Lecture 8. Poincaré duality

Last time we proved that if M is a oriented manifold of dim n and $A \subseteq M$ compact that

- $H_i(M, M \setminus A) = 0$ for $i > n$
- there is a class $\mu_A \in H_n(M, M \setminus A)$ such that
$$(\rho_x)_*(\mu_A) = \mu_x \quad \text{for all } x \in A,$$
where $\mu_x \in H_n(M, M \setminus x)$ and $\rho_x : (M, M\setminus A) \longrightarrow (M, M \setminus x)$.

To formulate Poincaré duality we need another product called cap product

$$\cap : H_n(X;R) \otimes H^k(X;R) \longrightarrow H_{n-k}(X;R)$$

defined on chains and cochains by the formula

$$\underbrace{\sigma \cap \varphi}_{} = \underset{\in R}{\varphi(\sigma/[v_0, \dots v_k])} \cdot \underset{\in C_{n-k}(X)}{\sigma/[v_k, \dots v_n]}$$

One can prove that

$$\partial(\sigma \cap \varphi) = (-1)^k (\partial\sigma \cap \varphi - \sigma \cap \delta\varphi)$$

It enables to define cap product on the level of homologies and cohomologies:

standard PD
- $\cap$: $H_n(X) \otimes H^k(X) \longrightarrow H_{n-k}(X)$

$H_n(X, A) \otimes H^k(X) \longrightarrow H_{n-k}(X, A)$

used in generalized PD
- $H_n(X, A) \otimes H^k(X, A) \longrightarrow H_{n-k}(X)$

$H_n(X, A \cup B) \otimes H^k(X, A) \longrightarrow H_{n-k}(X, B)$

for A, B open in X.

Naturality For $f : X \to Y$ we get

$$f_*(a \cap f^*(\beta)) = f_*(a) \cap \beta$$

$$\begin{array}{ccc} \overset{a}{H_n(X)} \otimes \overset{\epsilon}{H^k(X)} & \to & H_{n-k}(X) \\ f_* \downarrow \quad & \uparrow f^* \quad \beta & \downarrow f_* \\ H_n(Y) \otimes H^k(Y) & \to & H_{n-k}(Y) \end{array}$$

Theorem (Poincaré duality) + without boundary

Let M be a closed (= compact) R-oriented manifold of dim n. Then the map

$$D : H^k(M;R) \longrightarrow H_{n-k}(M;R)$$
$$D(\varphi) = [M] \cap \varphi$$

is an isomorphism.

$H^0(M) \cong H_n(M)$
$H^1(M) \cong H_{n-1}(M)$
$H^n(M) \cong H_0(M)$
$\cong \mathbb{Z}$

As for the proof. It shows up that it is better to formulate a more general statement (without assumption that M is compact) and prove this. („more difficult" is sometimes easier.) To it we need the notion of cohomology with compact support

Consider a space X with a directed system of compact sets (ordering by inclusions). For each pair $K \subseteq L$, the inclusion

$$(X, X-L) \hookrightarrow (X, X\setminus K)$$

induces in cohomology homomorphism

$$H^k(X, X\setminus K) \longrightarrow H^k(X, X-L).$$

So we can define cohomology groups of compact support as

$$H^k_c(X) = \varinjlim_{K} H^k(X, X-K)$$

If X is compact then

$$H^k_c(X) = H^k(X).$$

Example: We know $H^i(\mathbb{R}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{otherwise.} \end{cases}$ $\mathbb{R}^n \simeq *$

$$H^k_c(\mathbb{R}^n;\mathbb{Z}) = \lim_{r\to\infty} H^k(\mathbb{R}^n, \mathbb{R}^n \setminus D(0,r)) \cong H^k(D^n;\partial D^n) \cong \begin{matrix} 0 \\ \mathbb{Z} \end{matrix}$$

where $D(0,r) = \{x \in \mathbb{R}^n, \|x\| \leq r\}$.

