Global Analysis

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Motivation

Analysis in $\mathbb{R}^n$:

- It is concerned with the study of differentiable/smooth functions
$$f : U \to \mathbb{R}^m, \qquad U \subseteq \mathbb{R}^n \text{ open.}$$
- Sometimes already other domains than open subsets $U \subseteq \mathbb{R}^n$ occurred:
  - Method of Lagrange multipliers to find local extrema of functions $f : \mathbb{R}^2 \to \mathbb{R}$ subject to the condition that $(x, y) \in g^{-1}(0)$ for $g : \mathbb{R}^2 \to \mathbb{R}$.
  - Theorems of Gauß, Green and Stokes: domains called curves and surfaces appear.

Such domains are called submanifolds (with or without boundary) in $\mathbb{R}^n$.

Plan of the Course:

- Generalise the differential and integral calculus from open subsets of $\mathbb{R}^n$ to submanifolds of $\mathbb{R}^n$, which leads also naturally to the notion of abstract manifolds.

- Manifolds can be equipped with various geometric structures and as such they become objects of modern differential geometry:
  - Hypersurfaces in $\mathbb{R}^n$ inherit from the inner product in $\mathbb{R}^n$ a Riemannian metric. $\rightsquigarrow$ Riemannian submanifolds of $\mathbb{R}^n$.
  - Riemannian manifolds
  - Symplectic manifolds
  - Other geometric structures

- Lie Groups
  - appear as symmetry groups of geometric structures
  - appear in the study of PDEs

These lecture notes are mainly based/follow [**2**], [**6**], [**7**], [**8**] and [**9**].

CHAPTER 1

Smooth Manifolds

1.1. Submanifolds of $\mathbb{R}^n$

Submanifolds of $\mathbb{R}^n$ are sufficiently nice/regular subsets of $\mathbb{R}^n$, on which we can develop a differential and integral calculus as on open subsets of $\mathbb{R}^n$. What are some nice subsets?

For $m \leq n$ consider the inclusion

$$\mathbb{R}^m = \mathbb{R}^m \times \{0\} \hookrightarrow \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n. \tag{1.1}$$

Recalling that differentiability is a local concept, we may consider subsets of $\mathbb{R}^n$ that locally have the form of (1.1).

Definition 1.1. A subset $M \subset \mathbb{R}^n$ admits **local m-dimensional trivialisations**, if for every $x \in M$ there exists an open neighbourhood U of x in $\mathbb{R}^n$, an open subset V of $\mathbb{R}^n$ and a diffeomorphism $\phi : U \to V$ such that

$$\phi(U \cap M) = V \cap \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n.$$

We may also consider graphs of smooth functions $g : \mathbb{R}^k \to \mathbb{R}^{n-m}$:

$$\mathrm{gr}(g) := \{(x, g(x)) : x \in \mathbb{R}^m\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n. \tag{1.2}$$

Localising (1.2) yields:

Definition 1.2. A subset $M \subset \mathbb{R}^n$ is **locally the m-dimensional graph of a smooth function**, if for every $x \in M$ there exists an open neighbourhood U of x in $\mathbb{R}^n$, an m-dimensional subspace $W \subset \mathbb{R}^n$, an open subset $V \subset W$ and a smooth function $g : V \to W^\perp$ such that

$$U \cap M = \mathrm{gr}(g) \subset W \oplus W^\perp = \mathbb{R}^n,$$

where $W^\perp = \{x \in \mathbb{R}^n : \langle x, w\rangle = 0 \ \forall w \in W\}$ is the orthogonal compliment of W in $\mathbb{R}^n$ with respect to the standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

We may also consider zero sets of smooth regular functions. A smooth function

$$f : U \to \mathbb{R}^{n-m}, \quad U \subset \mathbb{R}^n \text{ open },$$

is called **regular at** $y \in U$, if the derivative $D_y f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is surjective. It is called **regular**, if f is regular at all points of U. Note that if f is regular at y, then it is so locally around y, since the rank of $D_y f$ is locally constant.

Definition 1.3. A subset $M \subset \mathbb{R}^n$ is **locally the m-dimensional zero set of a regular smooth function**, if for every $x \in M$ there exists an open neighbourhood U of x in $\mathbb{R}^n$ and smooth function $f : U \to \mathbb{R}^{n-m}$ that is regular at x such that

$$M \cap U = f^{-1}(0) = \{y \in U : f(x) = 0\}.$$

Yet another nice class of subsets arise as images of open subsets of $\mathbb{R}^m$ under immersions into $\mathbb{R}^n$:

DEFINITION 1.4. A subset $M \subset \mathbb{R}^n$ admits **local m-dimensional parametrisations**, if for every $x \in M$ there exists an open neighbourhood U of x in $\mathbb{R}^n$, an open subset $V \subset \mathbb{R}^m$ and a smooth map $\psi : V \to U$ such that

- $D_y\psi : \mathbb{R}^k \to \mathbb{R}^n$ is injective for all $y \in V$, and
- ψ induces a homeomorphism onto its image: $\psi : V \cong M \cap U = \mathrm{Im}(\psi)$.

THEOREM 1.5. *Assume $M \subset \mathbb{R}^n$ is a subset of $\mathbb{R}^n$, $1 \leq m \leq n$. Then the following are equivalent:*

(a) *M admits local m-dimensional trivialisations.*
(b) *M is locally the m-dimensional zero set of a regular smooth function.*
(c) *M is locally the m-dimensional graph of a smooth function.*
(d) *M admits local m-dimensional parametrisations.*

The proof is based on the Inverse Function Theorem, which we recall now:

THEOREM 1.6 (Inverse Function Theorem). *Let $U \subset \mathbb{R}^n$ be an open subset, $F : U \to \mathbb{R}^n$ a smooth map, and $x \in U$. If the derivative $D_xF : \mathbb{R}^n \to \mathbb{R}^n$ of F at x is a linear isomorphism, then there exist open neighbourhoods V of x and W of $F(x)$ such that $F(V) = W$ and*

$$F|_V : V \to W$$

is a diffeomorphism.

PROOF. See Analysis/Calculus class. □

An immediate corollary is:

COROLLARY 1.7 (Implicit Function Theorem). *Assume $m \leq n$. Suppose*

$$f : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$$

is a smooth function with $f(0,0) = 0$ and

$$\partial_2 f(0,0) := D_{(0,0)}f|_{\mathbb{R}^{n-m}} : \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$$

is a linear isomorphism. Then there exists locally a unique solution $g(x)$ of $f(x, g(x)) = 0$ and $x \mapsto g(x)$ is smooth.

PROOF. Consider $F : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ given by $F(x,y) = (x, f(x,y))$. Note that F is smooth, $F(0,0) = (0,0)$ and

$$D_{(0,0)}F = \begin{pmatrix} \mathrm{Id}_k & 0 \\ * & \partial_2 f(0,0) \end{pmatrix}$$

is invertible. By Theorem 1.6, F^{-1} exists locally around $(0,0)$ and is smooth. By construction of F, the local inverse F^{-1} is of the form $F^{-1}(u,v) = (u, G(u,v))$ with G smooth. Hence,

$$\begin{aligned} f(x,y) = 0 &\iff F(x,y) = (x,0) \\ &\iff (x,y) = F^{-1}(x,0) = (x, G(x,0)) \\ &\iff y = G(x,0) =: g(x). \end{aligned}$$

□

PROOF OF THEOREM 1.5.

(a) $\implies$ (b) Assume $x \in M$, $U, V \subset \mathbb{R}^n$ open and $\phi : U \to V$ a diffeomorphism as in Definition 1.1. Set $f := \pi \circ \phi : U \to \mathbb{R}^{n-m}$, where $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ is the natural projection. By construction, $f^{-1}(0) = U \cap M$ and f is smooth. Moreover,

$$D_y f = D_{\phi(y)} \pi \circ D_y \phi = \pi \circ D_y \phi : \mathbb{R}^n \cong \mathbb{R}^n \to \mathbb{R}^{n-m}$$

is surjective for all $y \in U$.

(b) $\implies$ (c) Assume $x \in M$ and $f : U \to \mathbb{R}^{n-m}$ as in Definition 1.3. Then $D_x f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is surjective and $\ker(D_x f) =: W \subset \mathbb{R}^n$ an m-dimensional subspace. Identify $\mathbb{R}^n = W \oplus W^\perp$ and write $x = w + w^\perp$. Then $D_x f|_{W^\perp} : W^\perp \to \mathbb{R}^{n-m}$ is a linear isomorphism. Hence, by Corollary 1.7, there exists open neighbourhoods $V \subset W$ and $V' \subset W^\perp$ of w respectively $w^\perp$ and a smooth function $g : V \to V' \subset W^\perp$ such that

$$M \cap (V \times V') = f^{-1}(0) \cap (V \times V') = \{(v, g(v)); v \in V\}.$$

(c) $\implies$ (d) Assume $x \in M$, U, $V \subset W$, and $g : V \to W^\perp$ as in Definition 1.2. Now consider the map $\psi : V \to W \oplus W^\perp = \mathbb{R}^n$ given by $\psi(v) = (v, g(v))$. It is smooth and $\psi(V) = M \cap U$. Moreover, since the natural projection $\pi_W : W \oplus W^\perp \to W$ is a continuous left-inverse of ψ, i.e. $\pi_W \circ \psi = \mathrm{Id}$, ψ is a homeomorphism onto its image. Also, for $D_v \psi : W \to W \oplus W^\perp$ one has

$$D_v \psi(w) = (w, D_v g\, w) = (0,0) \iff w = 0.$$

(d) $\implies$ (a) Assume $x \in M$, $V \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^n$ open and $\psi : V \to U$ as in Definition 1.4. Without loss of generality we may assume $0 \in V$ and $\psi(0) = x$. Then $W := \mathrm{Im}(D_0 \psi) \subset \mathbb{R}^n$ is an m-dimensional subspace and we identify $\mathbb{R}^n = W \oplus W^\perp$. Now define

$$\begin{aligned} &\Phi : V \times W^\perp \to \mathbb{R}^n \\ &\Phi(v, w) := \psi(v) + w. \end{aligned}$$

Note that $\Phi(0,0) = x$ and with respect to the identification $\mathbb{R}^n = W \oplus W^\perp$ the derivative of Φ at $(0,0)$ has the form

$$D_{(0,0)} \Phi = \begin{pmatrix} D_0 \psi & 0 \\ 0 & \mathrm{Id}_{W^\perp} \end{pmatrix}.$$

Hence, $D_{(0,0)} \Phi : W \oplus W^\perp \to \mathbb{R}^n$ is a linear isomorphism and, by Theorem 1.6, there exist open subsets $V_1 \subset V$, $V_2 \subset W^\perp$ and $S \subset \mathbb{R}^n$ with $x \in S$ such that $\Phi : V_1 \times V_2 \to S$ is a diffeomorphism.

Since $\psi : V \to U \cap M$ is a homeomorphism, there exists an open subset $\widetilde{S} \subset \mathbb{R}^n$ with $\psi(V_1) = \widetilde{S} \cap M$. Set $\widetilde{U} := U \cap S \cap \widetilde{S} \subset \mathbb{R}^n$, which is an open neighbourhood of x by construction, and define

$$\phi := (\Phi^{-1})|_{\widetilde{U}} : \widetilde{U} \to \phi(\widetilde{U}) := \widetilde{V}.$$

Then ϕ is a diffeomorphism between the open subsets $\widetilde{U} \subset \mathbb{R}^n$ and $\widetilde{V} \subset V_1 \times V_2 \subset V \times W^\perp \subset \mathbb{R}^n$. Moreover, if $y \in M \cap \widetilde{U}$, then in particular $y \in M \cap \widetilde{S}$, which implies that there exists $v_1 \in V_1$ such that $\psi(v_1) = y$. Since $y \in S$, this shows $\phi(y) = (v_1, 0)$. Conversely, if $(v_1, 0) \in \widetilde{V} \cap W$, then $\Phi(v_1, 0) = \psi(v_1) \in \widetilde{U} \cap M$ by definition of ψ. Hence, $\phi(\widetilde{U} \cap M) = \widetilde{V} \cap W$.

□

Definition 1.8. Assume $1 \leq m \leq n$ are integers. A subset $M \subset \mathbb{R}^n$ is called a **(smooth) submanifold of $\mathbb{R}^n$ of dimension** m, if M satisfies any of the equivalent conditions in Theorem 1.5.

Note that as a subset of $\mathbb{R}^n$ a submanifold $M \subset \mathbb{R}^n$ inherits a topology from $\mathbb{R}^n$, namely the subspace topology:

$$U \subset M \text{ is open} \iff U = \widetilde{U} \cap M \text{ for some open subset } \widetilde{U} \subset \mathbb{R}^n.$$

Remark 1.9.

- If one replaces smooth/C^∞ everywhere by C^r for $1 \leq r < \infty$ or by C^ω, one obtains the notion of C^r-submanifolds respectively real analytic submanifolds of $\mathbb{R}^n$.
- Similarly, if one replaces $\mathbb{R}$ by $\mathbb{C}$ and smooth by holomorphic, one obtains complex submanifolds of $\mathbb{C}^n$.
- Replacing C^∞ in Definition 1.1 by C^0 leads to topological submanifolds of $\mathbb{R}^n$. In this case, not all the definitions 1.1–1.4 are equivalent! Definition 1.2 is stronger than 1.1.

Some trivial examples and natural constructions:

Example 1.1 (Open subsets). Any open subset $U \subset \mathbb{R}^n$ is an n-dimensional submanifold of $\mathbb{R}^n$ and all n-dimensional submanifolds of $\mathbb{R}^n$ are of this form. More generally, any open subset of a submanifold in $\mathbb{R}^n$ is again a submanifold (of the same dimension). Note also that of course any open subset of $\mathbb{R}^n$ can be seen as an n-dimensional submanifold of $\mathbb{R}^d$ via the standard inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^d$ for $n \leq d$.

Example 1.2 (Products). If $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$ are submanifolds of dimensions m respectively k of $\mathbb{R}^n$ respectively $\mathbb{R}^\ell$, then

$$M \times K \subset \mathbb{R}^n \times \mathbb{R}^\ell = \mathbb{R}^{n+\ell}$$

is an $m + k$ dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^\ell$.

Some non-trivial examples:

Example 1.3. Consider $\mathbb{R}^{m+1}$ equipped with its standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$. Then the **$n$-dimensional (unit) sphere**

$$S^m := \{x \in \mathbb{R}^m : ||x|| = 1\} \subset \mathbb{R}^{m+1}$$

is the prototypical example of an m-dimensional submanifold of $\mathbb{R}^{m+1}$. For $m = 1$, one gets the unit circle S^1 in $\mathbb{R}^2$. To see this, note that S^m can be described globally as the zero set of the smooth function $f : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \langle x, x\rangle - 1$, i.e. $f^{-1}(0) = S^m$. Since for any $x \in \mathbb{R}^{m+1} \setminus \{0\}$ and $v \in \mathbb{R}^{m+1}$ one has

$$\begin{aligned} D_x f\, v &= \frac{d}{dt}|_{t=0}\langle x + tv, x + tv\rangle - 1 = \frac{d}{dt}|_{t=0}\langle x, x\rangle + 2t\langle x, v\rangle + t^2\langle v, v\rangle \\ &= 2\langle x, v\rangle, \end{aligned}$$

the derivative $D_x f : \mathbb{R}^{m+1} \to \mathbb{R}$ is surjective by non-degeneracy of $\langle\cdot,\cdot\rangle$. Hence, f is regular.

EXAMPLE 1.4. For fixed positive real numbers $a_1, ..., a_{n+1} \in \mathbb{R}_{>0}$ consider the function

$$\begin{aligned} f &: \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R} \\ f(x_1, ..., x_{n+1}) &:= \sum_{i=1}^{d} \frac{x_i^2}{a_i^2} - \sum_{i=d+1}^{m+1} \frac{x_i^2}{a_i^2} - 1. \end{aligned}$$

It is smooth and regular. Hence, $f^{-1}(0) := M$ is an m-dimensional submanifold of $\mathbb{R}^{m+1}$. Depending on d, these submanifolds are **m-dimensional ellipsoids or hyperboloids**.

EXAMPLE 1.5. Consider $\mathbb{C}^m \cong \mathbb{R}^{2m}$ as real vector space. Then

$$T^n := \{z \in \mathbb{C}^n : |z_1| = ... = |z_n| = 1\} \subset \mathbb{R}^{2m}$$

is an m-dimensional submanifold of $\mathbb{R}^{2m}$, since $f^{-1}(0) = T^m$, where $f : \mathbb{C}^m \setminus \{0\} \to \mathbb{R}^m$ is the smooth regular function given by

$$f(z_1, ..., z_m) = (|z_1| - 1, ..., |z_m| - 1).$$

Of course, also

$$T^m \cong \underbrace{S^1 \times ... \times S^1}_{m-times} \subset \underbrace{\mathbb{R}^2 \times ... \times \mathbb{R}^2}_{m-times} = \mathbb{R}^{2m},$$

so T^m is an m-dimensional submanifold of $\mathbb{R}^{2m}$ by Examples 1.3 and 1.2. It is called the **m-dimensional torus**.

EXAMPLE 1.6. Consider the vector space $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Via a choice of basis of $\mathbb{R}^n$,

$$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cong M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2},$$

where $M_{n\times n}(\mathbb{R})$ denotes the vector space of real $n \times n$ matrices. Since the determinant $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is continuous (polynomial in the eneries of the matrix), the subset

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in M_{n\times n}(\mathbb{R}) : \det(A) \neq 0\} \subset M_{n\times n}(\mathbb{R}) \qquad (1.3)$$

is open and as such an n^2-dimensional submanifold of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Note that $\mathrm{GL}(n, \mathbb{R})$ is also a group with respect to matrix multiplication. It is called the **general linear group**.

In fact, $\det : \mathrm{GL}(n,\mathbb{R}) :\to \mathbb{R}$ is smooth and also regular, since for any $A \in \mathrm{GL}(n,\mathbb{R})$ one has

$$\begin{aligned}(D_A \det)(A) &= \frac{d}{dt}|_{t=0} \det(A+tA)\\ &= \frac{d}{dt}|_{t=0} \det((1+t)A) = \frac{d}{dt}|_{t=0}(1+t)^n \det(A) = n\det(A) \neq 0,\end{aligned}$$

which shows that $D_A \det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is surjective for all $A \in \mathrm{GL}(n,\mathbb{R})$. Hence, also $f := \det -1 : \mathrm{GL}(n,\mathbb{R}) :\to \mathbb{R}$ is a smooth regular function. Therefore,

$$\mathrm{SL}(n,\mathbb{R}) := f^{-1}(0) = \{A \in \mathrm{GL}(n,\mathbb{R}) : \det A = 1\} \subset M_{n\times n}(\mathbb{R})$$

is an (n^2-1)-dimensional submanifold of $M_{n\times n}(\mathbb{R})$. It is also a group with respect to matrix multiplication, called the **special linear group**.

