V^*) \rightarrow S(V^*)$. Show that $S(V^*)$ is an unital, associative, commutative, graded algebra with respect to the symmetric product \odot .

27. Suppose $p : E \rightarrow M$ and $q : F \rightarrow M$ are vector bundles over M . Show that their direct sum $E \oplus F := \sqcup_{x \in M} E_x \oplus F_x \rightarrow M$ and their tensor product $E \otimes F := \sqcup_{x \in M} E_x \otimes F_x \rightarrow M$ are again vector bundles over M .

28. Suppose $E \subset TM$ is a smooth distribution of rank k on a manifold M of dimension n and denote by $\Omega(M)$ the vector space of differential forms on M .

(a) Show that locally around any point $x \in M$ there exists (local) 1-forms $\omega^1, \dots, \omega^{n-k}$ such that for any (local) vector field ξ one has: ξ is a (local) section of $E \iff \omega_i(\xi) = 0$ for all $i = 1, \dots, n - k$.

(b) Show that E is involutive \iff whenever $\omega^1, \dots, \omega^{n-k}$ are local 1-forms as in (a) then there exists local 1-forms $\mu^{i,j}$ for $i, j = 1, \dots, n - k$ such that

$$d\omega^i = \sum_{j=1}^{n-k} \mu^{i,j} \wedge \omega^j.$$

(c) Show

$$\Omega_E(M) := \{\omega \in \Omega(M) : \omega|_E = 0\} \subset \Omega(M)$$

is an ideal of the algebra $(\Omega(M), \wedge)$. Here, $\omega|_E = 0$ for a ℓ -form ω means that $\omega(\xi_1, \dots, \xi_\ell) = 0$ for any sections ξ_1, \dots, ξ_ℓ of E .

(d) An ideal \mathcal{J} of $(\Omega(M), \wedge)$ is called differential ideal, if $d(\mathcal{J}) \subset \mathcal{J}$. Show that $\Omega_E(M)$ is a differential ideal $\iff E$ is involutive.

29. Suppose M is a manifold and $D_i : \Omega^k(M) \rightarrow \Omega^{k+r_i}(M)$ for $i = 1, 2$ a graded derivation of degree r_i of $(\Omega(M), \wedge)$.

(a) Show that

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$$

is a graded derivation of degree $r_1 + r_2$.

(b) Suppose D is a graded derivation of $(\Omega(M), \wedge)$. Let $\omega \in \Omega^k(M)$ be a differential form and $U \subset M$ an open subset. Show that $\omega|_U = 0$ implies $D(\omega)|_U = 0$.

Hint: Think about writing 0 as $f\omega$ for some smooth function f and use the defining properties of a graded derivation.

(c) Suppose D and \tilde{D} are two graded derivations such that $D(f) = \tilde{D}(f)$ and $D(df) = \tilde{D}(df)$ for all $f \in C^\infty(M, \mathbb{R})$. Show that $D = \tilde{D}$.

30. Suppose M is a manifold and $\xi, \eta \in \Gamma(TM)$ vector fields.

(a) Show that the insertion operator $i_\xi : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ is a graded derivation of degree -1 of $(\Omega(M), \wedge)$.

(b) Recall from class that $[d, d] = 0$. Verify (the remaining) graded-commutator relations between $d, \mathcal{L}_\xi, i_\eta$:

(i) $[d, \mathcal{L}_\xi] = 0$.

(ii) $[d, i_\xi] = d \circ i_\xi + i_\xi \circ d = \mathcal{L}_\xi$.

(iii) $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]}$.

(iv) $[\mathcal{L}_\xi, i_\eta] = i_{[\xi, \eta]}$.

(v) $[i_\xi, i_\eta] = 0$.

Hint: Use (c) from 11.

31. Prove the **Poincaré Lemma**: Suppose $\omega \in \Omega^k(\mathbb{R}^m)$ is a closed k -form, where $k \geq 1$. Show that there exists $\tau \in \Omega^{k-1}(\mathbb{R}^m)$ such that $d\tau = \omega$.

Hint: Show that for any k -form $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ on \mathbb{R}^m ,

$$P(\omega) = \sum_{\alpha=1}^k \sum_{i_1 < \dots < i_k} (-1)^{\alpha-1} \left[\int_0^1 t^{k-1} \omega_{i_1 \dots i_k}(tx) dt \right] x^{i_\alpha} dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_\alpha}} \wedge \dots \wedge dx^{i_k}.$$

is a $(k-1)$ -form on \mathbb{R}^m satisfying

$$\omega = d(P(\omega)) + P(d\omega).$$

Here, $\widehat{dx^{i_\alpha}}$ means that this term is omitted.