Then $\displaystyle H_c^n(\mathbb{R}^n) = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \left(\mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to \dots \right) = \mathbb{Z}$

$k \neq n$ $\displaystyle H_c^k(\mathbb{R}^n) = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \left(0 \to 0 \to 0 \to \dots \right) = 0.$

Generalized Poincaré duality

Let M be an R-oriented manifold of dimension n.
Let $K \subseteq M$ be compact. Let $\mu_K \in H_n(M, M\setminus K; R)$ is a class such that $(\rho_x)_* \mu_K = \mu_x$ for all $x \in K$.
Then we define

$$D_K : H^k(M, M\setminus K; R) \longrightarrow H_{n-k}(M; R),$$
$$D_K(\varphi) = \mu_K \cap \varphi.$$

If $K \subseteq L$ are two compact sets, we can prove that

$$D_L(\rho^* \varphi) = D_K(\varphi) \qquad \varphi \in H^k(M, M\setminus K; R)$$

for $\rho : (M, M\setminus L) \hookrightarrow (M, M\setminus K)$.
It enables us to define

$$D_M : H^k_c(M; R) \longrightarrow H_{n-k}(M; R)$$
$$D(\varphi) = \mu_K \cap \varphi$$

since every $\varphi \in H^k_c(M; R)$ comes from an element in $H^k(M, M\setminus K; R)$ for some $K \subseteq M$ compact.

THEOREM (Poincaré duality for all manifolds)

If M is an R-oriented manifold of dimension n, then

$$D_M : H^k_c(M; R) \longrightarrow H_{n-k}(M; R)$$

is an isomorphism.

The proof is based on: If $M = U \cup V$ where U and V are open, then the following diagram with LES commutes:

$$\begin{tikzcd}
H_c^k(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H_c^k(U)\oplus H_c^k(V) \arrow[r] \arrow[d, "D_U \oplus D_V"] & H_c^k(M) \arrow[r] \arrow[d, "D_M"] & H_c^{k+1}(U\cap V) \arrow[d, "D_{U\cap V}"] \\
H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U\cap V)
\end{tikzcd}$$

From this diagram we can prove

(A) If $M = U\cup V$, U, V open and $D_U, D_V, D_{U\cap V}$ are isos, then D_M is an iso.

Using definition of cohomology with compact support we can prove:

(B) If $M = \bigcup_{i=1}^{\infty} U_i$ where U_i are open, $U_1 \subset U_2 \subset U_3 \subset \dots$ and D_{U_i} are isos, then D_M is an iso.

$0 \to H_c^k(U_i) \xrightarrow{\cong} H_{n-k}(U_i) \to 0$

The proof of duality itself can be carried out in 4 steps.

$H_c^k(M) \xrightarrow{\cong} H_{n-k}(M)$

(1) $M = \mathbb{R}^n$ We have $H_c^k(\mathbb{R}^n) \cong H^k(\Delta^n, \partial\Delta^n)$ $(D^n, \partial D^n)$

$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \Delta^n) \cong H_n(\Delta^n, \partial\Delta^n)$

The generator $\omega \in H_n(\Delta^n, \partial\Delta^n)$ is represented by $id: \Delta^n \to \Delta^n$. Take $\varphi \in H^n(\Delta^n, \partial\Delta^n) \cong Hom(H_n(\Delta^n, \partial\Delta^n), R)$

Then $C_n(\Delta^n, \partial\Delta^n)$ generator of $H_n(\Delta^n, \partial\Delta^n)$. $C_0(\Delta^0, \partial\Delta^0)$

$D \cdots \omega \cap \varphi = \varphi(\omega)\cdot 1 = \pm 1 .$ ✓

For $\varphi \in H^k(\Delta^n, \partial\Delta^n) = 0$, $k \neq n$, the statement is trivial.

$M = \bigcup V_i \quad V_1 \subseteq V_2 \subseteq \dots$

(2) $M \subseteq \mathbb{R}^n$ open. M is a union of countably many open convex sets which are homeomorphic to $\mathbb{R}^n$. The statement follows from (A) and (B)

(3) M is a manifold which is a countable union of open sets homeomorphic to $\mathbb{R}^n$. Use again (A) and (B).