Now consider the map

$$\begin{aligned}f &: \mathrm{GL}(n,\mathbb{R}) \to M_{n\times n}(\mathbb{R})\\ f(A) &:= AA^t - \mathrm{Id}\end{aligned}$$

and set

$$\mathrm{O}(n) := f^{-1}(0) = \{A \in \mathrm{GL}(n,\mathbb{R}) : AA^t = \mathrm{Id}\}. \tag{1.4}$$

Note that $f(A)^t = f(A)$. Hence, f has values in the subspace $M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$ of symmetric $n\times n$-matrices. The function

$$f : \mathrm{GL}(n,\mathbb{R}) \to M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$$

is obviously smooth. To see that it is also regular, note that $(A,B) \mapsto AB^t$ is bilinear as a map $M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$. Therefore, for any $A \in \mathrm{GL}(n,\mathbb{R})$ and $B \in M_{n\times n}(\mathbb{R})$, one has $D_A f B = AB^t + BA^t$. So, if $A \in \mathrm{O}(n)$ and $S \in M^{\mathrm{sym}}_{n\times n}(\mathbb{R})$ is arbitrary, then for $B := \frac{1}{2}SA$ one has

$$D_A f B = \frac{1}{2}(\underbrace{AA^t}_{=\mathrm{Id}} S^t + S\underbrace{AA^t}_{=\mathrm{Id}}) = \frac{1}{2}(S^t + S) = S,$$

which shows that $D_A f : M_{n\times n}(\mathbb{R}) \to M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$ is surjective for any $A \in \mathrm{O}(n)$. Therefore, the set $\mathrm{O}(n)$ of orthogonal $n\times n$-matrices is a submanifold of $\mathbb{R}^{n^2}$ of dimension $\frac{n(n-1)}{2}$. It is also closed under matrix multiplication and hence a group, called the **orthogonal group**.

For submanifolds of $\mathbb{R}^n$, we have an obvious notion of defining smooth maps between them:

Definition 1.10. Suppose $M \subset \mathbb{R}^n$ is an m-dimensional submanifold.

- A map $f : M \to \mathbb{R}^\ell$ is **smooth**, if for every point $x \in M$ there exists an open neighbourhood $\widetilde{U}$ of x in $\mathbb{R}^n$ and a smooth function $\tilde{f} : \widetilde{U} \to \mathbb{R}^\ell$ such that $\tilde{f}|_{M\cap\widetilde{U}} = f|_{M\cap\widetilde{U}}$.
- For a k-dimensional submanifold $K \subset \mathbb{R}^\ell$ a map $f : M \to K$ is **smooth**, if it is is smooth as a map $M \to \mathbb{R}^\ell$.

It follows immediately that constant maps, the identity map and composition of smooth maps are smooths.

DEFINITION 1.11. Suppose $M \subset \mathbb{R}^n$ is an m-dimensional and $K \subset \mathbb{R}^\ell$ a k-dimensional submanifold.

- A map $f : M \to K$ is called a **diffeomorphism**, if f is a smooth bijection with smooth inverse. We call M and N **diffeomorphic**, if there exists a diffeomorphism between them.
- A **local diffeomorphism** between M and K is a smooth map $f : M \to K$ such that for any $x \in M$ and $f(x) \in K$ there exist open neighbourhoods $U \subset M$ and $V \subset K$ of x respectively $f(x)$ such that $f|_U : U \to V$ is a diffeomorphism.

Note that diffeomorphic manifolds have necessarily the same dimension, i.e. $m = k$.

EXAMPLE 1.7. Note that matrix multiplication

$$\mu : M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R}) \qquad \mu(A, B) = AB$$

is smooth, since it is just polynomial in the entries of the matrices. Hence, if $G = \mathrm{GL}(n, \mathbb{R}), \mathrm{SL}(n, \mathbb{R})$ or $\mathrm{O}(n)$, then also $\mu : G \times G \to G$ is smooth as a restriction of a smooth map. Therefore, (G, μ) is a **Lie group**, that is, a (sub-)manifold with a smooth group structure.

To understand smooth maps better we introduce the concept of charts, which will be also key for the notion of abstract manifolds.

DEFINITION 1.12. Suppose $M \subset \mathbb{R}^n$ is an m-dimensional submanifold. A **(local) chart** (or **coordinate chart**) for M is a diffeomorphism

$$u : U \to V,$$

where U is an open subset of M and V an open subset of $\mathbb{R}^m$. Note that such a chart $u : U \to u(U) = V \subset \mathbb{R}^m$ associates to each point $x \in U$ coordinates in $\mathbb{R}^m$:

$$u(x) = (u^1(x), ..., u^m(x)) \in V \subset \mathbb{R}^m.$$

The functions $u^i : U \to \mathbb{R}$ are smooth and called the **local coordinates** associated with the chart (U, u).

LEMMA 1.13. *Suppose $M \subset \mathbb{R}^n$ is an m-dimensional submanifold and $\psi : V \to \widetilde{U}$ a local parametrisation for M, where $\widetilde{U} \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open subsets. Then,*

$$u := \psi^{-1} : U \to V, \quad U := \widetilde{U} \cap M,$$

defines a chart for M. Conversely, given a chart $u : U \to V$, then $U = \widetilde{U} \cap M$ for an open subset $\widetilde{U} \subset \mathbb{R}^n$ and $u^{-1} : V \to U \hookrightarrow \widetilde{U}$ defines a local parametrisation.

PROOF. The map $\psi : V \to U$ is bijective and smooth, since it is smooth as a function $V \to \widetilde{U}$. It remains to show that the inverse ψ^{-1} is smooth. By $(d) \implies (a)$ in the proof of Theorem 1.5: For any $x \in U = \widetilde{U} \cap M$ there exists an open neighbourhood $U' \subset \mathbb{R}^n$ of x and an $(n-m)$-dimensional subspace $W^\perp \subset \mathbb{R}^n$ and an open neighbourhood V' of $(\psi^{-1}(x), 0)$ in $V \times W^\perp$ such that $\Phi : V' \to U'$ given by $\Phi(y, w) = \psi(y) + w$ is a diffeomorphism. Hence, $\psi^{-1} : U' \cap M \to \mathbb{R}^m$ is given by $\mathrm{pr}_1 \circ \Phi|^{-1}_{U' \cap M}$, where $\mathrm{pr}_1 : V \times W^\perp \to V$, which is the restriction of a smooth map $\mathrm{pr}_1 \circ \Phi^{-1}$ from $U' \subset \mathbb{R}^n$ to V. $\square$

Suppose $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$ are submanifolds of dimension m and k respectively and let $f : M \to K$ be a continuous map. Fix $x \in M$ and let (U, u) be a chart for M with $x \in U$ and (V, v) a chart for K with $f(x) \in V$. Then $u(f^{-1}(V) \cap U) \subset \mathbb{R}^m$ is open and

$$v \circ f \circ u^{-1} : u(f^{-1}(V) \cap U) \to v(V)$$

is a continuous map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^k$. With respect to (U, u) and (V, v) we can therefore write it as

$$f = (f_1, ..., f_k),$$

where $v^j(f(y)) = f_j(u^1(y), ..., u^m(y))$ or short $v^j = f_j(u^1, ..., u^m)$.

DEFINITION 1.14. We call $(f_1, ..., f_k)$ the **local coordinate expression** of f with respect to (U, u) and (V, v).

THEOREM 1.15. *Suppose $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$ are submanifolds of dimension m and k respectively and $f : M \to K$ a map. Then the following are equivalent:*

(a) *f is smooth.*
(b) *f is continuous and for every $x \in M$ there exist charts (U, u) for M with $x \in U$ and (V, v) for K with $f(x) \in V$ such that $v \circ f \circ u^{-1} : u(f^{-1}(V) \cap U) \to v(V)$ is smooth.*
(c) *f is continuous and for every $x \in M$ and every chart (U, u) for M with $x \in U$ and every chart (V, v) of K with $f(x) \in V$ the map $v \circ f \circ u^{-1} : u(f^{-1}(V) \cap U) \to v(V)$ is smooth.*
(d) *f is continuous and has smooth local coordinate expressions with respect to some charts.*
(e) *f is continuous and has smooth local coordinate expressions with respect to arbitrary charts.*

PROOF. Evidently, $(b) \iff (d)$ and $(c) \iff (d)$ and $(c) \implies (b)$. Moreover, since compositions of smooth functions are smooth, we also have $(a) \implies (c)$. It remains to show that $(b) \implies (a)$: assume $v \circ f \circ u^{-1}$ is smooth. Since $u : U \to u(U) \subset \mathbb{R}^m$ is smooth, for every $x \in U$ there exist an open neighbourhood $\widetilde{U}$ in $\mathbb{R}^n$ and a smooth map $\tilde{u} : \widetilde{U} \to \mathbb{R}^m$ such that $\tilde{u}|_{U \cap \widetilde{U}} = u|_{U \cap \widetilde{U}}$. Note that $\tilde{u}^{-1}(u(U)) \subset \mathbb{R}^n$ is open and hence $\tilde{u} : \tilde{u}^{-1}(u(U)) \to u(U)$ smooth. Set

$$\tilde{f} := v^{-1} \circ (v \circ f \circ u^{-1}) \circ \tilde{u} : \tilde{u}^{-1}(u(U)) \to V.$$

Then $\tilde{f}$ is smooth as a composition of smooth functions and for any $y \in \tilde{u}^{-1}(u(U)) \cap U$, one has $\tilde{f}(y) = f(y)$. □

1.2. Abstract Manifolds

As we shall see it is useful to introduce the concept of abstract manifolds, which is based on the notion of charts.

DEFINITION 1.16. Suppose M is a topological space.

(a) A **chart with values in** $\mathbb{R}^m$ for M is a homeomorphism $u : U \to u(U)$ from an open subset $U \subset M$ onto an open subset $u(U) \subset \mathbb{R}^m$.

(b) A C^∞**-atlas of charts with values in** $\mathbb{R}^m$ for M is a family of charts with values in $\mathbb{R}^m$,

$$\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\},$$

such that
- $M = \bigcup_{\alpha \in I} U_\alpha$, and
- for any two charts (U_α, u_α) and (U_β, u_β) in $\mathcal{A}$ the transition map (or corresponding coordinate change)

$$u_{\beta\alpha} := u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to u_\beta(U_\alpha \cap U_\beta) \tag{1.5}$$

is smooth.

Note that $u_{\beta\alpha}$ is a map between open subsets of $\mathbb{R}^m$, hence it makes sense to require it to be smooth. Moreover, $u_{\beta\alpha}$ is inverse to $u_{\alpha\beta}$, hence the transition maps are diffeomorphisms.

There is a natural notion of equivalence for atlases.

Definition 1.17. Two atlases for a topological space M are called **equivalent** (or **compatible**), if their union is again an atlas for M.

Note that any atlas $\mathcal{A}$ is contained in a unique maximal atlas $\mathcal{A}_{\max}$ given by the union of all atlases compatible with $\mathcal{A}$.

Definition 1.18. A **(smooth) manifold** of dimension m is a second countable Hausdorff topological space M equipped with a maximal C^∞-atlas of charts with values in $\mathbb{R}^m$, or equivalently, with an equivalence class of C^∞-atlases of charts with values in $\mathbb{R}^m$.

Remark 1.19.
- Similarly, one may define C^k-atlases for $0 \leq k \leq \infty$ or $k = \omega$ and C^k-manifolds respectively real analytic manifolds.
- Atlases wit values in $\mathbb{C}^n$ and requiring the transition maps to be holomorphic leads to holomorphic atlases and the notion of complex manifolds.

Remark 1.20. Suppose M is a topological space such that every point admits an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^m$, i.e. M is a topological manifold. Then the following are equivalent:
- M is Hausdorff and second countable.
- M is metrisable and separable (i.e. there exists a countable dense subset).
- M is Hausdorff, paracompact and has only countably many connected components.

Example 1.8. By Lemma 1.13, any submanifold M of $\mathbb{R}^n$ of dimension m is in a natural way an m-dimensional manifold.

Example 1.9. Let us construct a smooth atlas for the m-sphere $S^m \subset \mathbb{R}^{m+1}$. Fix $p_1 \in S^m$ as ,north pole' and denote by $p_2 := -p_1$ the corresponding ,south pole'. Then $U_i := S^m \setminus \{p_i\}$ for $i = 1, 2$ are open subsets of S^m such that

$$S^m = U_1 \cup U_2.$$

Now stereographic projection gives rise to a chart $u_i : U_i \to \mathbb{R}^m = p_i^\perp$ by mapping $x \in U_i$ to the point of the intersection $u_i(x)$ of the line $(1-\lambda)p_i + \lambda x$ through x and p_i with the hyperplane $p_i^\perp \subset \mathbb{R}^{m+1}$. Since

$$0 = \langle p_1, (1-\lambda)p_i + \lambda x\rangle = ||p_i||^2 - \lambda\langle p_i, (p_i - x)\rangle = 1 - \lambda\langle p_i, (p_i - x)\rangle,$$

we get that $\lambda = \frac{1}{\langle p_i, (p_i - x)\rangle} = \frac{1}{1 - \langle p_i, x\rangle}$. Hence, for $i = 1, 2$ stereographic projection is given by

$$u_i : U_i \to \mathbb{R}^m = p_i^\perp$$
$$u_i(x) = \frac{1}{1 - \langle p_i, x\rangle}(x - \langle p_i, x\rangle p_i),$$

which is obviously continuous. Moreover, it has an inverse given by mapping a point $y \in p_i^\perp$ to the point of intersection $u_i^{-1}(y) = x$ of the line through y and p_i with S^m. If we write $x = p_i + \mu(y - p_i)$, then

$$1 = \langle x, x\rangle = \underbrace{\langle p_i, p_i\rangle}_{=1} + 2\mu\langle p_i, (y - p_i)\rangle + \mu^2\langle (y - p_i), (y - p_i)\rangle,$$

which implies that $\mu = 0$, i.e. $x = p_i$, or $\mu = \frac{2\langle p_i, (p_i - y)\rangle}{\langle (y - p_i), (y - p_i)\rangle} = \frac{2}{||y||^2 + 1}$. Hence, one has

$$u_i^{-1} : \mathbb{R}^m \to U_i$$
$$u_i^{-1}(y) = \frac{1}{1 + ||y||^2}(2y - (||y||^2 - 1)p_i),$$

which is also continuous. Therefore, the maps u_i are homeomorphisms. To see that they define a smooth atlas for S^m we have to compute their transition map. Note that $u_1(U_1 \cap U_2) = u_2(U_1 \cap U_2) = \mathbb{R}^m \setminus \{0\}$ and one easily verifies that

$$u_2 \circ u_1^{-1} : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$$
$$y \mapsto \frac{y}{||y||^2},$$

which is smooth.

Motivated by Theorem 1.15 we define:

Definition 1.21. Suppose M and N are smooth manifolds with maximal atlases $\mathcal{A}_M$ and $\mathcal{A}_N$ respectively. Let $f : M \to N$ be a map.

(a) f is **smooth** (or C^∞) at $x \in M$, if there exist charts $(U, u) \in \mathcal{A}_M$ with $x \in U$ and $(V, v) \in \mathcal{A}_N$ with $f(x) \in V$ such that

$$v \circ f \circ u^{-1} : u(U \cap f^{-1}(V)) \to v(V)$$

is smooth. Moreover, f is called **smooth**, if it is smooth at all points.

(b) f is a **diffeomorphism**, if f is a smooth bijection with smooth inverse.

(c) We say that M is **diffeomorphic** to N, if there exists a diffeomorphism between them.

Note that (a) is independent of the choice of charts, since the transition maps are smooth, cf. Theorem 1.15: if $(U_i, u_i) \in \mathcal{A}_M$ and $(V_i, v_i) \in \mathcal{A}_N$ for $i = 1, 2$, then

$$v_2 \circ f \circ u_2^{-1} = (v_2 \circ v_1^{-1}) \circ (v_1 \circ f \circ u_1^{-1}) \circ u_1 \circ u_2^{-1}.$$

DEFINITION 1.22. Suppose M and N are smooth manifolds with maximal atlases $\mathcal{A}_M$ and $\mathcal{A}_N$ respectively. Let $f : M \to N$ be a smooth map.

(a) f is an **immersion** (respectively a **submersion**) at $x \in M$, if there exist charts $(U, u) \in \mathcal{A}_M$ with $x \in U$ and $(V, v) \in \mathcal{A}_N$ with $f(x) \in V$ such that

$$v \circ f \circ u^{-1} : u(U \cap f^{-1}(V)) \to v(V)$$

is an immersion (respectively submersion) at $u(x)$.

(b) f is called **of constant rank** r on an open subset $W \subset M$, if for every $x \in W$ there exist charts $(U, u) \in \mathcal{A}_M$ with $x \in U$ and $(V, v) \in \mathcal{A}_N$ with $f(x) \in V$ such that the derivative of

$$v \circ f \circ u^{-1} : u(U \cap f^{-1}(V)) \to v(V)$$

at $u(x)$ is of rank r.

As we already noticed any submanifold of $\mathbb{R}^n$ is natural a smooth manifold. In fact, also the converse is true by a Theorem of Whitney: Any m-dimensional manifold is diffeomorphic to a (smooth) submanifold of $\mathbb{R}^{2m}$. The notion of abstract manifolds is however nevertheless useful, since is often easier to construct an atlas to show that a topological space can be given the structure of a smooth manifold then to realise that space as a submanifold in some $\mathbb{R}^n$. An example that demonstrates that well is the following:

EXAMPLE 1.10 (Projective space). Consider the set of lines through 0 in $\mathbb{R}^{m+1}$ given by the quotient

$$\mathbb{R}P^m := \mathbb{R}^{m+1} \setminus \{0\} / \sim,$$

where $x, y \in \mathbb{R}^{m+1}$ are equivalent, denoted by $x \sim y$, $\iff$ there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$. Denote by

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}P^m$$
$$\pi(x) =: [x] =: \underbrace{[x^1 : \ldots : x^{m+1}]}_{\text{homog. coordinates}}$$

the natural projection, where $x = (x^1, \ldots, x^{m+1})$ denotes the standard coordinates in $\mathbb{R}^{m+1}$. Then we may equip $\mathbb{R}P^m$ with the quotient topology with respect to π:

$$U \subset \mathbb{R}P^m \text{ is open } \iff \pi^{-1}(U) \subset \mathbb{R}^{m+1} \setminus \{0\} \text{ is open }.$$

Recall that for this topology a map $f : \mathbb{R}P^m \to X$ for some topological space X is continuous $\iff$ $f \circ \pi : \mathbb{R}^{m+1} \setminus \{0\} \to X$ is continuous.