32. Show that any manifold with a parallelizable tangent bundle is orientable.

33. Suppose $M \subset N$ is a submanifold of codimension 1 (i.e. $\dim M = \dim N - 1$) of an oriented manifold N . Suppose there exists a smooth vector field along M that is transverse everywhere to M , that is, a smooth map $\nu : M \rightarrow TN$ such that for all $x \in M$ one has

- (i) $\nu(x) \in T_x N$ and
- (ii) $\nu(x)$ and $T_x M$ span $T_x N$.

Show that M is orientable. Deduce that a hypersurface

$$(M, g) \subset (\mathbb{R}^{m+1}, g) = (\mathbb{R}^{m+1}, g^{\text{euc}})$$

in Euclidean space is orientable if and only if M admits a globally defined unit normal vector field.

34. Consider $S^m \subset \mathbb{R}^{m+1}$ the unit sphere and the global unit normal vector field $\nu(x) = \sum_{i=1}^{m+1} x^i \frac{\partial}{\partial x^i}$ for S^m . Show that for the nowhere vanishing $m+1$ -form

$$\Omega = dx^1 \wedge \dots \wedge dx^{m+1}$$

on \mathbb{R}^{m+1} ,

$$\omega(x) := (i_\nu \Omega)(x) = \Omega(x)(\nu(x), -, \dots, -) \text{ for } x \in S^m$$

restricts to a nowhere vanishing m -form on S^m that satisfies

$$A^* \omega = (-1)^{m+1} \omega,$$

where $A : S^m \rightarrow S^m$ is the antipodal map $A(x) = -x$.

35. Show that n -dimensional projective space $\mathbb{R}P^n$ is orientable $\iff n$ is odd.

Hint: For \implies consider the natural projection $\pi : S^n \rightarrow \mathbb{R}P^n$, given by $\pi(x) = [x]$, and use the previous exercise. For \impliedby construct an oriented atlas.

36. Suppose M and N are connected, compact, oriented manifolds of the same dimension m . Let $f_0, f_1 : M \rightarrow N$ be smooth maps that are homotopic to each other, i.e. there exists a smooth map $F : M \times [0, 1] \rightarrow N$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. Show that for any $\omega \in \Omega^m(N)$ one has

$$\int_M f_0^* \omega = \int_M f_1^* \omega.$$

Hint: $M \times [0, 1]$ is an oriented manifold with boundary $\partial M = -(M \times \{0\}) \cup M \times \{1\}$, where the minus indicates that the orientation on $M \times \{0\}$ is reversed. Use Stokes' Theorem.

37. Use the previous exercise to show that, if the antipodal map $A : S^m \rightarrow S^m$ on the sphere S^m is homotopic to the identity Id_{S^m} on S^m , then m is odd.
38. Show that on a sphere S^{2m} of even dimension any smooth vector field $\xi \in \mathfrak{X}(S^{2m})$ has a zero.

Hint: Show that if $\xi \in \mathfrak{X}(S^{2m})$ is nowhere vanishing, then there exists a homotopy between the antipodal map and the identity.

39. Suppose $(M, g) \subset (\mathbb{R}^3, g) = (\mathbb{R}^3, g^{\text{euc}}) = (\mathbb{R}^3, \langle -, - \rangle)$ is a surface in Euclidean space. Let $u : U \rightarrow u(U)$ be a local chart for M with corresponding local parametrization

$$v = u^{-1} : u(U) \rightarrow U.$$

With respect to the frame $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ of $T\mathbb{R}^2$, we can write v^*g and v^*II as matrices

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix},$$

where

$$E = g\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}\right) \circ v \quad F = g\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right) \circ v \quad G = g\left(\frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2}\right) \circ v,$$

and

$$\tilde{E} = II\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}\right) \circ v \quad \tilde{F} = II\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right) \circ v \quad \tilde{G} = II\left(\frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2}\right) \circ v.$$

Compute in terms of $E, F, G, \tilde{E}, \tilde{F}$ and \tilde{G} , the Weingarten map $L \circ v$, the Gauß curvature $K \circ v$, the mean curvature $H \circ v$, and the principal curvatures $\kappa_1 \circ v$ and $\kappa_2 \circ v$.