(4) General M (see Hatcher, page 248). $\square$

Corollary: Euler characteristic of odd dimensional orientable manifold is zero.

Euler characteristic of even dimensional oriented manifold is even number.

Proof: $\operatorname{rank} H_{n-k}(M;\mathbb{Z}) = \operatorname{rank} H^k(M;\mathbb{Z}) =$

$= \operatorname{rank} \operatorname{Hom}(H_k(M),\mathbb{Z})$?

$= \operatorname{rank} H_k(M)$.

Then $\sum_{i=0}^{n} (-1)^i \underset{\text{rank}}{} H_i(M;\mathbb{Z}) = 0$ for n odd

$\in 2\mathbb{Z}$ for n even

Example: Real projective spaces of even dimensions are not orientable.

We have computed that $H_n(\mathbb{R}P^n;\mathbb{Z}) \cong 0$ but $H^0(\mathbb{R}P^n;\mathbb{Z}) \cong \mathbb{Z}$. So $\mathbb{R}P^n$ for n even cannot satisfy assumption of Poincaré duality theorem. $\square$

Duality and cup product:

For $c \in C_n(X;R)$ and $\varphi \in C^k(X;R)$ and $\psi \in C^{n-k}(X;R)$ we have

$$\psi(c \cap \varphi) = (\varphi \cup \psi)(c)$$

Left hand side is

$$\psi(\varphi(c/[v_0,\dots v_k]) \cdot c/[v_k,\dots v_n]) = \varphi(c/[v_0\dots v_k]) \cdot \psi(c/[v_k,\dots v_n])$$

For closed R-oriented manifolds we define bilinear form

$$(*) \qquad \begin{aligned} H^k(M;R) \otimes H^{n-k}(M;R) &\longrightarrow R \\ \varphi \otimes \psi &\longmapsto (\varphi \cup \psi)[M] \end{aligned}$$

The form $A \otimes B \longrightarrow R$ is regular if induced maps

$$A \longrightarrow \mathrm{Hom}(B,R)$$
$$B \longrightarrow \mathrm{Hom}(A,R)$$

are isomorphisms.

$R = \mathbb{Q}$
$R = \mathbb{Z}/p$

Theorem (Modified Poincaré duality)

Let M be a closed R-orientable manifold and let R be a field. Then the bilinear form $(*)$ is regular.

Let M be a closed $\mathbb{Z}$-orientable manifold. Then the bilinear form

$$H^k(M;\mathbb{Z})/\mathrm{Tor}\, H^k(M;\mathbb{Z}) \otimes H^{n-k}(M;\mathbb{Z})/\mathrm{Tor}\, H^{n-k}(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

is regular.

Example: Using the theorem above we will prove that as a graded ring

$H^{2k}(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$ $\quad 0 \le 2k \le 2n$
0
$0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z}$

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[w]/\langle w^{n+1}\rangle$$

where $w \in H^2(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$ is a generator. $\quad w^i \in H^{2i}(\mathbb{C}P^n)$

Proof by induction: $\boxed{n=1}$, $\mathbb{C}P^1 = e_0 \cup e_2 = S^2$ and hence

$$H^*(\mathbb{C}P^1;\mathbb{Z}) \cong \mathbb{Z}[w]/\langle w^2\rangle .$$

Let $n \geq 2$. Then the inclusion

$$\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$$

$$\begin{array}{ccccccccc} & & & & & & n-1 & & \\ 0 & & 0 & & \mathbb{Z} & \leftarrow & 0 & \leftarrow & \mathbb{Z} \ \cdots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbb{Z} & \leftarrow & 0 & \leftarrow & \mathbb{Z} & \leftarrow & 0 & \leftarrow & \mathbb{Z} \end{array}$$

induces isomorphism $\quad H^k(\mathbb{C}P^{n-1};\mathbb{Z}) \to H^k(\mathbb{C}P^n;\mathbb{Z})$ iso

for $k \le 2n-2$.