Note that for $i = 1, \ldots, m + 1$ the subset

$$U_i := \{[x^1 : \ldots : x^{m+1}] \in \mathbb{R}P^m : x^i \neq 0\} \subset \mathbb{R}P^m$$

is open, since $\pi^{-1}(U_i)$ is, and $\mathbb{R}P^m = \bigcup_{i=1}^{m+1} U_i$. For $i = 1, ..., m+1$ define now

$$u_i : U_i \to \mathbb{R}^m$$
$$u_i([x^1 : ... : x^{m+1}]) = (\frac{x^1}{x^i}, ..., \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, ..., \frac{x^{m+1}}{x^i}).$$

Evidently, u_i is evidently continuous, since $u_i \circ \pi$ on $\pi^{-1}(U_i)$ is. Moreover,

$$(x^1, ..., x^m) \mapsto [x^1 : ... : x^{i-1} : 1 : x^i : ... : x^m]$$

defines a continuous inverse. Hence, $u_i : U_i \to \mathbb{R}^m$ is a homeomorphism. To show that the u_i's define a smooth atlas for $\mathbb{R}P^m$ it remains to verify that the transition maps are smooth. Note that for $i < j$ one has

$$u_i(U_i \cap U_j) = \{x \in \mathbb{R}^m : x^{j-1} \neq 0\}.$$

For $i + 1 < j$ the transition map is give by

$$u_j \circ u_i^{-1} : u_i(U_i \cap U_j) \to u_j(U_i \cap U_j) \tag{1.6}$$
$$(x^1, ..., x^m) \mapsto (\frac{x^1}{x^{j-1}}, ..., \frac{x^{i-1}}{x^{j-1}}, \frac{1}{x^{j-1}}, \frac{x^i}{x^{j-1}}, ..., \frac{x^{j-2}}{x^{j-1}}, \frac{x^j}{x^{j-1}}, ..., \frac{x^m}{x^{j-1}}), \tag{1.7}$$

and for $i + 1 = j$ it is given by

$$(x^1, ..., x^m) \mapsto (\frac{x^1}{x^i}, ..., \frac{x^{i-1}}{x^i}, \frac{1}{x^i}, \frac{x^{i+1}}{x^i}, ..., \frac{x^m}{x^i}).$$

In either case it is smooth. Similarly, one verifies that $u_j \circ u_i^{-1}$ is smooth for $j < i$. Hence,

$$\mathcal{A} = \{(U_i, u_i) : i = 1, ..., m+1\}$$

defines a smooth atlas for $\mathbb{R}P^m$. Therefore, $\mathbb{R}P^m$ is an m-dimensional manifold, called **m-dimensional projective space**.

The category of smooth manifolds respects the operations of taking finite products and restrictions to open open subsets.

EXAMPLE 1.11. Suppose M is a smooth manifold with atlas $\mathcal{A} = \{(U_i, u_i) : i \in I\}$ and $U \subset M$ an open subset. Then $\mathcal{A}|_U := \{(U \cap U_i, u_i|_{U \cap U_i}) : i \in \mathcal{I}\}$ defines an atlas for U.

EXAMPLE 1.12. Suppose $(M_i, \mathcal{A}_i)$ are smooth manifolds for $i = 1, ..., n$. Equip $M := M_1 \times ... \times M_n$ with the product topology. Then

$$\mathcal{A} := \{(U_1 \times ... \times U_n, u_1 \times ... \times u_n) : (U_i, u_i) \in \mathcal{A}_i\}$$

defines an atlas on M for which the natural projections $\mathrm{pr}_i : M \to M_i$ are smooth. The product M has the following universal property: If N is a manifold and $f_i : N \to M_i$ are smooth functions, then there exists a unique smooth map $f : M \to N$ such that $\mathrm{pr}_i \circ f = f_i$. This characterises the manifold structure on M uniquely.

We also have a natural notion of submanifolds of manifolds.

DEFINITION 1.23. Suppose $(N, \mathcal{A}^{\max})$ is a manifold of dimension n and $m \leq n$. A subset $M \subset N$ is a submanifold of N of dimension m, if for any $x \in M$ and any chart $(U, u) \in \mathcal{A}^{\max}$ with $x \in U$ the subset $u(U \cap M) \subset \mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$ of dimension m.

Note that in Definition 1.23 it is enough to ask for one such chart around point $x \in M$ and that for $N = \mathbb{R}^n$ the definition coincides with Definition 1.8. Given a submanifold M of a manifold $(N, \mathcal{A}^{\max})$, then for any $x \in M$ we can find $(U, u) \in \mathcal{A}^{\max}$ such that $u|_{U\cap M}$ has values in $\mathbb{R}^m = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$. Then the tuples $(U \cap M, u|_{U\cap M})$ for such charts $(U, u) \in \mathcal{A}^{\max}$ form an atlas for M making M into an m-dimensional manifold. This is called the **standard manifold structure** of the submanifold $M \subset N$. It has the following universal property:

PROPOSITION 1.24. *Suppose $(N, \mathcal{A}^{max})$ is a manifold and $M \subset N$ a submanifold. Then the inclusion $i : M \hookrightarrow N$ is a smooth injective immersion and the standard manifold structure on M is the unique one satisfying the following universal property: for any manifold P a map $f : P \to M$ is smooth $\iff$ $i \circ f : P \to N$ is smooth.*

PROOF. Note first that the standard manifold structure $M \subset N$ satisfies the universal property: with respect to charts $i \circ f$ has locally the form

$$\mathbb{R}^p \xrightarrow{f} \mathbb{R}^m \overset{i}{\hookrightarrow} \mathbb{R}^n,$$

which is smooth $\iff$ $\mathbb{R}^p \xrightarrow{f} \mathbb{R}^m$ is smooth. Applied to $f = \mathrm{Id}_M$ this shows that $i : M \to N$ is smooth. It remains to shows that this property characterises the manifold-structure on M uniquely. Suppose M is equipped with two different manifold structures satisfying the universal property, the standard one $\mathcal{B}^{\max}$ and another one $\mathcal{C}^{\max}$. Then $\mathrm{Id} : (M, \mathcal{B}^{\max}) \to (M, \mathcal{C}^{\max})$ and its inverse are smooth $\iff$ $i : (M, \mathcal{B}^{\max}) \hookrightarrow (N, \mathcal{A}^{\max})$ and $i : (M, \mathcal{C}^{\max}) \hookrightarrow (N, \mathcal{A}^{\max})$ are. The latter are smooth, since $(M, \mathcal{B}^{\max})$ and $(M, \mathcal{C}^{\max})$ satisfy the universal property and hence, as we observed, this implies that the inclusion into $(N, \mathcal{A}^{\max})$ is smooth. ☐

DEFINITION 1.25. Suppose M and N are manifolds. Then a smooth map $f : M \to N$ is called a (smooth) embedding, if the following holds:

(a) $f : M \to f(M)$ is a homeomorphism
(b) $f : M \to N$ is an immersion.

Images of embeddings are submanifolds:

PROPOSITION 1.26. *Suppose M and N are manifolds of dimension m and n respectively. Then $f : M \to N$ is an embedding $\iff$*

(a) *$f(M) \subset N$ is a submanifold of N.*
(b) *$f|_M : M \to f(M)$ is a diffeomorphism.*

PROOF.
$' \Longleftarrow '$ By Proposition 1.24 and (b), the map $f : M \to N$ is a smooth immersion.
$' \Longrightarrow '$ Since $f : M \to f(M)$ is a homeomorphism, it remains to show that $f(M) \subset N$ is a submanifold and that $f : M \to f(M)$ is a local diffeomorphism. Both properties are of a local nature, so we only need to verify it locally around any point $x \in M$ and $f(x) \in N$. Fix $x \in M$, a chart (U, u) for M with $x \in U$ and a chart (V, v) for N with $f(x) \in V$. Since f is a homeomorphism onto $f(M)$, we may assume $f(U) \subset V \cap f(M)$. Replacing f by $v \circ f \circ u^{-1}$ reduces the statement to the case where M and N are open

subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and the result follows from Theorem 1.5 (i.e. existence of local parametrisations). □

PROPOSITION 1.27. *Suppose M and N are manifolds of dimension m and n respectively and $f : M \to N$ a smooth map of constant rank r. Then for any $y \in f(M)$, $f^{-1}(y) \subset M$ is a submanifold of dimension $m - r$ in M.*

PROOF. Being a submanifold is a local property, hence taking charts we can reduce the problem to open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$. The results then follows from an exercise in the tutorial. □

REMARK 1.28. View $M = \mathbb{R}$ as a topological manifold. Then $\mathcal{A}_1 = \{\text{id} : \mathbb{R} \to \mathbb{R}\}$ and $\mathcal{A}_2 = \{u(x) = x^3 : \mathbb{R} \to \mathbb{R}\}$ are not compatible C^∞-atlases, since $u^{-1} \circ \text{id}$ is not smooth, but $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ are diffeomorphic via $f = \sqrt[3]{x} : (M, \mathcal{A}_1) \to (M, \mathcal{A}_2)$.

REMARK 1.29.

- If $k \leq k'$ are elements of $\mathbb{Z}_{>0} \cup \{\infty, \omega\}$, then any C^k-manifold is C^k-diffeomorphic to a $C^{k'}$-manifold. If two $C^{k'}$-manifolds are C^k-diffeomorphic, then they are $C^{k'}$-diffeomorphic.
- Any topological manifold of dimension ≤ 3 admits a unique C^ω-structure. If two C^ω-manifolds of dimension ≤ 3 are homeomorphic, they are C^ω-diffeomorphic.
- There are topological manifolds without any C^1-structure and there exist some with many different differentiable structures. For S^m the diffeomorphism classes of C^∞-structures are known in some dimensions:

m	≤ 3	4	5	6	7	8	9	10	11	12	13	14	15 ...
Diffeo. classes	1	?	1	1	28	2	8	6	992	1	3	2	16256 ...

  In dimension 4, the classification of topological and smooth manifold differs. For $m \neq 4$, $\mathbb{R}^m$ has unique smooth structure, but $\mathbb{R}^4$ has uncountably many!
- The classification of topological manifolds of dimension 1 and 2 is known:
  - Any connected 1-dimensional manifold is homeomorphic to $\mathbb{R}$ or S^1.
  - Any 2-dimensional connected compact manifold is homeomorphic to the connected sum of $g \geq 0$ copies of T^2 or $g \geq 1$ copies of $\mathbb{R}P^2$, and any of them are not homeomorphic. They admit a unique smooth structure, but many different holomorphic structures $\rightsquigarrow$ Theory of Riemann surfaces.

1.3. Partitions of Unity

To extend local constructions and locally defined objects to global ones we need a natural way to 'glue' them. For that we need functions which only locally do not vanish, are ≥ 0 and sum up to one. Such functions are called partitions of unity. In particular, the existence of partitions of unity implies the existence of globally defined smooth functions on manifold.

DEFINITION 1.30. Suppose M is a manifold.

(a) For a map $f : M \to \mathbb{R}^k$ the **support of** f is defined by
$$\operatorname{supp}(f) := \overline{\{x \in M : f(x) \neq 0\}}.$$

(b) A **(smooth) partition of unity** on M is a family
$$\mathcal{F} := \{f_i : M \to \mathbb{R} : i \in I\}$$
of smooth real-valued functions satisfying:

(i) $\mathcal{F}$ is locally finite: for any $x \in M$ there exists an open neighbourhood $U_x \subset M$ of x such that the set $\{i \in I : \operatorname{supp}(f_i) \cap U_x \neq \emptyset\}$ is finite;

(ii) Any $f \in \mathcal{F}$ has values in $[0, 1]$;

(iii) For any $x \in M$, $\sum_{i \in I} f_i(x) = 1$.

Note hat the sum in (iii) is finite by (i).

DEFINITION 1.31. Suppose M is a manifold.

(a) An **open cover** of M is a family $\mathcal{U} := \{U_j : j \in J\}$ of open subsets $U_j \subset M$ such that $M = \bigcup_{j \in J} U_j$.

(b) A partition of unity $\mathcal{F} := \{f_i : M \to \mathbb{R} : i \in I\}$ on M is **subordinate** to an open cover $\mathcal{U} := \{U_j : j \in J\}$ of M, if for every $i \in I$ there exists $j \in J$ such that $\operatorname{supp}(f_i) \subset U_j$.

THEOREM 1.32. *Suppose M is a (smooth) manifold and $\mathcal{U} := \{U_j : j \in J\}$ an open cover of it. Then there exists a (smooth) partition of unity of countably many functions $\mathcal{F} := \{f_k : M \to \mathbb{R} : k \in \mathbb{N}\}$ subordinate to $\mathcal{U}$.*

For a proof of this theorem see e.g. [**7**, Theorem 2.18], but let us just mention that, apart from the topological assumptions we made in our definition of a manifold, key to Theorem 1.32 is the following:

LEMMA 1.33. *For any $x_0 \in \mathbb{R}^m$ and any open neighbourhood $U \subset \mathbb{R}^m$ of x_0, there exists a smooth function $f : \mathbb{R}^m \to \mathbb{R}$ with supp$(f) \subset U$, $f \geq 0$ and $f(x_0) > 0$.*

PROOF. Let $\rho : \mathbb{R} \to \mathbb{R}$ be the function
$$\rho(t) = \begin{cases} e^{-\frac{1}{t^2}}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases} \tag{1.8}$$
It is smooth (but not real-analytic around 0). There exists $\epsilon > 0$ such that $B_{2\epsilon}(x_0) = \{x \in \mathbb{R}^m : ||x - x_0|| < 2\epsilon\} \subset U$. Then define the following smooth function
$$f : \mathbb{R}^m \to \mathbb{R} \tag{1.9}$$
$$f(x) = \rho(\epsilon^2 - ||x - x_0||^2). \tag{1.10}$$
Note that $f(x) \geq 0$ for all $x \in \mathbb{R}^m$, since $\rho \geq 0$, and that $f(x) > 0$ if and only if $x \in B_\epsilon(x_0)$. In particular, $f(x_0) > 0$. Moreover,
$$\operatorname{supp}(f) = \{x \in \mathbb{R}^m : ||x - x_0|| \leq \epsilon\} \subset U.$$
□

REMARK 1.34. On complex manifold there exist no holomorphic partitions of unity.

Typical applications of partitions of unity are:

Corollary 1.35. *Suppose M is a manifold, $U \subset M$ an open subset, $A \subset M$ a closed subset with $A \subset U$. Then the following holds:*

(a) *There exists a smooth function $\phi : M \to [0,1]$ such that supp(ϕ) $\subset U$ and $\phi(x) = 1$ for all $x \in A$ ('Bump function').*
(b) *If $f : U \to \mathbb{R}^k$ is a smooth function, then there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f|_A$.*
(c) *Suppose $M \subset \mathbb{R}^n$ is a submanifold and $f : M \to \mathbb{R}^k$ a smooth function. Then there exist an open subset $\widetilde{U} \subset \mathbb{R}^n$ with $M \subset \widetilde{U}$ and a smooth function $\tilde{f} : \widetilde{U} \to \mathbb{R}^k$ such that $\tilde{f}|_M = f$.*

Proof.

(a) Set $V := M \setminus A$, then $\{U, V\}$ is an open cover of M. By Theorem 1.32, there exists a partition of unity $\mathcal{F} := \{f_k : M \to \mathbb{R} : k \in \mathbb{N}\}$ subordinate to $\{U, V\}$. Hence, for $f_k \in \mathcal{F}$ either $\operatorname{supp}(f_k) \subset U$ or $\operatorname{supp}(f_k) \cap A = \emptyset$. Let $\phi : M \to [0,1]$ be the sum of all f_k such that $\operatorname{supp}(f_k) \subset U$. Then ϕ has the required properties.
(b) Choose ϕ as in (a) and define

$$\tilde{f}(x) = \begin{cases} f(x)\phi(x), & \text{if } x \in U, \\ 0, & \text{if } x \in M \setminus U. \end{cases} \tag{1.11}$$

Since $\operatorname{supp}(\phi) \subset U$, the open subsets U and $M \setminus \operatorname{supp}(\phi)$ form an open cover of M and $\tilde{f}$ is smooth on both, hence on M. By construction, $\tilde{f}|_A = f$, since ϕ equals 1 on A.
(c) By definition of smoothness, for every $x \in M$ there exist an open neighbourhood $\widetilde{U}_x \subset \mathbb{R}^n$ and a smooth function $\tilde{f}_x : \widetilde{U}_x \to \mathbb{R}^k$ such that $\tilde{f}_x|_{\widetilde{U}_x \cap M} = f|_{\widetilde{U}_x \cap M}$. Consider the open subset $\tilde{U} := \bigcup_{x \in M} \widetilde{U}_x \subset \mathbb{R}^n$. Then $\mathcal{U} = \{\widetilde{U}_x : x \in M\}$ is an open cover of $\tilde{U}$. By Theorem 1.32, there exists a partition of unity of $\{\phi_\ell : \ell \in \mathbb{N}\}$ of $\widetilde{U}$ subordinate to $\mathcal{U}$. For each $\ell \in \mathbb{N}$ choose $\widetilde{U}_\ell \in \mathcal{U}$ such that $\operatorname{supp}(\phi_\ell) \subset \widetilde{U}_\ell$ and write $\tilde{f}_\ell$ for the corresponding function. As in (b) we can extend $\tilde{f}_\ell \phi_\ell$ by 0 from a smooth function on $\widetilde{U}_\ell$ to a smooth function $\widetilde{U} \to \mathbb{R}^k$. Then

$$\tilde{f} := \sum_{\ell \in \mathbb{N}} \tilde{f}_\ell \phi_\ell$$

defines a smooth function $\widetilde{U} \to \mathbb{R}^k$. Moreover, for $x \in M$,

$$\tilde{f}(x) = \sum_{\ell \in \mathbb{N}} \tilde{f}_\ell(x)\phi_\ell(x) = \sum_{\ell \in \mathbb{N}} f(x)\phi_\ell(x) = f(x) \underbrace{\sum_{\ell \in \mathbb{N}} \phi_\ell(x)}_{=1} = f(x).$$

□

CHAPTER 2

The Tangent Bundle

For a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$, the derivative $D_x f : \mathbb{R}^m \to \mathbb{R}^n$ of f at a point $x \in M$ is the best linear approximation of f near x. To generalise this to maps $f : M \to N$ between manifolds we need first a linear approximation of M at x, which is called the tangent space $T_x M$ of M at x. The derivative of f at x will be a linear map $T_x M \to T_{f(x)} N$.

2.1. The tangent space of a submanifold of $\mathbb{R}^n$

For an open subset $U \subset \mathbb{R}^m$ and $x \in U$ we set

$$T_x U := \{(x, v) : v \in \mathbb{R}^m\}.$$

Note that $T_x U$ is a vector space: $(x, v) + \lambda(x, w) = (x, v + \lambda w)$ for $\lambda \in \mathbb{R}$, which is called the **tangent space** of U at x. It is a copy of $\mathbb{R}^m$ with origin at x.

For a smooth map $f : U \to \mathbb{R}^n$ the **tangent map** of f at $x \in U$ is given by the linear map

$$T_x f : T_x U \to T_{f(x)} \mathbb{R}^m$$
$$T_x f(x, v) = (f(x), D_x f v).$$

Under the identification $T_x U \cong \mathbb{R}^m$ and $T_{f(x)} \mathbb{R}^n \cong \mathbb{R}^n$, $T_x f$ equals $D_x f$. Sometimes we also simply write $T_x f v = D_x f v$.

Proposition 2.1. *Suppose $M \subset \mathbb{R}^n$ is a submanifold of dimension $m \leq n$ and fix $x \in M$. Then the following subset of $T_x \mathbb{R}^n$ coincide:*

(a) $\{(c(0), c'(0)) : c : (-\epsilon, \epsilon) \to M$ *smooth curve* $, \epsilon > 0, c(0) = x\}$, *where the derivative of c is taken as a curve in* $\mathbb{R}^n$.

(b) $\mathrm{Im}(T_y \psi)$*, where* $\psi : V \to U \subset \mathbb{R}^n$ *is a local parametrisation for* M *with* $\psi(y) = x$.

(c) $\ker(T_x f)$*, where* $f : U \to \mathbb{R}^{n-m}$ *is a local presentation of* M *as the zero set of a regular smooth function* f *(i.e.* $f^{-1}(0) = M \cap U$*).*

In particular, the subset of $T_x \mathbb{R}^n$ given by any of these equivalent descriptions is an m-dimensional subspace of $T_x \mathbb{R}^n$.

