# Lecture 10.5: CW-complexes and homotopy

X is simply connected, X is path connected
$\pi_0(X, x_0)$ = trivial

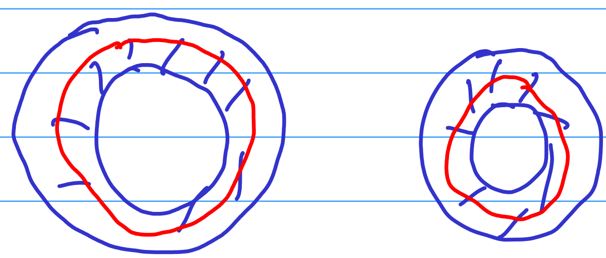
every closed curve can be deformed into constant curve
$\pi_1(X, x_0) = 0$

X is n-connected if $\pi_i(X, x_0) = 0$ for all $0 \le i \le n$ and some $x_0 \in X$.
(That is why $\pi_i(X, x_1)$ for all $x_1 \in X$.)

A pair $(X, A)$ is n-connected if

- every component of path connectivity of X contains a point from A
- $\pi_i(X, A, x_0) = 0$ $\quad 1 \le i \le n$ and all $x_0 \in A$

Example



X blue set
A red set

$$\pi_n(A) \xrightarrow{\cong} \pi_n(X) \xrightarrow{0} \pi_n(X, A; x_0) \xrightarrow{0} \pi_{n-1}(A) \xrightarrow{\cong} \pi_{n-1}(X)$$

$$\Rightarrow \pi_n(X, A, x_0) = 0$$

$f: X \to Y$ is an n-equivalence if

$f_*: \pi_i(X, x_0) \to \pi_i(Y, f(x_0))$ is an iso

for all $x_0$ and all $0 \le i < n$
and an epimorphism for $i = n$

Exercise: $(X, A)$ is $n$-connected if and only if the inclusion $i : A \hookrightarrow X$ is an $n$-equivalence.

$$\begin{array}{c}\pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X,A) \longrightarrow \pi_{n-1}(A) \longrightarrow \pi_{n-1}(X) \\ \hfill \downarrow \\ \hfill \pi_{n-1}(X,A)\end{array}$$

$\pi_n(X,A) = 0 \iff \pi_n(A) \xrightarrow{i_*} \pi_n(X)$ is an epi

$\pi_{i+1}(X,A) = 0 \wedge \pi_i(X,A) = 0 \iff \pi_i(A) \xrightarrow{i_*} \pi_i(X)$
is an iso.

$$\pi_1(X,A) \longrightarrow \pi_0(A) \longrightarrow \pi_0(X)$$

In every path comp. of $X$ there is a point of $A$ + $\pi_1(X,A) = 0$
$\iff \pi_0(A) \xrightarrow{\text{iso}} \pi_0(X)$

Compression Lemma Let $(X, A)$ be a pair of CW-complexes and let $(Y, B \neq \emptyset)$ be a pair of spaces. Suppose that

$$\pi_n(Y, B, y_0) = 0 \quad \forall y_0 \in B$$

whenever there is an $n$-cell in $X - A$.

Then every map $f:(X,A)\longrightarrow(Y,B)$ is homotopic to a map $g: X\longrightarrow B$ rel $A$.

$$\begin{tikzcd} A \arrow[r, "f/A"] \arrow[d, hook] & B \arrow[d, hook] \\ X \arrow[r, "f"'] \arrow[ur, "g"] \arrow[ur, phantom, shift right=3, "\sim"] & Y \end{tikzcd}$$

**Proof:** For every $n$ we find a map

$$f_n : X \longrightarrow Y$$

$$f_n(X^n \cup A) \subseteq B, \quad f_n \sim f_{n-1} \text{ rel } X^{n-1}\cup A$$

We start with

$$f_{-1} = f : X \to Y \quad f(A)\subseteq B$$

and proceed by induction.

Let us have $f_{n-1} : X \to Y$, $f_{n-1}(X^{n-1}\cup A) \subseteq B$

We will define $f_n$ first on $n$-cells.

$$\varphi : (D^n, \partial D^n) \longrightarrow (X^n, X^{n-1})$$

$$f_{n-1}\circ\varphi : (D^n, \partial D^n) \longrightarrow (Y, B)$$

according to the assumption this map is homotopic to a constant map in $(Y,B)$ which means that it is homotopic to a map

$$f_{n-1}\circ\varphi \sim h_n : (D^n, \partial D^n) \longrightarrow (B,B) \text{ rel } \partial D^n$$

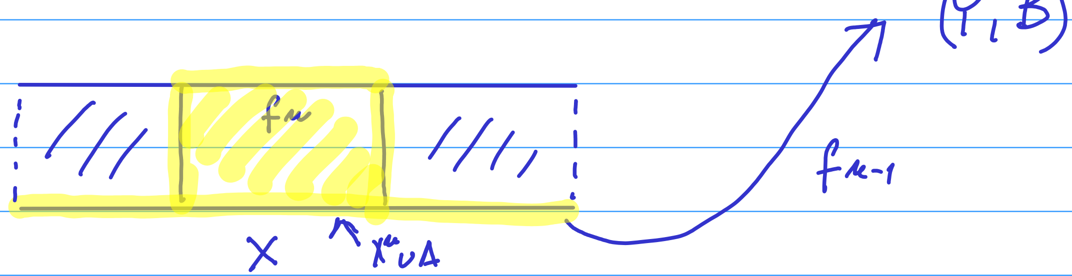
Define $f_n = f_{n-1}$ on $X^{n-1}\cup A$

$f_n$ in the interior of the $n$ cell is defined by $h_n$

And we define $f_n$ in the way on all $n$-cells.

Now $f_n : X^n \longrightarrow (B, B)$ $f_n \sim f_{n-1}$ rel $(X^{n-1} \cup A)$.

Extend $f_n$ on the whole $X$ using HEP.




We can extend the homotopy of $f_{n-1}/X^n \cup A \sim f_n/X^n \cup A$ onto $X$, $h(-,1) = f_n : X \longrightarrow Y$

$f_n \sim f_{n-1}$, rel $X^{n-1} \cup A$

$f_n$ defined on $X$. We are finished if $X = X^n$

Now we define $g = f_n$ for $x \in X^n \cup A$

One can prove that $g \sim f_{-1} = f$ rel $A$.