40. Let us write (x^1, x^2, x^3) for the coordinates in \mathbb{R}^3 . Take a circle of radius $r > 0$ in the (x^1, x^3) -plane and rotate it around a circle of radius $R > r$ in the (x^1, x^2) -plane. The result is a 2-dimensional torus M in \mathbb{R}^3 . If $I \subset \mathbb{R}$ is an open interval of length $< 2\pi$ the map $v : I \times I \rightarrow \mathbb{R}^3$ given by

$$v(\phi, \theta) = ((R + r \cos \theta) \cos \phi, (R + r \cos \theta) \sin \phi, r \sin \theta)$$

defines a local parametrization of M . With respect to v , compute, using the previous exercise, the metric g on M induced by the Euclidean metric on \mathbb{R}^3 , the 2nd fundamental form, the Gauß and the mean curvature, the principal curvatures and the principal curvature directions of the surface (M, g) in \mathbb{R}^3 .

Hint: Note that $\nu(\phi, \theta) = (\cos \phi \cos \theta, \sin \phi \cos \theta, \sin \theta)$ defines a local unit normal vector field for M .

41. Suppose $(M, g) \subset (\mathbb{R}^{m+1}, g) = (\mathbb{R}^{m+1}, g^{\text{euc}})$ is a connected oriented hypersurface in Euclidean space. Show that all points in M are umbilic if and only if M is part of an affine hyperplane or a sphere.

Hint: For \implies' show the following:

- Fix a global unit normal vector field $\nu : M \rightarrow \mathbb{R}^{m+1}$. Then, by assumption, for any $x \in M$ there exists $\lambda(x) \in \mathbb{R}$ such that

$$L_x = \lambda(x) \text{Id}_{T_x M}.$$

Since $\lambda = \frac{g(L(\xi), \xi)}{g(\xi, \xi)}$ for any local vector field ξ on M , $\lambda : M \rightarrow \mathbb{R}$ is smooth. Show that λ is constant, by, for instance, picking a chart and computing the left-hand-side of $[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] \cdot \nu = 0$.

- If $\lambda = 0$, show that any curve in M is contained in an affine hyperplane with (constant) normal vector ν .
- If $\lambda \neq 0$, show that $f : M \rightarrow \mathbb{R}^{m+1}$, given by $f(x) = x - \frac{1}{\lambda}\nu(x)$, is constant.

42. Suppose ∇ is an affine connection on a manifold M .

(a) Show that its curvature, given by,

$$R(\xi, \eta)(\zeta) = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi, \eta]} \zeta,$$

for vector fields $\xi, \eta, \zeta \in \mathfrak{X}(M)$ defines a $\binom{1}{3}$ -tensor on M .

(b) Show that, if ∇ is torsion-free, the Bianchi identity holds:

$$R(\xi, \eta)(\zeta) + R(\eta, \zeta)(\xi) + R(\zeta, \xi)(\eta) = 0,$$

for any $\xi, \eta, \zeta \in \mathfrak{X}(M)$.

43. Suppose $E \rightarrow M$ is a vector bundle over a manifold M equipped with a linear connection ∇ , that is, a \mathbb{R} -bilinear map

$$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(E) &\rightarrow \Gamma(E) \\ (\xi, s) &\mapsto \nabla_\xi s \end{aligned}$$

such that for $\xi \in \Gamma(TM)$, $s \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$ one has

- $\nabla_{f\xi} s = f \nabla_\xi s$
- $\nabla_\xi f s = f \nabla_\xi s + (\xi \cdot f) s$.

(a) Show that $\nabla : \Gamma(TM) \times \Gamma(E^*) \rightarrow \Gamma(E^*)$ (typically also denoted by ∇) given by

$$(\nabla_\xi \mu)(s) = \xi \cdot \mu(s) - \mu(\nabla_\xi s), \text{ for } \mu \in \Gamma(E^*), \xi \in \Gamma(TM), s \in \Gamma(E)$$

defines a linear connection on the dual vector bundle $E^* \rightarrow M$.

(b) Suppose $\tilde{E} \rightarrow M$ is another vector bundle equipped with a linear connection $\tilde{\nabla}$. Show the vector bundle $E \otimes \tilde{E} \rightarrow M$ admits a linear connection characterized by

$$\nabla_\xi (s \otimes \tilde{s}) = \nabla_\xi s \otimes \tilde{s} + s \otimes \tilde{\nabla}_\xi \tilde{s}$$

for $\xi \in \Gamma(TM)$, $s \in \Gamma(E)$ and $\tilde{s} \in \Gamma(\tilde{E})$.