So using inductive assumption we get that $\underline{H^{2k}(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}}$ is generated by w^k where $w \in H^2(\mathbb{C}P^n;\mathbb{Z})$, for $k \le n-1$.

It suffices to show that w^n $= w \cup w \cup \dots \cup w$ is a generator of $H^{2n}(\mathbb{C}P^n;\mathbb{Z})$. $\mathbb{C}P^n$ is orientable (it is simply connected), so

$e_0 \cup e_2 \cup e_4$

Modified PD: $\quad \mathbb{Z} \otimes \mathbb{Z} \longrightarrow \mathbb{Z}$

$$H^2(\mathbb{C}P^n) \otimes H^{2n-2}(\mathbb{C}P^n) \longrightarrow \mathbb{Z} \quad \text{regular}$$

is regular, which means that

$$w \cup w^{n-1}[\mathbb{C}P^n] = \pm 1 \quad \Rightarrow$$

And this means that w^n is a generator of $H^{2n}(\mathbb{C}P^n)$.

We know $H_*(\mathbb{C}P^n)$ as a graded group. $H^*(\mathbb{C}P^n)$ as a graded ring.

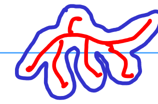
Poincaré duality for manifolds with boundary

Let M be a compact R-orientable manifold with boundary $\partial M = A \cup B$, where A, B are $(n-1)$-dim manifolds such that $\partial A = \partial B = A \cap B$. Then the cap product defines Poincaré duality isomorphism

$$D : H^k(M, A; R) \longrightarrow H_{n-k}(M, B; R).$$

$H^k(M,\partial M) \to H_{n-k}(M) \qquad H^k(M) \to H_{n-k}(M,\partial M)$

ALEXANDER DUALITY



Let $K \subsetneq S^n$ be compact subset of S^n which is a deform. retract of an open neighbourhood. Then

$$\tilde{H}_i(S^n \setminus K ; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K ; \mathbb{Z})$$

S^2

$K = S^1 \subseteq S^2$

$S^2 \setminus S^1 = \{x_1, x_2\} = S^0$

$i = 0$

$\tilde{H}_0(S^2 \setminus S^1) = \tilde{H}^{2-0-1}(S^1)$

$\tilde{H}_0(S^0) \cong \mathbb{Z}$

$\tilde{H}^1(S^1) \cong \mathbb{Z}$

Proof: For $i > 0$ and a neighbourhood U of K

$$H_i(S^n \setminus K) \cong H_c^{n-i}(S^n \setminus K) \quad \text{by Poincaré duality}$$

$S^n \setminus K,\ S^n \setminus K \setminus (S^n \setminus U) = U \setminus K$

$$\cong \varinjlim_U H^{n-i}(S^n \setminus K, U \setminus K) \quad \text{by definition}$$

U open, $S^n \setminus U$ compact in $S^n \setminus K$

$$\cong \varinjlim_U H^{n-i}(S^n, U) \quad \text{by excision}$$

$H^{n-i}(S^n) \to H^{n-i}(S^n, U) \xrightarrow{\cong} H^{n-i-1}(U)$

$$\cong \varinjlim_U H^{n-i-1}(U) \quad \text{by connecting homomorphism}$$

not true for $i = 0$

$$\cong H^{n-i-1}(K) \quad K \text{ is a def retract of some small } U$$

For $i = 0$ we can use the first three isos:

$$\tilde{H}_0(S^n \setminus K) \cong \operatorname{Ker}\left(H_0(S^n \setminus K) \to H_0(pt)\right)$$

$$\cong \operatorname{Ker}\left(H_0(S^n \setminus K) \to H_0(S^n)\right)$$

$$\cong \operatorname{Ker}\left(\varinjlim H^n(S^n, U) \to H^n(S^n)\right)$$

$$\cong \varinjlim \left(\operatorname{Ker}\left(H^n(S^n, U) \to H^n(S^n)\right)\right)$$