Proof.

$(b) \subset (a)$ By definition of ψ, $T_y \psi : T_y V \to T_x U$ is an injective linear map. Hence, $\mathrm{Im}(T_y \psi) \subset T_x U$ is an m-dimensional subspace of the n-dimensional vector space $T_x U = T_x \mathbb{R}^n$. For any $(y, v) \in T_y V$ there exists $\epsilon > 0$ such that $y + tv \in V$ for $|t| < \epsilon$, since V is open. Hence,

$$c : (-\epsilon, \epsilon) \to M$$
$$c(t) = \psi(y + tv)$$

is a well-defined smooth curve with $c(0) = x$. Moreover, $c'(0) = D_y\psi v$. Hence, $T_y\psi(y, v) = (c(0), c'(0))$.

$(a) \subset (c)$ For any smooth curve $c : (-\epsilon, \epsilon) \to M$ with $c(0) = x$ there exists $\epsilon' > 0$ such that $c((-\epsilon', \epsilon')) \subset M \cap U$, since U is open. Then the smooth map $f \circ c : (-\epsilon', \epsilon') \to \mathbb{R}^{n-m}$ is identically zero. Therefore, $0 = D_0(f \circ c) = D_x f c'(0)$ and so $(c(0), c'(0)) \in \ker(T_x f)$.

In summary, $(b) \subset (a) \subset (c)$. Moreover, since $D_x f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is surjective, $\ker(T_x f) \subset T_x\mathbb{R}^n$ is an m-dimensional subspace. Hence, by dimensional reasons, we must have $(a) = (b) = (c)$. □

Definition 2.2. Suppose $M \subset \mathbb{R}^n$ is an m-dimensional submanifold of $\mathbb{R}^n$. For $x \in M$ the **tangent space of** M **at** x, denoted by T_xM, is the m-dimensional subspace of $T_x\mathbb{R}^n$ defined by any of the three equivalent descriptions in Proposition 2.1.

Example 2.1. If $U \subset \mathbb{R}^n$ is an open subset, then $\mathrm{Id}_U : U \to U$ is a global parametrisation of U and so $T_xU = T_x\mathbb{R}^n$, which justifies our definition of T_xU for open subset sets at the beginning of Section 2.1.

Example 2.2. Consider the m-sphere $S^m \subset \mathbb{R}^{m+1}$. Recall that $f^{-1}(0) = S^m$, where $f : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}$ is the smooth function $f(x) = \langle x, x\rangle - 1$. Then for $x \in S^m$, the tangent map $T_x f : T_x\mathbb{R}^{m+1} \to T_0\mathbb{R} = \mathbb{R}$ of f at x is given by

$$T_x f(x, v) = (0, D_x f v), \quad v \in \mathbb{R}^{m+1}.$$

Since $D_x f v = 2\langle x, v\rangle$, one has

$$T_xS^m = \ker(T_x f) = \{(x, v) \in T_x\mathbb{R}^{m+1} : \langle x, v\rangle = 0\}$$

Example 2.3. Since $\mathrm{GL}(n, \mathbb{R}) \subset M_{n\times n}(\mathbb{R})$ is an open subset, for any $A \in \mathrm{GL}(n, \mathbb{R})$ the tangent space of the general linear group at A is given by

$$T_A\mathrm{GL}(n, \mathbb{R}) = \{(A, X) : X \in M_{n\times n}(\mathbb{R})\} \cong M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}.$$

Example 2.4. Consider the orthogonal group

$$\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) : A^{-1} = A^t\}.$$

Recall that $\mathrm{O}(n) = f^{-1}(0)$ for the smooth function

$$\begin{aligned} &f : \mathrm{GL}(n, \mathbb{R}) \to M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{\frac{(n+1)n}{2}} \\ &f(A) = AA^t - \mathrm{Id} \end{aligned}$$

and that for $A \in \mathrm{O}(n)$ one has

$$\begin{aligned} &T_A f : T_A\mathrm{GL}(n, \mathbb{R}) \to T_0M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) = M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) \\ &T_A f(A, X) = (0, AX^t + XA^t). \end{aligned}$$

Hence, for $A \in \mathrm{O}(n)$, one obtains

$$T_A\mathrm{O}(n) = \{(A, X) \in T_A\mathrm{GL}(n, \mathbb{R}) : X^t = -A^{-1}XA\}.$$

In particular,

$$T_{\mathrm{Id}}\mathrm{O}(n) = \{(\mathrm{Id}, X) \in T_{\mathrm{Id}}\mathrm{GL}(n, \mathbb{R}) : X^t = -X\} \cong M^{\mathrm{skew}}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{\frac{(n-1)n}{2}}.$$

Remark 2.3. Let us remark that in the previous two examples the group structure on $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{O}(n)$ induces on the tangent spaces at the identity of $\mathrm{GL}(n,\mathbb{R})$ respectively $\mathrm{O}(n)$ the structure of a so-called Lie algebra, where the Lie bracket is given by the commutator of matrices. The tangent spaces at the identity of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{O}(n)$ are usually denoted by $\mathfrak{gl}(n,\mathbb{R})$ respectively $\mathfrak{o}(n)$ and are called the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$ respectively $\mathrm{O}(n)$. We will discuss this more in the tutorial.

Example 2.5. Suppose $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$ are submanifolds. Then $M \times K \subset \mathbb{R}^n \times \mathbb{R}^\ell$ is a submanifold of $\mathbb{R}^{n+\ell}$ and

$$T_{(x,y)}(M \times K) \cong T_xM \times T_yK \subset T_x\mathbb{R}^n \times T_y\mathbb{R}^\ell.$$

Note that if $\psi_1 : V_1 \to U_1$ and $\psi_2 : V_2 \to U_2$ are local parametrisations of M respectively K, then $\psi_1 \times \psi_2 : V_1 \times V_2 \to U_1 \times U_2$ is a local parametrisation for $M \times K$.

For example, for $T^m \cong S^1 \times ... \times S^1$ one therefore has

$$T_zT^m = T_{z_1}S^1 \times ... \times T_{z_m}S^1,$$

where $z = (z_1, ..., z_m) \in S^1 \times ... \times S^1 \cong T^m$.

Suppose $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$ are submanifolds and $f : M \to K$ a smooth map. The tangent map of f at a point $x \in M$ should be a linear map

$$T_xf : T_xM \to T_{f(x)}K.$$

If the chain rule should hold, the description of the tangent space in Proposition 2.1[(a)] suggests the following definition:

$$T_xf(c(0), c'(0)) = (f(c(0)), (f \circ c)'(0)), \quad (c(0), c'(0)) \in T_xM, \tag{2.1}$$

where $c : (-\epsilon, \epsilon) \to M$ is a smooth curve with $c(0) = x$.

Lemma 2.4. *The map* (2.1) *is well-defined and linear.*

Proof. Smoothness of f implies that there exists an open neighbourhood $\widetilde{U}_x \subset \mathbb{R}^n$ of x and a smooth map $\tilde{f} : \widetilde{U}_x \to \mathbb{R}^\ell$ such that $f|_{M\cap\widetilde{U}_x} = \tilde{f}|_{M\cap\widetilde{U}_x}$. Without loss of generality we may assume that $c : (-\epsilon, \epsilon) \to M$ with $c(0) = x$ satisfies $c((-\epsilon, \epsilon)) \subset M \cap \widetilde{U}_x$. Then $\tilde{f} \circ c = f \circ c : (-\epsilon, \epsilon) \to \mathbb{R}^\ell$ is a smooth curve with values in K and

$$(f \circ c)'(0) = (\tilde{f} \circ c)'(0) = D_{c(0)}\tilde{f}\, c'(0).$$

Hence, (2.1) just depends on $c'(0)$ and not on the extension $\tilde{f}$ of f, which proves that it is well-defined. It is also linear as it is the restriction of the linear map $T_x\tilde{f} : T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^\ell$ to the linear subspace $T_xM \subset T_x\mathbb{R}^n$. □

Definition 2.5. Suppose $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$ are submanifolds and $f : M \to K$ a smooth map. Then the **tangent map of f at $x \in M$** is denoted by

$$T_xf : T_xM \to T_{f(x)}K$$

and is given by (2.1).

From the chain rule for functions between the ambient vector spaces it follows:

COROLLARY 2.6. *Suppose $f : M \to K$ and $g : K \to P$ are smooth maps between submanifolds of $M \subset \mathbb{R}^n$, $K \subset \mathbb{R}^\ell$ and $P \subset \mathbb{R}^r$.*

(a) *$T_x(g \circ f) = T_{f(x)}g \circ T_x f : T_x M \to T_{g(f(x))}P$ for any $x \in M$.*
(b) *If $f : M \to K$ is a diffeomorphism, then for any $x \in M$ its tangent map at x,*
$$T_x f : T_x M \to T_{f(x)}K,$$
is linear isomorphism with inverse $(T_x f)^{-1} = T_{f(x)}f^{-1}$.

PROOF.

(a) Locally around x and $f(x)$ we can find smooth extensions $\tilde{f}$ and $\tilde{g}$ of f respectively g to smooth functions defined on open subsets of $\mathbb{R}^n$ respectively $\mathbb{R}^\ell$. It follows
$$\begin{aligned} T_x(g \circ f) &= (g(f(x)), D_x(\tilde{g} \circ \tilde{f})|_{T_x M}) \\ &= (g(f(x)), D_{f(x)}\tilde{g}|_{T_{f(x)}K} \circ D_x \tilde{f}|_{T_x M}) = T_{f(x)}g \circ T_x f. \end{aligned}$$
(b) We have $f^{-1} \circ f = \mathrm{Id}_M$, $f \circ f^{-1} = \mathrm{Id}_K$ and $T_x \mathrm{Id}_M = \mathrm{Id}_{T_x M}$ for any $x \in M$. By (a) it thus follows
$$\mathrm{Id}_{T_x M} = T_x \mathrm{Id}_M = T_x(f^{-1} \circ f) = T_{f(x)}f^{-1} \circ T_x f.$$
Similarly, $\mathrm{Id}_{T_{f(x)}K} = T_{f(x)}(f \circ f^{-1}) = T_x f \circ T_{f(x)}f^{-1}$.

□

COROLLARY 2.7. *Let $f : M \to K$ be a smooth map between submanifolds $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$.*

(a) *If the tangent map $T_x f : T_x M \to T_{f(x)}K$ at $x \in M$ is an isomorphism, then there exist open neighbourhoods $W_1 \subset M$ and $W_2 \subset K$ of x respectively $f(x)$ such that*
$$f|_{W_1} : W_1 \to W_2$$
is a diffeomorphism.
(b) *$f : M \to K$ is a local diffeomorphism if and only if $T_x f : T_x M \to T_{f(x)}K$ is an isomorphism for all $x \in M$.*

PROOF.

(a) Let (U, u) be a chart of M with $x \in M$ and (V, v) be a chart of K with $f(x) \in K$. Then $v \circ f \circ u^{-1} : u(U \cap f^{-1}(V)) \to v(V)$ is a smooth map between open subsets of $\mathbb{R}^n$ and $\mathbb{R}^\ell$ respectively. We have
$$D_{u(x)}(v \circ f \circ u^{-1}) = D_{f(x)}v \circ D_x f \circ D_{u(x)}u^{-1},$$
which is the composition of three linear isomorphisms. By the inverse function theorem there exist an open neighbourhood $\widetilde{W_1}$ of $u(x)$ in $\mathbb{R}^m$ such that $(v \circ f \circ u^{-1})(\widetilde{W_1}) =: \widetilde{W_2}$ is open and a smooth map $g : \widetilde{W_2} \to \widetilde{W_1}$ inverse to $(v \circ f \circ u^{-1})|_{\widetilde{W_1}}$. Then $W_1 := u^{-1}(\widetilde{W_1})$ is open in M, $W_2 = f(W_1) = v^{-1}(\widetilde{W_2})$ is open in K and $u^{-1} \circ g \circ v : W_2 \to W_1$ is inverse to $f|_{W_1}$.
(b) Follows from (a) and Corollary 2.6.

□

2.2. The tangent bundle of a submanifold in $\mathbb{R}^n$

Ordinary differential equations (of first order) on (sub)manifolds M are described by vector fields. To be able to speak about the smoothness of them, it is convenient to form the disjoint union of the tangent spaces T_xM as x varies over points in M and equip it with a smooth structure. More precisely, recall that a first oder differential equation is given by

$$x'(t) = f(x(t)), \tag{2.2}$$

where $f : U \to \mathbb{R}^n$ is a smooth map and $U \subset \mathbb{R}^n$ open. For any initial condition $x(0) = x_0$ there exists a unique maximal (solution) smooth curve $x : (a, b) \to U$ with $x(0) = x_0$ satisfying (2.2).

If we replace U by a submanifold $M \subset \mathbb{R}^n$, then a solution of (2.2) is a smooth curve $x : (a, b) \to M$, which implies that $x'(t) \in T_{x(t)}M$. So f has to be a map of the form

$$f : M \to \bigsqcup_{x\in M} T_xM$$

such that $f(x) \in T_xM$ for all $x \in M$. To speak about the smoothness of f we need to equip $\bigsqcup_{x\in M} T_xM$ with the structure of a manifold.

Definition 2.8. Suppose $M \subset \mathbb{R}^n$ is a submanifold.

(a) Set

$$TM := \bigsqcup_{x\in M} T_xM := \bigcup_{x\in M} \{x\}\times T_xM = \{(x, v) : x \in M, v \in T_xM\} \subset \mathbb{R}^n\times\mathbb{R}^n$$

and denote by $p : TM \to M$ the natural projection $p(x, v) = x$. Then TM is called the **tangent space** of M and $p : TM \to M$ the **tangent bundle** of M.

(b) If $K \subset \mathbb{R}^\ell$ is another submanifold and $f : M \to K$ a smooth map, the the **tangent map** of f is given by

$$Tf : TM \to TK$$
$$Tf(x, v) = T_xf(x, v).$$

We also simply write $Tfv = T_xv$ for any $v \in T_xM$.

Theorem 2.9. *Suppose $M \subset \mathbb{R}^n$, $K \subset \mathbb{R}^\ell$ and $P \subset \mathbb{R}^r$ are submanifolds.*

(a) *$TM \subset \mathbb{R}^{2n}$ is a submanifold of $\mathbb{R}^{2n}$ of dimension $2\dim(M)$ and $p : TM \to M$ is smooth.*

(b) *For a smooth map $f : M \to K$, the tangent map $Tf : TM \to TK$ is smooth.*

(c) *If $g : K \to P$ is another smooth map, then $T(g \circ f) = Tg \circ Tf$. In particular, if f is a diffeomorphism, then Tf is a diffeomorphism with $(Tf)^{-1} = Tf^{-1}$.*

Proof.

(a) Assume $\dim M = m$ and fix $x \in M$. Let $\psi : \widetilde{U} \to \mathbb{R}^{n-m}$ be a regular smooth function such that $\psi^{-1}(0) = \widetilde{U} \cap M$, where $\widetilde{U} \subset \mathbb{R}^n$ is an open neighbourhood of x. Then

$$\widetilde{V} := \{(y, v) \in \mathbb{R}^n \times \mathbb{R}^n : y \in \widetilde{U}\} \subset \mathbb{R}^n \times \mathbb{R}^n$$

is open and

$$\Psi : \widetilde{V} \to \mathbb{R}^{n-m} \times \mathbb{R}^{n-m}$$
$$\Psi(y,v) = (\psi(y), D_y\psi\, v)$$

is smooth. Moreover, $\Psi(y,v) = 0 \iff (y,v) \in TM$ by Proposition 2.1. Hence, $\Psi^{-1}((0,0)) = TM \cap \widetilde{V}$. To see that Ψ is regular, note that for $v \in T_xM$ one has

$$D_{(x,v)}\Psi = \begin{pmatrix} D_x\psi & 0 \\ * & D_x\psi, \end{pmatrix}$$

which is a surjective linear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n-m} \times \mathbb{R}^{n-m}$ by regularity of ψ. Hence, $TM \subset \mathbb{R}^{2n}$ is a $2m$-dimensional submanifold of $\mathbb{R}^{2n}$. Moreover, the projection $p : TM \to M$ is smooth as it is the restriction to TM of the smooth projection $p_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ to the first $\mathbb{R}^n$-component.

(b) Smoothness of f implies that for $x \in M$ there exists an open neighbourhood $\widetilde{U}_x \subset \mathbb{R}^n$ and a smooth map $\tilde{f} : \widetilde{U}_x \to \mathbb{R}^\ell$ such that $\tilde{f}|_{\widetilde{U}_x \cap M} = f|_{\widetilde{U}_x \cap M}$. Set $\widetilde{V} := \widetilde{U}_x \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ and define

$$F : \widetilde{V} \to \mathbb{R}^\ell \times \mathbb{R}^\ell$$
$$F(y,v) = (\tilde{f}(y), D_y\tilde{f}\, v).$$

Then for $(y,v) \in TM \cap \widetilde{V}$ we have $f(y) = \tilde{f}(y)$ and $F(y,v) = Tf(y,v) = T_yf(y,v)$. Hence, F is a smooth local extension around $(x,0)$ of Tf and so Tf is smooth.

(c) By Corollary 2.6 one has

$$T(g \circ f)(x,v) = T_x(g \circ f)(x,v) = T_{f(x)} \circ T_xf(x,v) = Tg \circ Tf(x,v),$$

which implies the statement about diffeomorphism similarly as in Corollary 2.6[(b)].

□

Distinguished charts for TM from charts from M

Suppose (U,u) is a chart for an m-dimensional submanifold $M \subset \mathbb{R}^n$. Then

- $T(u(U)) = u(U) \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$ is an open subset;
- $TU = p^{-1}(U) \subset TM$ is open, since p is continuous (by openness of U, $T_xU = T_xM$ $x \in U$);
- $Tu : TU \to T(u(U))$ is a diffeomorphism by Theorem 2.9[(c)].

Hence, (TU, Tu) is a chart for TM.

Suppose now (U_α, u_α) and (U_β, u_β) are two charts for M with $U_\alpha \cap U_\beta \neq \emptyset$. Then their transition map

$$u_{\beta\alpha} := u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to u_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism between open subsets of $\mathbb{R}^m$. Hence, its tangent map is a diffeomorphism between open subset of $\mathbb{R}^{2m}$, given by

$$\begin{aligned} Tu_{\beta\alpha} = Tu_\beta \circ Tu_\alpha^{-1} : (u_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^m &\to u_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^m \\ (y, v) &\mapsto (u_{\beta\alpha}(y), D_y(u_{\beta\alpha})v). \end{aligned} \tag{2.3}$$

Therefore, an atlas of M induces in a natural way an atlas for TM.

Suppose $f : M \to K$ is a smooth map between submanifolds $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$ of dimension m and k respectively. Let (U, u) be a chart for M and (V, v) be a chart for K. With respect to the induced charts (TU, Tu) and (TV, Tv) of TM and TK the local coordinate expresssion of Tf has the form

$$\begin{aligned} Tv \circ Tf \circ Tu^{-1} = T(v \circ f \circ u^{-1}) : T(u(U \cap f^{-1}(V))) &\to Tv(V) \\ (y, v) &\mapsto (f^1(y), ..., f^k(y), D_y(f^1, .., f^k)v) \\ &= ((f^1(y), ..., f^k(y), \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(y) & ... & \frac{\partial f^1}{\partial x^m}(y) \\ \vdots & \vdots & \vdots \\ \frac{\partial f^k}{\partial x^1}(y) & ... & \frac{\partial f^k}{\partial x^m}(y) \end{pmatrix} v), \end{aligned}$$

where $(f^1, ..., f^k)$ denotes the local coordinate expression of f with respect to (U, u) and (V, v). Recall that $f^i : u(U \cap f^{-1}(V)) \to \mathbb{R}$ is characterised by $v^i(f(x)) = f^i(u(x))$ for all $x \in U \cap f^{-1}(V)$.