44. Suppose ∇ is an affine connection on a manifold M . Then the previous exercise shows that ∇ induces a linear connection $\nabla : \Gamma(TM) \times \mathcal{T}_q^p(M) \rightarrow \mathcal{T}_q^p(M)$ on all tensor bundles. Show that it also induces a linear connection on the bundles $\Lambda^k T^*M$ for $k = 1, \dots, \dim(M)$ characterized by

$$\nabla_\xi (\omega \wedge \mu) = \nabla_\xi \omega \wedge \mu + \omega \wedge \nabla_\xi \mu$$

for $\omega \in \Gamma(\Lambda^k T^*M)$ and $\mu \in \Gamma(\Lambda^\ell T^*M)$ and give a formula.

45. Suppose (M, g) is a Riemannian manifold.

- (a) For vector fields $\xi, \eta \in \mathfrak{X}(M)$, let $\nabla_{\xi}\eta \in \mathfrak{X}(M)$ be the unique vector field such that

$$g(\nabla_{\xi}\eta, \zeta) = \frac{1}{2} \left(\xi \cdot g(\eta, \zeta) + \eta \cdot g(\zeta, \xi) - \zeta \cdot g(\xi, \eta) + g([\xi, \eta], \zeta) - g([\xi, \zeta], \eta) - g([\eta, \zeta], \xi) \right)$$

for all $\zeta \in \mathfrak{X}(M)$. Show that ∇ defines a torsion-free affine connection satisfying

$$\xi \cdot g(\eta, \zeta) = g(\nabla_{\xi}\eta, \zeta) + g(\eta, \nabla_{\xi}\zeta)$$

for $\xi, \eta, \zeta \in \mathfrak{X}(M)$.

- (b) The connection ∇ in (a) is called the Levi-Civita connection of (M, g) . Show that its curvature satisfies:

- $g(R(\xi, \eta)(\zeta), \mu) = -g(R(\xi, \eta)(\mu), \zeta)$,
- $g(R(\xi, \eta)(\zeta), \mu) = g(R(\zeta, \mu)(\xi), \eta)$,

for $\xi, \eta, \zeta, \mu \in \mathfrak{X}(M)$.

- (c) Suppose (U, u) is a chart for M and let R be the Riemann curvature, i.e. the curvature of the Levi-Civita connection of (M, g) . Compute

$$R\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)\left(\frac{\partial}{\partial u^k}\right)$$

in terms of the Christoffel symbols.

46. Suppose $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is the upper-half plane and equip it with the Riemannian metric

$$g = \frac{1}{y^2} dx \otimes dx + \frac{1}{y^2} dy \otimes dy.$$

- (a) Compute the Christoffel symbols of g .
 (b) Compute the geodesics of g .
 (c) Compute the Riemann curvature.

47. Identifying $H = \{(x, y) \in \mathbb{R}^2 : y > 0\} = \{z = x + iy \in \mathbb{C} : y > 0\}$ in the previous example, we may write g as

$$g = \frac{1}{\text{Im}(z)^2} \text{Re}(dz \otimes d\bar{z}) = \frac{4}{|z - \bar{z}|^2} \text{Re}(dz \otimes d\bar{z}),$$

where $dz = dx + idy$, $d\bar{z} = dx - idy$, Im and Re denote imaginary and real part, and $|\cdot|$ is the absolute value of complex numbers.

Consider $\text{SL}(2, \mathbb{R}) = \{A \in \text{GL}(2, \mathbb{R}) : \det A = 1\}$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ and $z \in \mathbb{C}$ let

$$f_A(z) = \frac{az + b}{cz + d}.$$

- (a) Show that f_A is a diffeomorphism from H to itself for any $A \in \mathrm{SL}(2, \mathbb{R})$ and that $f_{AB} = f_A \circ f_B$.
- (b) Show that f_A is an isometry of H .
- (c) Show that for any two points $z, z' \in H$ there exists $A \in \mathrm{SL}(2, \mathbb{R})$ such that $f_A(z) = z'$.
- (d) Characterize the elements $A \in \mathrm{SL}(2, \mathbb{R})$ such that $f_A(i) = i$.