$$\cong \varinjlim \tilde{H}^{n-1}(U) \cong \tilde{H}^{n-1}(K)$$

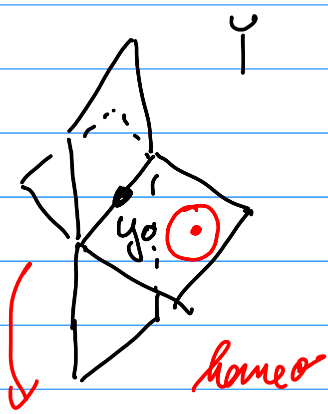
$H^{n-1}(S^n) \to H^{n-1}(U) \to H^n(S^n, U) \to H^n(S^n)$, with $H^{n-1}(S^n) = 0$

$\boxed{X \cong \mathbb{R}^2}$ $\quad$ $\boxed{Y}$ union of two planes with the intersection a line

$$H_2(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\}) \cong H_2(D^2, \partial D^2) \cong H_1(S^1) \cong \mathbb{Z}$$

$$H_2(Y, Y \setminus \text{point}) \cong \cdots\cdots \not\cong \mathbb{Z}$$

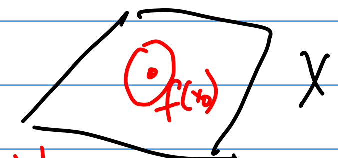
↑
line



Suppose that there is a homeo

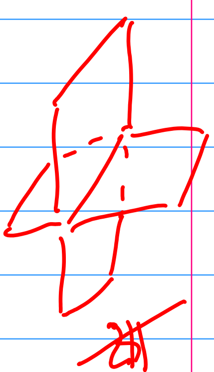
$$f : Y \longrightarrow X = \mathbb{R}^2$$

$y_0 \in$ ~~three~~ intersection



$f(y_0) \in X$ $\qquad$ $f : Y \to X$ homeo

$$f : (Y, Y \setminus y_0) \longrightarrow (X, X - f(y_0)) \text{ homeo}$$

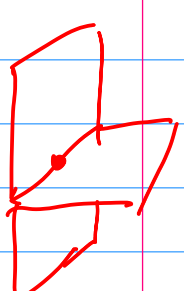


$$f_* : H_2(Y, Y \setminus y_0) \longrightarrow H_2(X, X \setminus f(y_0)) \quad \text{iso}$$

not the case y_0

local homology groups X, $x \in X$

$$H_*(X, X \setminus x)$$

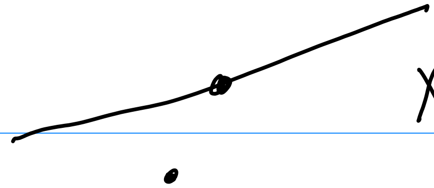


X and Y are not homeomorphic

you can use local hom. groups.

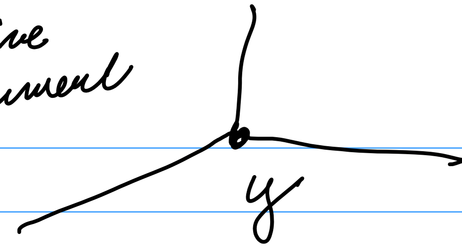
$x \in X$ $\qquad H_*(X, X \setminus x)$ $\qquad H_*(Y, Y \setminus y)$ $\quad y \in Y$

are different



X

naive argument



Y

$X \setminus \{x\}$ ... 2 components

$\tilde{H}_1(X, X \setminus x) \cong \tilde{H}_0(X \setminus x) \cong \mathbb{Z}$

$Y \setminus \{y\}$ 3 components

$\tilde{H}_1(Y, Y \setminus y) \cong \tilde{H}_0(Y \setminus y) \cong \mathbb{Z} \oplus \mathbb{Z}$

$X \not\cong Y$