2.3. Vector fields

Definition 2.10. Suppose M is a manifold. A **(smooth) vector bundle** of rank r over M is a manifold E together with a (smooth) surjective submersion $p : E \to M$ such that

(a) for any $x \in M$ the fiber $p^{-1}(x); = E_x$ over x is endowed with the structure of a real vector space of dimension r;
(b) for any $x \in M$ there exists an open neighbourhood $U \subset M$ and a diffeomorphism $\phi : p^{-1}(U) \to U \times \mathbb{R}^r$ such that $\mathrm{pr}_1 \circ \phi = p|_{p^{-1}(U)}$ and such that $\phi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^r$ is a linear isomorphism for all $y \in U$. Such a map ϕ is called a **local trivialisation** of E around x.

Here, E is called the total space and M the base of $p : E \to M$.

Remark 2.11. If $p : E \to M$ is a vector bundle and $U \subset M$ an open set, then $E|_U := p^{-1}(U) \xrightarrow{p} U$ is a vector bundle over U.

Definition 2.12. Two vector bundles $p_1 : E^1 \to M$ and $p_2 : E^2 \to M$ are called **isomorphic**, if there exists a diffeomorphism $F : E^1 \to E^2$ such that $p_2 \circ F = p_1$ and $F|_{E^1_x} : E^1_x \to E^2_x$ is a linear isomorphism for all $x \in M$.

Example 2.6. For any manifold M the natural projection $p = \mathrm{pr}_1 : M \times \mathbb{R}^r \to \mathbb{R}^r$ is a vector bundle of rank r over M. It is called the **trivial vector bundle** over M of rank r. Note that (b) in Definition 2.10 says that locally any vector bundle is isomorphic to the trivial one.

EXAMPLE 2.7. Suppose $M \subset \mathbb{R}^n$ is an m-dimensional submanifold. Then its tangent bundle $p : TM \to M$ is a vector bundle of rank m over M, hence the name. Indeed, take a chart (U, u) for M centered at $x \in M$, then $Tu : TU \to u(U) \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m$ is a chart for TM. Define

$$\phi := u^{-1} \times \mathrm{Id} \circ Tu : p^{-1}(U) \xrightarrow{Tu} u(U) \times \mathbb{R}^m \xrightarrow{u^{-1}\times \mathrm{Id}} U \times \mathbb{R}^m.$$

Evidently,

- ϕ is a diffeomorphism as the composition of two diffeomorphisms and $\mathrm{pr}_1(\phi(y, v)) = y = p(y, v)$.
- $\phi|_{T_yU} : T_yU \to \{y\} \times \mathbb{R}^m \cong \mathbb{R}^m$ equals $T_yU \xrightarrow{T_yu} T_{u(y)}u(U) \cong \mathbb{R}^m$, which is a linear isomorphism.

An important notion in the context of vector bundles is:

DEFINITION 2.13. Suppose $p : E \to M$ is a vector bundle over a manifold M.

- A **(smooth) section of** p is a smooth map $s : M \to E$ such that $p \circ s = \mathrm{Id}_M$ (i.e. $s(x) \in E_x$ for all $x \in M$).
- If $U \subset M$ an open subset, then a section of $E|_U := p^{-1}(U) \xrightarrow{p} U$ is called a **local section of** p defined on U.

LEMMA 2.14. *Suppose $p : E \to M$ is a vector bundle over a manifold M. Then the set of sections, usually denoted by $\Gamma(E)$, is a (infinite-dimensional) real vector space, where for $s, t \in \Gamma(E)$ and $\lambda \in \mathbb{R}$ one has:*

$$(s + \lambda t)(x) := s(x) + \lambda t(x) \quad \textit{for all } x \in M.$$

Moreover, it is a modul over the ring $C^\infty(M, \mathbb{R})$ of smooth real-valued functions: $(fs)(x) = f(x)s(x)$ for $f \in C^\infty(M, \mathbb{R})$.

DEFINITION 2.15. Suppose $M \subset \mathbb{R}^n$ is a submanifold.

- A **(smooth) vector field** on M is a (smooth) section $\xi : M \to M$ of the tangent bundle $p : TM \to M$.
- A local section of $p : TM \to M$ defined on an open subset $U \subset M$ is called a **local vector field** defined on U.

The vector space of sections of $p : TM \to M$ is either denoted by $\Gamma(TM)$ or classically also by $\mathfrak{X}(M)$.

DEFINITION 2.16. Suppose $M \subset \mathbb{R}^n$ is a submanifold of dimension m and (U, u) a chart for M with corresponding local trivialisation $\phi = u^{-1} \times \mathrm{Id} \circ Tu$ for TM. Then for $y \in U$ we set

$$\frac{\partial}{\partial u^i}(y) := \phi^{-1}(y, e^i) \in T_yM, \tag{2.4}$$

where e^i denotes the i-th vector in the standard basis of $\mathbb{R}^m$. Note that $\frac{\partial}{\partial u^1}(y)$,..., $\frac{\partial}{\partial u^m}(y)$ form a basis of T_yM for any $y \in U$.

Evidently, one has:

LEMMA 2.17. *In Definition 2.16, $\frac{\partial}{\partial u^i} : U \to TU$ defines a local vector field on U, called the **i-th coordinate vector field associated with** (U, u).*

Proof. The local coordinate expression of $\frac{\partial}{\partial u^i} : U \to TU$ with respect to (U, u) and (TU, Tu) equals

$$\begin{aligned} Tu \circ \frac{\partial}{\partial u^i} \circ u^{-1} : u(U) &\to u(U) \times \mathbb{R}^m \\ z &\mapsto (z, e^i), \end{aligned}$$

which is smooth. □

Suppose $M \subset \mathbb{R}^n$ is a submanifold of dimension m, (U, u) a chart for M, and $\xi^i \in C^\infty(U, \mathbb{R})$ for $i = 1, ..., m$. Then, by Lemma 2.14,

$$\xi := \sum_{i=1}^m \xi^i \frac{\partial}{\partial u^i}$$

is a local vector field on U. Hence, there are many local vector fields. Conversely, if $\xi \in \mathfrak{X}(U)$, then for any $y \in U$ we may write

$$\xi(y) = \sum_{i=1}^m \xi^i(y) \frac{\partial}{\partial u^i}(y) \in T_y M \tag{2.5}$$

for uniquely defined real numbers $\xi^i(y)$ depending on y. In fact, smoothness of ξ implies that the functions $\xi^i : U \to \mathbb{R}$ are smooth. Indeed, ξ is by definition smooth if and only if its local coordinate expressions are smooth. The latter are given by the map

$$\begin{aligned} Tu \circ \xi \circ u^{-1} &: u(U) \to u(U) \times \mathbb{R}^m \\ (u^1(y), ..., u^m(y)) &\mapsto (u^1(y), ..., u^m(y), \xi^1(y), ..., \xi^m(y)), \end{aligned}$$

which implies the claim.

Moreover, using partitions of unity, we can see that there are also many global vector fields on a submanifold $M \subset \mathbb{R}^n$: if $\xi \in \mathfrak{X}(U)$ and $x \in U$, then there exists an open neighbourhood V of x in M such that $\bar{V} \subset U$. By Corollary 1.35, there exists a smooth function $\phi : M \to \mathbb{R}$ such that $\operatorname{supp}(\phi) \subset U$ and $\phi|_{\bar{V}}$ equal to 1. Setting

$$\tilde{\xi}(y) := \begin{cases} \phi(x)\xi(y), & \text{if } y \in U, \\ 0, & \text{if } y \in M \setminus U, \end{cases}$$

then $\tilde{\xi} \in \mathfrak{X}(M)$ and $\tilde{\xi}|_V = \xi|_V$.

Definition 2.18. Suppose $M \subset \mathbb{R}^n$ is a submanifold of dimension m, (U, u) a chart for M and $\xi \in \mathfrak{X}(M)$ a vector field. Then $\xi|_U \in \mathfrak{X}(U)$ and

$$\xi := \sum_{i=1}^m \xi^i \frac{\partial}{\partial u^i} \quad \text{for } \xi^i \in C^\infty(U, \mathbb{R}). \tag{2.6}$$

(2.6) or also $(\xi^1, ..., \xi^m)$ are called the **local coordinate expression of ξ with respect to (U, u)**.

Let us now compute how the local coordinate expression of a vector field changes when we change the chart. Suppose (U_α, u_α) and (U_β, u_β) are local charts for a submanifold $M \subset \mathbb{R}^n$ of dimension m with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$. Consider

$$u_{\beta\alpha} := u_\beta \circ u_\alpha^{-1} : u_\alpha(U_{\alpha\beta}) \to u_\beta(U_{\alpha\beta}).$$

Then we know that

$$Tu_{\beta\alpha} = Tu_\beta \circ Tu_\alpha^{-1} : u_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to u_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$$
$$(y, v) \mapsto (u_{\beta\alpha}(y), D_y u_{\beta\alpha}\, v).$$

For $x \in U_{\alpha\beta}$ set $A_i^j(x) := \frac{\partial u^j_{\beta\alpha}}{\partial y^i}(u_\alpha(x))$, which defines a smooth map $U_{\alpha\beta} \to \mathrm{GL}(m, \mathbb{R})$. Now we have:

- $\frac{\partial}{\partial u^i_\alpha}(x) = Tu_\alpha^{-1}(u_\alpha(x), e^i)$
- $T_x u_\beta(\frac{\partial}{\partial u^i_\alpha}(x)) = (u_\beta(x), \underbrace{D_{u_\alpha(x)} u_{\beta\alpha}(e^i)}_{\text{i-th column of A}})$.

Hence, one has

$$\frac{\partial}{\partial u^i_\alpha} = \sum_{j=1}^m A_i^j \frac{\partial}{\partial u^j_\beta}.$$

Suppose $\xi \in \mathfrak{X}(M)$ and consider the local coordinate expressions $(\xi^1_\alpha, ..., \xi^m_\alpha)$ and $(\xi^1_\beta, ..., \xi^m_\beta)$ of ξ with respect to (U_α, u_α) and (U_β, u_β). Then one computes:

$$\xi|_{U_{\alpha\beta}} = \sum_i \xi^i_\alpha \frac{\partial}{\partial u^i_\alpha} = \sum_{i,j} \xi^i_\alpha A_i^j \frac{\partial}{\partial u^j_\beta} = \sum_j \xi^j_\beta \frac{\partial}{\partial u^j_\beta},$$

which implies $\xi^j_\beta = \sum_{i=1}^m \xi^i_\alpha A_i^j$.

EXAMPLE 2.8. Suppose $M = \mathbb{R}^2$. Let $u_\alpha : \mathbb{R}^2 \setminus \{0\} \to (0, \infty) \times [0, 2\pi)$ be the polar coordinates so that

$$u_\alpha^{-1}(r, \phi) = (r \cos\phi, r \sin\phi)$$

and $u_\beta = \mathrm{Id}_{\mathbb{R}^2}$ the standard coordinates, i.e. $u^1_\beta = x^1$ and $u^2_\beta = x^2$. The Jacobian of $\mathrm{Id} \circ u_\alpha^{-1} = u_\alpha^{-1}$ equals

$$\begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}.$$

Therefore, one has

- $\frac{\partial}{\partial r} = \cos\phi \frac{\partial}{\partial x^1} + \sin\phi \frac{\partial}{\partial x^2} = \frac{1}{r}(x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2})$
- $\frac{\partial}{\partial \phi} = -r\sin\phi \frac{\partial}{\partial x^1} + r\cos\phi \frac{\partial}{\partial x^2} = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$.

An important operation with vector fields is the pull-back via local diffeomorphism:

DEFINITION 2.19. Suppose $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$ are submanifolds, and $f : M \to K$ a local diffeomorphism. For any $\xi \in \mathfrak{X}(K)$,

$$f^*\xi : M \to TM$$
$$x \mapsto (T_x f)^{-1} \xi(f(x)),$$

defines a vector field, called the **pull-back of ξ via/with respect to f**.

One verifies directly that:

PROPOSITION 2.20. *Suppose $f : M \to K$ is a local diffeomorphism between submanifolds $M \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^\ell$.*

(a) *$f^* : \mathfrak{X}(K) \to \mathfrak{X}(M)$ is $\mathbb{R}$-linear and for $h \in C^\infty(K, \mathbb{R})$ one has*
$$f^*(h\xi) = (h \circ f) f^*\xi \quad \forall \xi \in \mathfrak{X}(K).$$
(b) *If $g : K \to P$ is another local diffeomorphism between submanifolds, then*
$$(g \circ f)^*\xi = f^*(g^*\xi) \quad \forall \xi \in \mathfrak{X}(P).$$
(c) *For any $\xi \in \mathfrak{X}(K)$ one has $Id_K^*\xi = \xi$ and if $M = U \subset K$ is an open subset of K and $i : U \hookrightarrow K$ the inclusion, then $i^*\xi = \xi|_U$*

Example 2.9. Suppose $M \subset \mathbb{R}^n$ is a submanifold of dimension m and (U, u) a chart for M. Then $u(U) \subset \mathbb{R}^m$ is an open subset and
$$\frac{\partial}{\partial x^i} : u(U) \to T(u(U)) = u(U) \times \mathbb{R}^m$$
$$x \mapsto (x, e^i)$$
is a vector field on $u(U)$ (which extends to $\mathbb{R}^m$),—the i-th coordinate vector field with respect to the standard coordinates on $\mathbb{R}^m$. Here, e^i denotes the i-the standard basis vector of $\mathbb{R}^m$. Then for any $y \in U$ one has
$$\begin{aligned} u^* \frac{\partial}{\partial x^i}(y) &= (T_y u)^{-1}\Big(\frac{\partial}{\partial x^i}(u(y))\Big) \\ &= (T_y u)^{-1}(u(y), e^i) = \frac{\partial}{\partial u^i}(y). \end{aligned}$$
In particular, if $\xi \in \mathfrak{X}(M)$ and $\xi|_U = \sum_{i=1}^m \xi^i \frac{\partial}{\partial u^i}$ its local coordinate expression, then
$$(u^{-1})^*\xi|_U = \sum_{i=1}^m \xi^i \circ u^{-1} \frac{\partial}{\partial x^i}.$$

Definition 2.21. Suppose $M \subset \mathbb{R}^n$ is a submanifold, $\xi \in \mathfrak{X}(M)$, and $I \subset \mathbb{R}$ an interval. A smooth curve $c : I \to M$ is called an **integral curve** of ξ, if
$$c'(t) = \xi(c(t)). \tag{2.7}$$

Note that for $M = U \subset \mathbb{R}^n$ an open subset, equation (2.7) defines a system of ordinary differential equations of first order, where its solutions are the integral curves of ξ.

Via charts the theorem about existence and uniqueness of solutions of a system of first order differential equations implies:

Theorem 2.22. *Suppose $M \subset \mathbb{R}^n$ is a submanifold and $\xi \in \mathfrak{X}(M)$.*

(a) *For any $x \in M$ there exists a unique maximal integral curve*
$$c_x : I_x \to M$$
of ξ, where $I_x \subset \mathbb{R}$ is an interval with $0 \in I_x$ and $c(0) = x$.

(b) *$\mathcal{D}(\xi) = \{(t, x) \in \mathbb{R} \times M : t \in I_x\} \subset \mathbb{R} \times M$ is an open subset containing $\{0\} \times M$ and the map*
$$\mathrm{Fl}^\xi : \mathcal{D}(\xi) \to M$$
$$(t, x) \mapsto c_x(t)$$
*is smooth, which is called the **local flow of** ξ.*

(c) *If* $y := \mathrm{Fl}^\xi(s, x)$ *exists, then* $\mathrm{Fl}^\xi(t+s, x)$ *exists* $\iff$ $\mathrm{Fl}^\xi(t, y)$ *exists. In this case,*

$$\mathrm{Fl}^\xi(t + s, x) = \mathrm{Fl}^\xi(t, \mathrm{Fl}^\xi(s, x)). \tag{2.8}$$

In particular, if $\mathcal{D}(\xi) = M \times \mathbb{R}$*, then* (2.8) *says that*

$$(\mathbb{R}, +) \to (\mathrm{Diff}(M), \circ)$$
$$t \mapsto \mathrm{Fl}^\xi(t, \_\,)$$

is a group homomorphism.

Notation: We also write $\mathrm{Fl}^\xi_t(x) := \mathrm{Fl}^\xi(t, x)$.

Note that (b) and (c) of Theorem 2.22 imply:

COROLLARY 2.23. *For any* $x \in M$ *there exists an open neighbourhood* $U \subset M$ *of* x *and* $\epsilon > 0$ *such that*

$$\mathrm{Fl}^\xi : (-\epsilon, \epsilon) \times U \to M$$

is defined and $\mathrm{Fl}^\xi_t : U \to M$ *is a local diffeomorphism for any* $t \in (-\epsilon, \epsilon)$*. Note moreover, that, wherever defined,* $(\mathrm{Fl}^\xi_t)^*\xi = \xi$*, which is equivalent to* $T_x\mathrm{Fl}^\xi_t\, \xi(x) = \xi(\mathrm{Fl}^\xi_t(x))$*.*

DEFINITION 2.24. Suppose $M \subset \mathbb{R}^n$ is a submanifold. A vector field $\xi \in \mathfrak{X}(M)$ is called **complete**, if $\mathcal{D}(\xi) = M \times \mathbb{R}$.

PROPOSITION 2.25. *Suppose* $M \subset \mathbb{R}^n$ *is a submanifold and* $\xi \in \mathfrak{X}(M)$ *a vector field.*

(a) *Suppose there exists* $\epsilon > 0$ *such that for any* $x \in M$ *there exists an open neighbourhood* $U_x \subset M$ *of* x *such that the local flow* Fl^ξ *of* ξ *is defined on* $(-2\epsilon, 2\epsilon) \times U_x$*. Then* ξ *is complete.*
(b) *If* M *is compact,* ξ *is complete.*

PROOF. (a) Set

$$\Psi_t(x) := ((\mathrm{Fl}^\xi_\epsilon)^{\circ k} \circ \mathrm{Fl}^\xi_{t-k\epsilon})(x) = (\underbrace{\mathrm{Fl}^\xi_\epsilon \circ ... \circ \mathrm{Fl}^\xi_\epsilon}_{k-times} \circ \mathrm{Fl}^\xi_{t-k\epsilon})(x),$$

where k is the integer part of t/ϵ. Note that this is defined for all $t \in \mathbb{R}$ and $x \in M$. By (c) of Theorem 2.22 we must have $\Psi_t = \mathrm{Fl}^\xi_t$.

(b) By Corollary 2.23, for any $x \in M$ there exist $\epsilon_x > 0$ and an open neighbourhood $U_x \subset M$ of x such that $\mathrm{Fl}^\xi : (-2\epsilon_x, 2\epsilon_x) \times U_x \to M$ is defined. Compactness of M implies that there exists finitely many points $x_1, ..., x_r$ such that $M = U_{x_1} \cup ... \cup U_{x_r}$. Then $\epsilon := \min_{i=1,...,r} \epsilon_{x_i}$ satisfies the assumption of (a).

□

EXAMPLE 2.10. Let $M = \mathbb{R}^2$ with coordinates (x, y) and corresponding coordinate vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Consider the vector fields

$$\xi = y\frac{\partial}{\partial x} \quad \text{and} \quad \eta = \frac{x^2}{2}\frac{\partial}{\partial y}.$$

Then their flows are given by

$$\mathrm{Fl}^{\xi}(t,(x,y)) = (x+ty, y)$$
$$\mathrm{Fl}^{\eta}(t,(x,y)) = (x, y+t\frac{x^2}{2}),$$

and so both vector fields are complete. Note however that their sum

$$\xi + \eta = y\frac{\partial}{\partial x} + \frac{x^2}{2}\frac{\partial}{\partial y}$$

is not: if we write $c(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \in \mathbb{R}^2$ for a curve in $\mathbb{R}^2$, then

$$c'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = (\xi+\eta)(c(t)) = \begin{pmatrix} y(t) \\ \frac{x(t)^2}{2} \end{pmatrix}$$

implies that $x''(t) = \frac{x(t)^2}{2}$, which in turn yields $(x'(t))^2 = \frac{x(t)^3}{3} + \mathrm{const}$. If one solves this for initial value $\frac{y_0^2 - x_0^3}{3} = 0$ with $x_0 > 0$, then the integral curve is not defined for all t.

2.4. Tangent vectors as derivations

Suppose $M \subset \mathbb{R}^n$ is a submanifold.

DEFINITION 2.26. A map $\partial : C^\infty(M,\mathbb{R}) \to \mathbb{R}$ is called a **derivation at** $x \in M$, if ∂ is $\mathbb{R}$-linear and

$$\partial(fg) = (\partial f)g(x) + f(x)\partial g \quad \forall f, g \in C^\infty(M,\mathbb{R}).$$

We set $\mathrm{Der}_x(C^\infty(M,\mathbb{R}),\mathbb{R}) := \{\partial : C^\infty(M,\mathbb{R}) \to \mathbb{R} : \partial \text{ is a derivation}\}$, which is a real vector space in the obvious way.

LEMMA 2.27. *Suppose $\partial : C^\infty(M,\mathbb{R}) \to \mathbb{R}$ is a derivation at $x \in M$.*

(a) $\partial(1) = 0$ *(which implies $\partial(f) = 0$ for all constant functions by linearity of ∂).*
(b) *If $f_1, f_2 \in C^\infty(M,\mathbb{R})$ coincide on an open neighbourhood $U \subset M$ of x, then $\partial(f_1) = \partial(f_2)$.*

PROOF.

(a) $\partial(1) = \partial(1 \cdot 1) = 1\partial(1) + \partial(1)1 = 2\partial(1)$ and hence $\partial(1) = 0$.
(b) Suppose $U \subset M$ is an open neighbourhood of x on which $f_1, f_2 \in C^\infty(M,\mathbb{R})$ coincide. Then $f := f_1 - f_2$ vanishes on U. By Corollary 1.35, there exists $g \in C^\infty(M,\mathbb{R})$ such that $\mathrm{supp}(g) \subset U$ and $g(x) = 1$. Since $\mathrm{supp}(g) \subset U$ and $f|_U = 0$, one has

$$0 = \partial(0) = \partial(fg) = \partial f \underbrace{g(x)}_{=1} + \underbrace{f(x)}_{=0} \partial g = \partial f = \partial f_1 - \partial f_2.$$

□

LEMMA 2.28. *Any tangent vector $\xi \in T_xM$ in induces a derivation at x given by*

$$\partial_\xi : f \mapsto \xi \cdot f := T_x f \xi \in T_{f(x)}\mathbb{R} \cong \mathbb{R}.$$

Proof. Let $c : I \to M$ be a C^∞-curve with $c(0) = x$ and $c'(0) = \xi$, and $f, g \in C^\infty(M, \mathbb{R})$ and $\lambda \in \mathbb{R}$. Then $(f + \lambda g) \circ c = f \circ c + \lambda(g \circ c)$, which implies

$$\partial_\xi(f + \lambda g) = ((f + \lambda g) \circ c)'(0) = (f \circ c)'(0) + \lambda(g \circ c)'(0) = \partial_\xi(f) + \lambda\partial_\xi(g).$$

Similarly, $fg \circ c = (f \circ c)(g \circ c)$ and the product rule give

$$(fg \circ c)'(0) = (f \circ c)'(0)g(x) + f(x)(g \circ c)'(0) = \partial_\xi(f)g(x) + f(x)\partial_\xi(g).$$

□

Let (U, u) be a chart for M with $x \in U$. Then $\frac{\partial}{\partial u^i}(x)$ for $i = 1, ..., m$ form a basis of T_xM. Moreover,

$$\begin{aligned}\frac{\partial}{\partial u^i}(x) \cdot f &= T_xf \circ T_{u(x)}u^{-1}(u(x), e^i) \\ &= T_{u(x)}(f \circ u^{-1})(u(x), e^i) = (f(x), D_{u(x)}(f \circ u^{-1})e^i).\end{aligned}$$

equals the i-th partial derivative at $u(x)$ of the local coordinate expression $f \circ u^{-1} : u(U) \to \mathbb{R}$ of f. We therefore write

$$\frac{\partial}{\partial u^i}(x) \cdot f =: \frac{\partial f}{\partial u^i}(x).$$

Since any $\xi \in T_xM$ can be written as

$$\xi = \sum_{i=1}^m \xi^i \frac{\partial}{\partial u^i}(x), \quad \xi^i \in \mathbb{R},$$

we have

$$\partial_\xi(f) = T_xf\xi = \sum_{i=1}^m \xi^i T_xf \frac{\partial}{\partial u^i}(x) = \sum_{i=1}^m \xi^i \frac{\partial f}{\partial u^i}(x).$$

Theorem 2.29. *Suppose $M \subset \mathbb{R}^n$ is a submanifold and $x \in M$ a point. Then the map*

$$\begin{aligned}\Psi_x : T_xM &\to \mathrm{Der}_x(C^\infty(M, \mathbb{R}), \mathbb{R}) \\ \xi &\mapsto \partial_\xi\end{aligned}$$

is a linear isomorphism. Moreover, for any smooth map $F : M \to K$ and $K \subset \mathbb{R}^\ell$ another submanifold, the following diagram commutes

$$\begin{array}{ccc} T_xM & \xrightarrow{\Psi_x} & \mathrm{Der}_x(C^\infty(M, \mathbb{R}), \mathbb{R}) \\ \downarrow{\scriptstyle T_xF} & & \downarrow{\scriptstyle F_*} \\ T_{F(x)}K & \xrightarrow{\Psi_{F(x)}} & \mathrm{Der}_{F(x)}(C^\infty(K, \mathbb{R}), \mathbb{R}), \end{array}$$

where $F_*(\partial)(g) := \partial(g \circ F)$ for all $g \in C^\infty(K, \mathbb{R})$.

Proof.

- Linearity of Ψ_x: this is clear, since T_xf is linear for any $f \in C^\infty(M, \mathbb{R})$.
- Commutativity of the diagram:

$$\begin{aligned}F_*(\Psi_x(\xi))(g) &= F_*(\partial_\xi)(g) = \partial_\xi(g \circ F) \\ &= T_x(g \circ F)\xi = (T_{F(x)}g \circ T_xF)\xi = \partial_{T_xF\xi}(g).\end{aligned}$$

- Injectivity of Ψ_x: If $0 \neq \xi \in T_xM$, then we need to show that there exists a function $f \in C^\infty(M, \mathbb{R})$ such that $\partial_\xi(f) \neq 0$.

  Let V be an open neighbourhood of x such that $\bar{V} \subset U$, where (U, u) is a chart. By Corollary 1.35, there exists $g \in C^\infty(M, \mathbb{R})$ such that $\mathrm{supp}(g) \subset U$ and $g|_{\bar{V}} \equiv 1$. Then gu^i can be extended by zero to a smooth function $\tilde{u}^i : M \to \mathbb{R}$, which locally around x coincides with u^i. By construction, $\tilde{u}^i \circ u^{-1}$ locally around $u(x)$ equals the i-th projection. If $\xi = \sum_{j=1}^m \xi^j \frac{\partial}{\partial u^j}(x)$, then

$$\partial_\xi(\tilde{u}^i) = \sum_{j=1}^m \xi^j \frac{\partial \tilde{u}^i}{\partial u^j}(x) = \xi^i.$$

  Since for $\xi \neq 0$ there is at least one nonzero coefficient ξ^i, we conclude that Ψ_x is injective.
- Surjectivity of Ψ_x: Let (U, u) be a chart with $x \in U$. Without loss of generality $u(x) = 0$ and $u(U) \supset B_1(0) := \{z \in \mathbb{R}^n : ||z|| < 1\}$. If $y \in U$ such that $u(y) \in B_1(0)$, then for $f \in C^\infty(M, \mathbb{R})$ we have

$$\begin{aligned} f(y) &= f(x) + \int_0^1 \frac{d}{dt}(f \circ u^{-1})(tu(y))dt \\ &= f(x) + \int_0^1 \sum_i \frac{\partial(f \circ u^{-1})}{\partial x^i}(tu(y))u^i(y)dt \\ &= f(x) + \sum_i u^i(y) \underbrace{\int_0^1 \frac{\partial(f \circ u^{-1})}{\partial x^i}(tu(y))dt}_{:=h_i(y)}, \end{aligned}$$

  where $h_i : u^{-1}(B_1(0)) \to \mathbb{R}$ is a smooth function. So we can write f locally around x as

$$f(y) = f(x) + \sum_i u^i(y)h_i(y).$$

  By Corollary 1.35, we can extend h_i and u^i to smooth functions on M without changing them locally around x. So locally around x we have

$$f = f(x) + \sum_i u^i h_i.$$

  If $\partial \in \mathrm{Der}_x(C^\infty(M, \mathbb{R}), \mathbb{R})$, then by Lemma 2.27 one has

$$\begin{aligned} \partial(f) &= \sum_i \partial(u^i) \underbrace{h_i(x)}_{\frac{\partial f}{\partial u^i}(x)} + \underbrace{u^i(x)}_{=0} \partial(h_i) \\ &= \sum_i \partial(u^i) \frac{\partial f}{\partial u^i}(x). \end{aligned}$$

  This shows that $\partial = \partial_\xi$ with $\xi = \sum_i \partial(u^i) \frac{\partial}{\partial u^i}(x)$.

∎

2.5. The tangent bundle of an abstract manifold

Suppose $(M, \mathcal{A}) = (M, \mathcal{A}_{\max})$ is an abstract manifold of dimension m. Then we define the **tangent space of M at $x \in M$** as the vector space

$$T_xM := \mathrm{Der}_x(C^\infty(M, \mathbb{R}), \mathbb{R}). \tag{2.9}$$

We use the notation $\xi_x(f) := \xi_x \cdot f$ for $\xi_x \in T_xM$ and $f \in C^\infty(M, \mathbb{R})$.

Remark 2.30. Alternatively, we could have defined T_xM as the set of equivalence classes of smooth curves $c : (-\epsilon, \epsilon) \to M$, where $c_1 \sim c_2$ if $c_1(x) = c_2(x)$ and for a (equivalently, any) chart (U, u) around x one has $(u \circ c_1)'(0) = (u \circ c_2)'(0)$. In contrast to (2.9), it is however not obvious that this is a vector space.

Definition 2.31.

- The **tangent bundle** of M is defined as
$$TM := \bigsqcup_{x \in M} T_xM = \bigcup_{x \in M} \{x\} \times T_xM,$$
and we write $p : TM \to M$ for the natural projection.
- For a smooth map between abstract manifolds $F : M \to K$ we define its **tangent map** by
$$\begin{aligned} Tf &: TM \to TK \\ Tf(x, \xi_x) &= (f(x), T_xf\xi_x), \end{aligned} \tag{2.10}$$
where $T_xf : T_xM \to T_{f(x)}K$ is given by
$$T_xf(\xi_x)(f) := (T_xf\xi_x) \cdot g := \xi_x(g \circ f) = \xi_x \cdot (g \circ f)$$
for all $g \in C^\infty(K, \mathbb{R})$.

It follows directly, cf. the analogues statements for submanifolds:

Proposition 2.32. *Suppose M, K and P are smooth manifolds, and $f : M \to K$ and $g : K \to P$ smooth maps. Then one has:*

(a) $T(g \circ f) = Tg \circ Tf$ *and* $TId_M = Id_{TM}$.
(b) *f is a local diffeomorphism if and only if $T_xf : T_xM \to T_{f(x)}K$ is a linear isomorphism for all $x \in M$.*

There is natural topology on TM: We may equip TM with the coarsest topology such that $TU \subset TM$ is open and $Tu : TU \to Tu(U) = u(U) \times \mathbb{R}^m$ is a homeomorphism for all $(U, u) \in \mathcal{A}$. It is again second countable and Hausdorff. Moreover,

$$\mathcal{A}_{TM} := \{(TU, Tu) : (U, u) \in \mathcal{A}\},$$

defines a C^∞-atlas of charts with values in $\mathbb{R}^{2m}$ (cf. the corresponding statement for submanifolds). Hence, $(TM, \mathcal{A}_{TM})$ has naturally the structure of an abstract manifold of dimension $2m$.

As for submanifolds, with respect to this smooth structure on TM, $p : TM \to M$ is a smooth vector bundle of rank m over M and for a smooth map $f : M \to K$ also $Tf : TM \to TK$ is smooth. Moreover, vector fields are again defined as (smooth) sections of the tangent bundle. Also, of course, the local coordinate expressions of Tf and vector fields remain valid. Similarly, all the definitions and statements about the pull-back of vector fields via local diffeomorphisms and about the local flow of vector fields remain valid without change.

2.6. Vector fields as derivations and the Lie bracket

Suppose $(M, \mathcal{A})$ is a manifold. For any $\xi \in \mathfrak{X}(M)$ and $f \in C^\infty(M, \mathbb{R})$,

$$\xi \cdot f : M \to \mathbb{R}$$
$$(\xi \cdot f)(x) := \xi_x \cdot f := T_x f \xi_x \qquad \xi(x) = (x, \xi_x),$$

defines a smooth functions, since $\xi \cdot f$ is the second component of $T \circ \xi : M \to TM \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$.

Definition 2.33. A **derivation** of the algebra $C^\infty(M, \mathbb{R})$ is a linear map $D : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ such that

$$D(fg) = D(f)g + fD(g) \quad \forall f, g \in C^\infty(M, \mathbb{R}). \tag{2.11}$$

We write $\mathrm{Der}(C^\infty(M, \mathbb{R}))$ for the vector space of derivations of $C^\infty(M, \mathbb{R})$.

Theorem 2.34. *The map $\Psi : \xi \mapsto (f \mapsto \xi \cdot f)$ defines a linear isomorphism*

$$\mathfrak{X}(M) \cong \mathrm{Der}(C^\infty(M, \mathbb{R})).$$

Proof. First, $\xi \cdot \_ : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ is evidently also linear and for $f, g \in C^\infty(M, \mathbb{R})$ one has

$$\begin{aligned} \xi \cdot (fg)(x) = \xi_x \cdot (fg) &= (\xi_x \cdot f)g(x) + f(x)(\xi_x \cdot g) \\ &= ((\xi \cdot f)g + f(\xi \cdot g))(x), \end{aligned}$$

which shows that Ψ maps indeed vector fields to derivations of $C^\infty(M, \mathbb{R})$. Moreover, Ψ is linear, since $T_x f$ is linear for all $x \in M$ and $f \in C^\infty(M, \mathbb{R})$.

- Injectivity of Ψ: If $\xi \neq 0$, then there exists $x \in M$ such that $\xi_x \neq 0$. By Theorem 2.29, we know that there exists $f \in C^\infty(M, \mathbb{R})$ such that $(\xi \cdot f)(x) = \xi_x \cdot f \neq 0$.
- Surjectivity of Ψ: Let $D \in \mathrm{Der}(C^\infty(M, \mathbb{R}))$ and $x \in M$. Then $f \mapsto D(f)(x)$ is a derivation at x by (2.11). Hence, by Theorem 2.29, there exists a unique $\xi_x \in T_x M$ such that $D(f)(x) = \xi_x \cdot f$. It remains to show that $\xi \mapsto \xi_x$ is a smooth vector field. To show smoothness fix $x \in M$ and a chart (U, u) with $x \in U$. As in the proof of Theorem 2.29 we may extend u^i for $i = 1, ..., m$ to smooth functions $\tilde{u}^i : M \to \mathbb{R}$ that coincide with u^i on some open neighbourhood $V \subset U$ of x. Then $D(\tilde{u}^i) : M \to \mathbb{R}$ is smooth and

$$\xi_y = \sum_i \underbrace{(\xi_y \cdot \tilde{u}^i)(y)}_{=\xi^i(y)} \frac{\partial}{\partial u^i}(y) \quad \forall y \in V \quad \text{(cf. Theorem 2.29).}$$

Hence, $\xi|_V = \sum_i D(\tilde{u}^i)|_V \frac{\partial}{\partial u^i}$ is a smooth vector field on V.

□

Recall that for a chart (U, u) the function

$$\frac{\partial}{\partial u^i} \cdot f = \frac{\partial f}{\partial u^i}$$

equals the i-th partial derivative of the local coordinate expression $f \circ u^{-1}$ of f. This implies that for $\xi \in \mathfrak{X}(M)$ with $\xi|_U = \sum_i \xi^i \frac{\partial}{\partial u^i}$ one has

$$\xi \cdot f = \sum_i \xi^i \frac{\partial f}{\partial u^i}.$$

LEMMA 2.35. *Suppose $\xi, \eta \in \mathfrak{X}(M)$ are vector fields on a manifold M. Then*

$$f \mapsto \xi \cdot (\eta \cdot f) - \eta \cdot (\xi \cdot f)$$

defines a derivation of $C^\infty(M, \mathbb{R})$.

PROOF.

$$\begin{aligned} \xi \cdot (\eta \cdot (fg)) &= \xi \cdot ((\eta \cdot f)g + f(\eta \cdot g)) \\ &= (\xi \cdot (\eta \cdot f))g + \underbrace{(\eta \cdot f)(\xi \cdot g) + (\xi \cdot f)(\eta \cdot g)}_{\text{symmetric in } \xi \text{ and } \eta} + f(\xi \cdot (\eta \cdot g)). \end{aligned}$$

□

DEFINITION 2.36. Suppose M is manifold. Then the **Lie bracket** of two vector fields $\xi, \eta \in \mathfrak{X}(M)$ is the unique vector field $[\xi, \eta] \in \mathfrak{X}(M)$ such that

$$[\xi, \eta] \cdot f = \xi \cdot (\eta \cdot f) - \eta \cdot (\xi \cdot f) \quad \forall f \in C^\infty(M, \mathbb{R}).$$

PROPOSITION 2.37. *Suppose M is manifold and $\xi, \eta, \zeta \in \mathfrak{X}(M)$ vector fields.*

(a) $[\xi, \eta] = -[\eta, \xi]$ *and*

$$[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0 \quad (\textit{Jacobi Identity}).$$

(b) *For all $f \in C^\infty(M, \mathbb{R})$ one has*

$$\begin{aligned} [\xi, f\eta] &= f[\xi, \eta] + (\xi \cdot f)\eta \\ [f\xi, \eta] &= f[\xi, \eta] - (\eta \cdot f)\xi. \end{aligned}$$

PROOF.

(a) Skew-symmetry is clear and the Jacobi identity follows from a mindless computation.
(b) Let $f, g \in C^\infty(M, \mathbb{R})$. Then for all $x \in M$ one has

$$((f\eta) \cdot g)(x) = f(x)(\eta_x \cdot g) = f(\eta \cdot g)(x),$$

which implies that

$$\xi \cdot ((f\eta) \cdot g) = (\xi \cdot f)(\eta \cdot g) = f(\xi \cdot (\eta \cdot g)).$$

Moreover, $(f\eta) \cdot (\xi \cdot g) = f(\eta \cdot (\xi \cdot g))$, which together gives

$$[\xi, f\eta] = f[\xi, \eta] + (\xi \cdot f)\eta.$$

The second identity follows from the first by skew-symmetry of the Lie bracket .

□

REMARK 2.38. The properties in [(a)] of Proposition 2.37 say that $(\mathfrak{X}(M), [\_,\_])$ is an infinite-dimensional Lie algebra, i.e. an infinite-dimensional vector space $\mathfrak{X}(M)$ equipped with a skew-symmetric bilinear bracket

$$[\_,\_] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

satisfying the Jacobi identity.

PROPOSITION 2.39. *Suppose M and N are manifolds and $f : M \to N$ is a local diffeomorphism.*

(a) $f^*[\xi, \eta] = [f^*\xi, f^*\eta]$ *for all* $\xi, \eta \in \mathfrak{X}(N)$, *(i.e.* $f^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$ *is a Lie algebra homomorphism).*

(b) *If $U \subset M$ is open and $i : U \to M$ the natural inclusion, then*

$$[\xi, \eta]|_U = i^*[\xi, \eta] = [i^*\xi, i^*\eta] = [\xi|_U, \eta|_U] \quad \forall \xi, \eta \in \mathfrak{X}(M).$$

In particular, if $\xi|_U = 0$, *then* $[\xi, \eta]|_U =$ *for all other vector fields* $\eta \in \mathfrak{X}(M)$.

(c) *Suppose (U, u) is a chart and $\xi, \eta \in \mathfrak{X}(M)$ vector fields with local coordinate expressions* $\xi|_U = \sum_i \xi^i \frac{\partial}{\partial u^i}$ *and* $\eta|_U = \sum_i \eta^i \frac{\partial}{\partial u^i}$. *Then*

$$[\xi, \eta]|_U = \sum_{i=1}^{m} [\xi, \eta]^i \frac{\partial}{\partial u^i},$$

where $[\xi, \eta]^i = \sum_{j=1}^m (\xi^j \frac{\partial \eta^i}{\partial u^j} - \eta^j \frac{\partial \xi^i}{\partial u^j})$.

PROOF.

(a) By definition $f^*\xi = (Tf)^{-1} \circ \xi \circ f$ for $\xi \in \mathfrak{X}(N)$. Now for $g \in C^\infty(N, \mathbb{R})$ one has

$$(f^*\xi) \cdot (g \circ f)(x) = (f^*\xi)_x \cdot (g \circ f) = (T_x f f^*\xi) \cdot g = \xi_{f(x)} \cdot g,$$

that is, $(f^*\xi) \circ (g \circ f) = (\xi \cdot g) \circ f$. Therefore, we have

$$\begin{aligned}[f^*\xi, f^*\eta] \cdot (g \circ f) &= f^*\xi \cdot (\underbrace{f^*\eta \cdot (g \circ f)}_{(\eta \cdot g) \circ f}) - f^*\eta \cdot (f^*\xi \cdot (g \circ f)) \\ &= \xi \cdot (\eta \cdot g) \circ f - \eta \cdot (\xi \cdot g) \circ f \\ &= ([\xi, \eta] \cdot g) \circ f = f^*[\xi, \eta] \cdot (g \circ f).\end{aligned}$$

(b) Follows directly from (a).

(c) By (b), we have

$$\begin{aligned}[\xi, \eta]|_U = [\xi|_U, \eta|_U] &= \sum_{i,j} [\xi^i \frac{\partial}{\partial u^i}, \eta^j \frac{\partial}{\partial u^j}] \\ &= \sum_{i,j} \eta^j [\xi^i \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] + \xi^i \frac{\partial \eta^j}{\partial u^i} \frac{\partial}{\partial u^j} = \sum_{i,j} \xi^i \frac{\partial \eta^j}{\partial u^i} \frac{\partial}{\partial u^j} - \eta^j \frac{\partial \xi^i}{\partial u^j} \frac{\partial}{\partial u^i},\end{aligned}$$

since $[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] = 0$. Indeed, by the symmetry of 2nd partial derivatives,

$$\begin{aligned}
[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] \cdot f &= \frac{\partial}{\partial u^i} \cdot (\frac{\partial}{\partial u^j} \cdot f) - \frac{\partial}{\partial u^j} \cdot (\frac{\partial}{\partial u^i} \cdot f) \\
&= \underbrace{\frac{\partial^2 f}{\partial u^i \partial u^j}}_{\text{2nd-partial derivative of } f \circ u^{-1}} - \frac{\partial^2 f}{\partial u^j \partial u^i} = 0.
\end{aligned}$$

□

The following proposition gives a geometric interpretation of the Lie bracket showing that it measures the behaviour of one vector field along the flow lines of the other.

Proposition 2.40. *Suppose $\xi, \eta \in \mathfrak{X}(M)$ are vector fields.*

(a) $\frac{d}{dt}|_{t=0}(\mathrm{Fl}^\xi_t)^*\eta)(x) = [\xi, \eta](x)$ *for all* $x \in M$.
(b) $[\xi, \eta] = 0 \iff (\mathrm{Fl}^\xi_t)^*\eta = \eta$, *whenever defined* $\iff \mathrm{Fl}^\xi_t \circ \mathrm{Fl}^\eta_s = \mathrm{Fl}^\eta_s \circ \mathrm{Fl}^\xi_t$, *whenever defined.*

Proof. See Tutorial. □

Definition 2.41. Suppose $f : M \to N$ is a smooth map between manifolds. Then $\xi \in \mathfrak{X}(M)$ and $\eta \in \mathfrak{X}(N)$ are called **f-related**, if

$$Tf_x\,\xi(x) = \eta(f(x)) \quad \forall x \in M.$$

Proposition 2.42. *Suppose $f : M \to N$ is a smooth map between manifolds. If two vector fields ξ and η on M are f-related to vector fields $\tilde{\xi}$ respectively $\tilde{\eta}$ on N, then $[\xi, \eta]$ is f-related to $[\tilde{\xi}, \tilde{\eta}]$.*

Proof. See Tutorial. □

2.7. Distributions and the Frobenius Theorem

Let us revisit the concept of a flow of a vector field $\xi \in \mathfrak{X}(M)$ on a manifold M. For any $x \in M$ there exists an integral curve $c : I \to M$, $0 \in I$, through $c(0) = x$. (i.e. $c(t) = \mathrm{Fl}^\xi_x(x)$).

- If $\xi(x) = 0$, then $c(t) = x$ is the the constant curve.
- If $\xi(x) \neq 0$, then $\xi(y) \neq 0$ for all y in some open neighbourhood $U \subset M$ of x. In this case, the integral curve through x defines a submanifold of U of dimension 1. Hence, ξ decomposes U into a union of 1-dimensional submanifolds, given by the images of the integral curves of ξ through the points $y \in U$.The tangent space of such a submanifold through y equals

$$\mathbb{R}\xi(y) \subset T_yM.$$

  Moreover, if we replace ξ by $f\xi$ for a nowhere vanishing function $f \in C^\infty(M, \mathbb{R})$, then the integral curves of $f\xi$ and ξ are just reparametrisations of each other. Hence, they define the same family of 1-dimensional submanifolds.

Suppose $E : x \mapsto E_x \subset T_xM$ is a map that assigns to each point $x \in M$ a 1-dimensional subspace $E_x \subset T_xM$ such that there exists an open cover $\{U_i\}$ of M and local vector fields $\xi_i \in \mathfrak{X}(U_i)$ such that $\mathbb{R}\xi_i(y) = E_y$ for all $y \in U_i$ and for all i. Then the existence of integral curves implies that for any $x \in M$ there exists a unique local smooth submanifold $K_x \subset M$ such that

$$T_yK_x = E_y \subset T_yM \quad \forall y \in K_x.$$

DEFINITION 2.43. Suppose M is a (smooth) manifold of dimension m.

(a) A **distribution E of rank k** is an assignment of a k-dimensional subspace $E_x \subset T_xM$ to every point $x \in M$.
(b) A (smooth) **section of a distribution** $E \subset TM$ is a (smooth) vector field $\xi \in \mathfrak{X}(M)$ such that $\xi(x) \in E_x$ for all $x \in M$.
(c) A distribution $E \subset TM$ of rank k is called **smooth**, if for every $x \in M$ there exists an open neighbourhood U of x and local sections $\xi_1, ..., \xi_k \in \mathfrak{X}(U)$ of E such that $\{\xi_1(y), ..., \xi_k(y)\}$ form a basis of E_y for all $y \in U$. A smooth distribution is also called a **vector subbundle** $E \subset TM$ of the vector bundle TM and such collection of local sections of E is called a **local frame**.
(d) A distribution $E \subset TM$ is called **involutive**, if for any local sections ξ and η of E their Lie bracket $[\xi, \eta]$ is also a local section of E.
(e) A distribution $E \subset TM$ is **integrable**, if for each $x \in M$ there exists a submanifold $K \subset M$ with $x \in K$ such that for any $y \in K$

$$T_yK = E_y \subset T_yM.$$

Such submanifolds are called **integral submanifolds**.

The existence of flows implies:

PROPOSITION 2.44. *Any smooth distribution of rank* 1 *on manifold is integrable.*

Distributions of higher rank are not anymore always integrable. A necessary condition for integrability is involutivity:

Let $E \subset TM$ be an integrable distribution and $K \subset M$ an integral submanifold, i.e. $T_xK = E_x$ for all $x \in K$. Assume ξ and η are local sections of E defined on some open neighbourhood U of $x \in K$ in M. Replacing, K by $K \cap U$, we may assume $K \subset U$. Then there exist vector fields $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(K)$ that are i-related to $\xi|_U$ respectively $\eta|_U$, where $i : K \hookrightarrow U \subset M$ is the natural inclusion. Hence, by Proposition 2.42, $[\tilde{\xi}, \tilde{\eta}] \in \mathfrak{X}(K)$ is i-related to $[\xi|_U, \eta|_U]$. This shows that

$$[\xi, \eta](y) \in \mathrm{Im}(T_yi) = E_y \quad \forall y \in K.$$

The Frobenius Theorem states that also the converse is true, i.e. any involutive smooth distribution is integrable.

Note that involutivity is easy to check:

LEMMA 2.45. *Suppose $E \subset TM$ is a smooth distribution on a manifold M. Then E is involutive $\iff$ locally around each point $x \in M$ there exists a local frame $\{\xi_1, ..., \xi_k\}$ such that $[\xi_i, \xi_j]$ is a local section of E for all i, j.*

PROOF. Follows from (b) of Proposition 2.37. □

Recall that the coordinate vector fields $\frac{\partial}{\partial u^1},\ldots,\frac{\partial}{\partial u^m}$ corresponding to a chart (U,u) of M define a local frame of TM and $[\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}]=0$ for all $i.j$. Note that for $k\le m$, the coordinate vector fields $\frac{\partial}{\partial u^1},\ldots,\frac{\partial}{\partial u^k}$ span an integrable distribution on U. Its integral submanifolds are given by

$$u^{-1}(y,a) \quad \text{for fixed } a\in u(U)\cap\mathbb{R}^{n-k},$$

where $u(U)\subset\mathbb{R}^m=\mathbb{R}^k\times\mathbb{R}^{m-k}$.

LEMMA 2.46. *Suppose M is a manifold of dimension m and $V\subset M$ an open subset. If $\xi_1,\ldots,\xi_k\in\mathfrak{X}(V)$ are local vector fields on V such that $\xi_1(y),\ldots,\xi_k(y)\in T_yV=T_yM$ are linearly independent for all $y\in V$, then the following are equivalent:*

(a) $[\xi_i,\xi_j]=0$.
(b) *For any $y\in V$ there exists a chart (U,u) with $y\in U\subset V$ such that*

$$\frac{\partial}{\partial u^1}=\xi_1|_U,\ldots,\frac{\partial}{\partial u^k}=\xi_k|_U.$$

PROOF.

$(b)\implies(a)$: This is clear.

$(a)\implies(b)$: Fix $y\in V$ and let $(\tilde U,\tilde u)$ be a chart with $y\in\tilde U\subset V$, $\tilde u(y)=0$ and $\xi_i(y)=\frac{\partial}{\partial\tilde u^i}(y)$ for $i=1,\ldots,k$. There exists open neighbourhoods $W\subset\mathbb{R}^k$ and $\tilde W\subset\mathbb{R}^{m-k}$ of zero such that

$$\phi(t^1,\ldots,t^k,t^{k+1},\ldots,t^m)=(\mathrm{Fl}^{\xi_1}_{t^1}\circ\ldots\circ\mathrm{Fl}^{\xi_k}_{t^k})(\tilde u^{-1}(0,\ldots,0,t^{k+1},\ldots,t^m))$$

is defined for all $(t^1,\ldots,t^k)\in W$ and all $(t^{k+1},\ldots,t^m)\in\tilde W$. It is a smooth map $\phi:W\times\tilde W\to M$ such that $\phi(0,0)=y$.

For $i\le k$ we have,

$$\begin{aligned}\frac{\partial\phi}{\partial t^i}(t)&=\frac{d}{ds}|_{s=0}\phi(t^1,..,t^i+s,\ldots,t^m)\\&=\mathrm{Fl}^{\xi_i}_s(\phi(t))=\xi_i(\phi(t)),\end{aligned}\tag{2.12}$$

since $\mathrm{Fl}^{\xi_i}_{t^i+s}=\mathrm{Fl}^{\xi_i}_s\circ\mathrm{Fl}^{\xi_i}_{t^i}$ and $\mathrm{Fl}^{\xi_i}_s$ commutes with all $\mathrm{Fl}^{\xi_j}_{t^j}$. In particular,

$$\frac{\partial\phi}{\partial t^i}(0)=\xi_i(y)=\frac{\partial}{\partial\tilde u^i}(y)\quad\text{for } i\le k.$$

For $i>k$ we have

$$\begin{aligned}\frac{\partial\phi}{\partial t^i}(0)&=\frac{d}{dt}|_{t=0}\phi(te^i)=\frac{d}{dt}|_{t=0}\phi(0,\ldots,t,\ldots,0)\\&=\frac{d}{dt}|_{t=0}\tilde u^{-1}(0,\ldots,0,\ldots,t,\ldots,0)\\&=T_{(0,0)}\tilde u^{-1}e^i=\frac{\partial}{\partial\tilde u^i}(y).\end{aligned}$$

This implies that $T_{(0,0)}\phi$ is invertible as it maps the basis $\{\frac{\partial}{\partial t^i}(0)\}$ to the basis $\{\frac{\partial}{\partial\tilde u^i}(y)\}$. Hence, by possibly shrinking W and $\tilde W$ we can assume that $\phi:W\times\tilde W\to U$ is a diffeomorphism, where U

is an open neighbourhood of y in M. Then (2.12) implies that $u := \phi^{-1} : U \to W \times \tilde{W} \subset \mathbb{R}^k \times \mathbb{R}^{m-k} = \mathbb{R}^m$ is the required chart.

□

COROLLARY 2.47. *If $\xi \in \mathfrak{X}(M)$ is a vector field, then for any $x \in M$ with $\xi(x) \neq 0$, there exists a chart (U, u) with $x \in U$ such that $\xi_1|_U = \frac{\partial}{\partial u^1}$.*

THEOREM 2.48 (Frobenius Theorem, local version). *Let M be a manifold of dimension m and $E \subset TM$ a smooth involutive distribution of rank $k \leq m$. Then for each $x \in M$ there exists a chart (U, u) with $x \in U$ such that*

- *$u(U) = W \times \tilde{W} \subset \mathbb{R}^k \times \mathbb{R}^{m-k} = \mathbb{R}^m$, where $W \subset \mathbb{R}^k$ and $\tilde{W} \subset \mathbb{R}^{m-k}$ are some open subsets; and*
- *for each $a \in \tilde{W}$ the subset $u^{-1}(W \times \{a\}) \subset M$ is an integral manifold for E.*

In particular, any involutive smooth distribution is integrable.

PROOF. We will show that around every point in M there exists a local frame for E that consists of pairwise commuting vector fields. Then the result follows from Lemma 2.46.

Fix $x \in M$ and a local frame $\{\xi_1, ..., \xi_k\}$ for E defined on some open neighbourhood $\tilde{U} \subset M$ of x. Without loss of generality we may assume $\tilde{U}$ is the domain of a chart $(\tilde{U}, \tilde{u})$ of M with $\tilde{u}(x) = 0$. Then for $j = 1, ..., k$ we have

$$\xi_j = \sum_{i=1}^{m} f_j^i \frac{\partial}{\partial \tilde{u}^i} \quad f_j^i \in C^\infty(\tilde{U}, \mathbb{R}).$$

Since $\{\xi_j(y)\}_{j=1}^k$ is basis of E_y for all $y \in \tilde{U}$, the $m \times k$ matrix $(f_j^i(y))_{j=1,...,k}^{i=1,...,m}$ has rank k for all $y \in \tilde{U}$. Renumbering the coordinates, we may assume that at x the first k rows of $(f_j^i(x))$ are linearly independent. By continuity, this holds locally around x, and so by possibly shrinking $\tilde{U}$, we may assume that it holds on $\tilde{U}$.

For $y \in \tilde{U}$ let $(g_j^i(y))$ be the inverse of $(f_j^i(y))_{j=1,...,k}^{i=1,...,k}$. Since inversion in $\mathrm{GL}(k, \mathbb{R})$ is smooth, the functions $g_j^i : \tilde{U} \to \mathbb{R}$ are smooth for all i, j. Now for $i = 1, ..., k$,

$$\eta_i := \sum_{j=1}^{k} g_i^j \xi_j$$

are local smooth sections of E defined on $\tilde{U}$. Since $(g_j^i(y))$ is invertible for all $y \in \tilde{U}$ and $\{\xi_1, ..., \xi_k\}$ is a local frame, also $\{\eta_1, ..., \eta_k\}$ is a local frame for E defined on $\tilde{U}$.

Claim: $[\eta_i, \eta_j] = 0$ for all i, j.
Indeed, note that

$$\eta_i = \sum_{j=1}^{k} g_i^j \xi_j = \sum_{1 \leq \ell \leq m, 1 \leq j \leq k} g_i^j f_j^\ell \frac{\partial}{\partial \tilde{u}^\ell} = \frac{\partial}{\partial \tilde{u}^i} + \sum_{\ell > k} h_i^\ell \frac{\partial}{\partial \tilde{u}^\ell}, \tag{2.13}$$

for some smooth functions h_i^ℓ. By involutivity,

$$[\eta_i, \eta_j] = \sum_{r=1}^{k} c_{ij}^r \eta_r \quad \text{for } c_{i,j}^r \in C^\infty(\tilde{U}, \mathbb{R}). \tag{2.14}$$

By (2.13), the right-hand side, RHS, of (2.14) equals

$$\text{RHS} = \sum_{r=1}^{k} c_{i,j}^r \Big(\frac{\partial}{\partial \tilde{u}^r} + \sum_{\ell > k} h_r^\ell \frac{\partial}{\partial \tilde{u}^\ell}\Big) = \sum_{r=1}^{k} c_{i,j}^r \frac{\partial}{\partial \tilde{u}^r} + \sum_{\ell > k} \tilde{h}_{i,j}^\ell \frac{\partial}{\partial \tilde{u}^\ell}$$

for some $\tilde{h}_{i,j}^\ell \in C^\infty(\tilde{U}, \mathbb{R})$. Now, by (2.13), the left-hand side, LHS, of (2.14) equals

$$\text{LHS} = \Big[\frac{\partial}{\partial \tilde{u}^i} + \sum_{\ell > k} h_i^\ell \frac{\partial}{\partial \tilde{u}^\ell}, \frac{\partial}{\partial \tilde{u}^j} + \sum_{\ell > k} h_j^\ell \frac{\partial}{\partial \tilde{u}^\ell}\Big] = \sum_{\ell > k} h_{i,j}^\ell \frac{\partial}{\partial \tilde{u}^\ell}$$

for some $h_{i,j}^\ell \in C^\infty(\tilde{U}, \mathbb{R})$. Hence, $h_{i,j}^\ell = \tilde{h}_{i,j}^\ell$ and $\sum_{r=1}^k c_{i,j}^r \frac{\partial}{\partial \tilde{u}^r} = 0$ on $\tilde{U}$. The latter in turn implies that $c_{i,j}^r = 0$ on $\tilde{U}$. By the proof of Lemma 2.46 there exists a chart

$$u : U \to u(U) = W \times \tilde{W} \subset \mathbb{R}^k \times \mathbb{R}^{m-k} = \mathbb{R}^m$$

such that

- $x \in U$, $u(x) = (0, 0)$ and $U \subset \tilde{U}$, and
- $\eta_i|_U = \frac{\partial}{\partial u^i}$.

Hence, for any $a \in \tilde{W}$, $u^{-1}(W \times \{a\})$ is an integral submanifold for E (described by the equations $u^{k+1} = a^{k+1}$, ..., $u^m = a^m$). ☐

Note that Theorem 2.48 says, that given an involutive smooth distribution $E \subset TM$ on a manifold M of dimension m, locally around point in M there exists a chart (U, u) such that U is filled up by integral submanifolds and in the corresponding coordinates they are given by affine horizontal subspaces $\mathbb{R}^k \times \{a\}$ of $\mathbb{R}^m$. → make a picture.

The charts in Theorem 2.48 are called **distinguished charts** for (M, E) and the integral submanifolds $u^{-1}(W \times \{a\})$ are called **plaques**. Note that, if (U_α, u_α) and (U_β, u_β) are two distinguished charts for (M, E) with $U_\alpha \cap U_\beta \neq \emptyset$, then the transition maps are of the form

$$\begin{aligned} u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) &\to u_\beta(U_\alpha \cap U_\beta) \\ (x, y) &\mapsto (f(x, y), g(y)) \end{aligned} \tag{2.15}$$

for smooth maps f, g. Hence, the transition maps map subsets $W_\alpha \times \{a\}$ to subsets $W_\beta \times \{b\}$.

Definition 2.49.

(a) A **foliated atlas of dimension** k on a manifold $(M, \mathcal{A})$ of dimension m is a subatlas $\mathcal{A}' \subset \mathcal{A}$ consisting of charts $(U, u) \in \mathcal{A}$ such that
   - $u(U) = W \times \tilde{W} \subset \mathbb{R}^k \times \mathbb{R}^{m-k} = \mathbb{R}^m$ for open subsets $W \subset \mathbb{R}^k$ and $\tilde{W} \subset \mathbb{R}^{m-k}$;
   - the transition maps are of the form (2.15).

(b) A **k-dimensional foliation** $\mathcal{F}$ on a manifold M is a maximal foliated atlas of dimension k on M.

The Frobenius Theorem shows that any involutive smooth distribution $E \subset TM$ of rank k defines a k-dimensional foliation $\mathcal{F}^E$ on M. Conversely, any such foliation $\mathcal{F}$ determines a smooth involutive distribution $E \subset TM$ given by

$$E_x = T_{u(x)} u^{-1}(T_w \mathbb{R}^k \times \{0\})$$

for a chart (U, u) of the foliation with $x \in U$ and $u(x) = w + \tilde{w} \in W \times \tilde{W}$ (by (2.15) E_x is well defined, i.e. independent of choice of chart around x).

Given a smooth involutive distribution $E \subset TM$, we know by the Frobenius Theorem that through any point $x \in M$ we have an integral submanifold.

Question: What about maximal integral submanifolds a given point?

These are in general not real submanifolds but so-called initial submanifolds as the following example shows: consider the 2-dimensional torus T^2 ($= \mathbb{R}^2/\mathbb{Z}^2$) and denote by $\pi : \mathbb{R}^2 \to T^2$ the natural projection given by $\pi(x, y) = (e^{ix}, e^{iy})$. Now consider the vector field $\xi = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y}$ on $\mathbb{R}^2$, whose integral curves are lines

$$\text{const} + t \begin{pmatrix} 1 \\ \alpha \end{pmatrix}.$$

The vector field ξ is π-related to a vector field on T^2, whose integral curves are the images under π of the integral curves of ξ. If α is rational, then

$$\pi(t(1, \alpha)) = (e^{it}, e^{it\alpha}) \subset T^2$$

is a submanifold. But, Ii α is irrational, then it is not, since it winds densely around the torus. In an appropriate chart around any point, $(e^{it}, e^{it\alpha})$ consists of countable many line segments. One can separate these segments however, since it is not possible to move from one to another by a smooth curve. This suggest the following definition:

Definition 2.50. Suppose M is a manifold of dimension m.

(a) For a subset $A \subset M$ and $x_0 \in A$ set

$$C_{x_0}(A) = \{x \in A : \exists C^\infty\text{−curve } c : [0, 1] \to M \text{ s.t. } c([0, 1]) \subset A, c(0) = x_0 \text{ and } c(1) = x\}.$$

(b) $K \subset M$ is called an **initial submanifold** of M of dimension k, if for any $x \in K$ there exists a chart (U, u) of M with $x \in U$, $u(x) = 0$ and

$$u(C_x(U \cap K)) = u(U) \cap (\mathbb{R}^k \times \{0\}) \subset \mathbb{R}^k \times \mathbb{R}^{m-k} = \mathbb{R}^m.$$

If $K \subset M$ is an initial submanifold, the there exists a unique C^∞-manifold structure on K such that properties (P) hold, that are:

- the inclusion $i : K \hookrightarrow M$ is an injective immersion,
- for any manifold N and a map $f : N \to K$, f is smooth $\iff$ $i \circ f : N \to M$ is smooth.

Note that the connected components are 2nd countable, but there might be uncountably many of them (so K might be not 2nd countable)!

The smooth structure on K is given by the atlas $\mathbb{B} = \{(C_x(U \cap K), u_x)_{x \in K}\}$ consisting of charts as in (b) of Definition 2.50.

- Equip K with final topology with respect to the inclusions $C_x(U \cap K) \hookrightarrow K$. This topology on K is usually finer than the subspace topology on K induced from M (hence, in particular still Hausdorff). (The set $C_x(U \cap K)$ is in general not open in subspace topology. If i is a homeomorphism onto its image, then it however is and $C_x(U \cap K) = V \cap K$ for an open subset $V \subset M$. In this case, $K \subset M$ is an actual submanifold.)
- The transition maps of elements in $\mathcal{B}$ are smooth as they are restrictions of smooth maps.
- Uniquenes follows from (P) (cf. submanifolds).

Conversely, one may show that images of injective immersions satisfying properties (P) are initial submanifolds.

Let us come to integrable distributions. Suppose $E \subset TM$ is an integrable distribution of rank k with corresponding foliation $\mathcal{F}^E$. For any $x \in M$ let $\mathcal{F}^E_x$ denote the set of points $y \in M$ such that there exists a smooth curve $c : [0,1] \to M$ connecting $c(0) = x$ and $c(1) = y$ satisfying $c'(t) \in E_{c(t)}$ for all $t \in [0,1]$. Then $\mathcal{F}^E_x$ is called the **leaf of the foliation** $\mathcal{F}^E$ through x.

Note that if a plaque intersects $\mathcal{F}^E_x$, then it must be contained in it. Hence, the plaques contained in $\mathcal{F}^E_x$ and the corresponding charts of the foliation can be used to give $\mathcal{F}^E_x$ the structure of a k-dimensional manifold (plaques in $\mathcal{F}^E_x$ form basis of the toplogy). Then one may show:

- $i : \mathcal{F}^E_x \hookrightarrow M$ is an initial submanifold (Hausdorff and second countable).
- $\mathcal{F}^E_x$ is an integral submanifold and any connected integral (initial) submanifold that intersects $\mathcal{F}^E_x$ is contained in $\mathcal{F}^E_x$. Hence, the leaves of $\mathcal{F}^E$ may be thought of as the maximal integral (initial) submanifolds of E.

Hence, a foliation $\mathcal{F}^E$ of dimension k divides M into k-dimensional initial submanifolds.
Given a manifold M equipped with a smooth involutive distribution E and corresponding foliation $\mathcal{F}^E$. Then one can equip M with a different manifold structure (and topology) M_E, where the atlas (inducing also the toplogy) is given by maps of the form

$$\mathrm{pr}_1 \circ u_\alpha : u_\alpha^{-1}(W_\alpha \times \{a\}) \to W_\alpha \subset \mathbb{R}^k$$

for charts (U_α, u_α) in $\mathcal{F}^E$. The topology on M_E is then finer than that of M and the connected components of M_E are the leaves of $\mathcal{F}^E$. Note that M_E is Hausdorff, but not second countable, since it has uncountably many connected components.

In summary, one has:

THEOREM 2.51 (Frobenius Theorem, Global Version). *Suppose M is a manifold, $E \subset TM$ a smooth involutive distribution of rank k and $\mathcal{F}^E$ the corresponding foliation. Then the following holds:*

(a) *If $E \neq TM$, the topology on M_E is finer than that on M.*
(b) *M_E has uncountably many connected components given by the leaves of $\mathcal{F}^E$.*
(c) *$Id: M_E \to M$ is a bijective smooth immersion.*
(d) *Each leaf of $\mathcal{F}^E$ is an initial submanifold of M and a maximal connected (initial) integral submanifold of E.*

2.8. Applications of the Frobenius Theorem and bracket-generating distributions

Let us discuss some applications of Theorem 2.48 to the study of PDEs:

EXAMPLE 2.11. Let us write $(x, y, z) \in \mathbb{R}^3$ for the standard coordinates in $\mathbb{R}^3$ and consider the following system of PDEs for a function $f : \mathbb{R}^3 \times \mathbb{R}$:

$$\begin{aligned} 2z^2\frac{\partial f}{\partial x} + 2x\frac{\partial f}{\partial z} &= 0 \\ 3z^3\frac{\partial f}{\partial y} + 2y\frac{\partial f}{\partial z} &= 0. \end{aligned} \tag{2.16}$$

It is a linear system of PDEs of first oder and it is overdetermined.

Question: Does (2.16) has any non-constant solution?

Consider the vector fields

$$\begin{aligned} X &= 2z^2\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z} \\ Y &= 3z^3\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}. \end{aligned}$$

Note that they span a distribution E of rank 2 on the open subset $V = \{(x, y, z \in \mathbb{R}^3) : z \neq 0\} \subset \mathbb{R}^3$. Moreover,

$$[X, Y] = -12xz\frac{\partial}{\partial y} + 8yz\frac{\partial}{\partial x} = \frac{4x}{z}Y - \frac{4y}{z}X,$$

which shows that E is an involutive, hence integrable distribution, on V. By Theorem 2.48 there exists locally around any $(x_0, y_0, z_0) \in V$ a chart (U, u) such that E is spanned by $\frac{\partial}{\partial u^1}$ and $\frac{\partial}{\partial u^2}$. Then (2.16) in coordinates $(u^1(x, y, z), u^2(x, y, z)), u^3(x, y, z))$ is equivalent to

$$\frac{\partial f}{\partial u^1} = \frac{\partial f}{\partial u^2} = 0.$$

Hence, $f = u^3$ is a solution and solutions in a sufficiently small neighbourhood of (x_0, y_0, z_0) are of the form $f(x, y, z) = g(u^3(x, y, z))$ for a smooth function g in one variable.

EXAMPLE 2.12. Consider the following system of PDEs for a function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \alpha(x,y,f(x,y)) \\ \frac{\partial f}{\partial y}(x,y) &= \beta(x,y,f(x,y)), \end{aligned} \tag{2.17}$$

where α, β are smooth functions defined on some open subset $V \subset \mathbb{R}^3$. This is again an overdetermined system of PDEs of possibly non-linear first oder equations.

Question: When does (2.17) have a solution?

Since $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, a necessary condition is

$$\frac{\partial}{\partial y}\alpha(x,y,f(x,y)) = \frac{\partial}{\partial x}\beta(x,y,f(x,y)),$$

which by the chain rule means that

$$\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z}. \tag{2.18}$$

must hold at any point $(x,y,z) \in V$ where there is a solution of (2.17) with $f(x,y) = z$.

Thus, (2.18) is a necessary condition for (2.17) to have a solution in a neighbourhood of any point (x_0, y_0) with arbitrary initial value $f(x_0, y_0) = z_0$. By the Frobenius Theorem, (2.18) is also sufficient: it implies that for any $(x_0, y_0, z_0) \in V$ there exists an open neighbourhood U of $(x_0, y_0) \in \mathbb{R}^2$ and a unique solution $f : U \to \mathbb{R}$ of (2.17) such that $f(x_0, y_0) = z_0$.

Why? Note that (2.17) describes the tangent plane to the graph of f in terms of coordinates of the graph. The collection of tangent planes defines a rank 2 distribution on V and (2.18) is equivalent to the involutivity of that distribution. Indeed, suppose $f : U \to \mathbb{R}$ were a solution (on some open subset $U \subset \mathbb{R}^2$) of (2.17). Then,

$$\begin{aligned} \psi &: U \to \mathbb{R}^3 \\ \psi(x,y) &= (x,y,f(x,y)) \end{aligned}$$

is a diffeomorphism onto $\mathrm{gr}(f)$ (ψ is a parametrisation of the submanifold $\mathrm{gr}(f) \subset \mathbb{R}^3$). Then, $T_{\psi(x,y)}\mathrm{gr}(f)$ is spanned by

$$T_{(x,y)}\psi\left(\frac{\partial}{\partial x}(x,y)\right) = \frac{\partial}{\partial x}(x,y,f(x,y)) + \underbrace{\frac{\partial f}{\partial x}(x,y)}_{=\alpha(x,y,f(x,y))} \frac{\partial}{\partial z}(x,y,f(x,y))$$

$$T_{(x,y)}\psi\left(\frac{\partial}{\partial y}(x,y)\right) = \frac{\partial}{\partial y}(x,y,f(x,y)) + \underbrace{\frac{\partial f}{\partial y}(x,y)}_{=\beta(x,y,f(x,y))} \frac{\partial}{\partial z}(x,y,f(x,y)).$$

Note that the vector fields

$$X = \frac{\partial}{\partial x} + \alpha(x, y, f(x,y))\frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y} + \beta(x, y, f(x,y))\frac{\partial}{\partial z}$$

span a rank 2 distribution E on V. It is straightforward to check that E is involutive (hence integrable) $\iff$ (2.18) holds. Moreover, in this case, f is a solution of (2.17) $\iff$ $\mathrm{gr}(f)$ is an integral submanifold of E.

In summary, if (2.18) holds, then through any point $(x_0, y_0, z_0) \in V$ there exists an integral submanifold $K \subset V \subset \mathbb{R}^3$ of E through (x_0, y_0, z_0), which locally (as a submanifold) must have the form $\mathrm{gr}(f)$ for a smooth function $f : U \to \mathbb{R}$, where U is an open neighbourhood of $(x_0, y_0) \in \mathbb{R}^2$ with $f(x_0, y_0) = z_0$.

On the opposite ending of integrable distributions within the world of distributions one has so-called bracket generating distributions, which are maximally non-integrable:

Definition 2.52. A (smooth) distribution $E \subset TM$ on a manifold M is called **bracket-generating**, if any local frame $\{\xi_1, ..., \xi_k\}$ of E together with its iterated Lie brackets, $[\xi_i, \xi_j]$, $[\xi_\ell, [\xi_i, \xi_j]]$ ... and so on, forms a local frame of TM.

Note that, if this is true for some local frame of E around a point in M, then this is true for any other local frame of E around that point.

Example 2.13. (Contact manifolds)
Consider $\mathbb{R}^3$ with coordinates $(x, y, z) \in \mathbb{R}^3$ and let $E \subset T\mathbb{R}^3$ be the rank 2 distribution spanned by the vector fields

$$X := \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \text{and } Y := \frac{\partial}{\partial y}.$$

Then $[X, Y] = \frac{\partial}{\partial z}$ is not a section of E and so E is not integrable. Moreover, X, Y and $[X, Y]$ span $T\mathbb{R}^3$. Hence, E is bracket-generating and called the **standard contact distribution** on $\mathbb{R}^3$.

More generally, a **contact distribution** on a manifold M of dimension $2m + 1$ is a rank $2m$ distribution $E \subset TM$ such that

$$\mathcal{L}_x : E_x \times E_x \to T_xM/E_x \cong \mathbb{R}$$
$$\xi_x \times \eta_x \mapsto q_x([\xi, \eta](x))$$

is non-degenerate for all $x \in M$, where ξ and η are extensions of ξ_x respectively η_x to local vector fields around x. It is easy to check that $\mathcal{L}_x$ is well-defined, i.e. independent of the choice of local extensions ξ and η.

Example 2.14. (Driving a car)
The configuration/phase space (or phase space) of a car consists of all points

$$(x, y, \alpha, \beta) \in \mathbb{R}^2 \times S^1 \times (-\pi/4, \pi/4) =: M,$$

where (x, y) is the position of the midpoint of the rear axle, α the angle of the chassis to the x-axis, and β the steering angle of the front wheels.
$\rightarrow$ make a picture.
Driving a car traverses a curve

$$c(t) = (x(t)), y(t), \alpha(t), \beta(t))$$

in M. There are non-holonomic constraints, that are constraints on position and velocity that can not be integrated to constraints on position only. We have:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \text{ is parallel to } \begin{pmatrix} \cos\alpha(t) \\ \sin\alpha(t) \end{pmatrix}$$

and

$$\frac{d}{dt}\begin{pmatrix} x(t) + \ell\cos(\alpha(t)) \\ y(t) + \ell\sin(\alpha(t)) \end{pmatrix} \text{ is parallel to } \begin{pmatrix} \cos(\alpha(t) - \beta(t)) \\ \sin(\alpha(t) - \beta(t)) \end{pmatrix},$$

where ℓ is the length from the midpoint (x, y) of the rear axis to the midpoint of the front axis (connecting the two front wheels). This means we have the following to linear equations for $(x'(t), y'(t), \alpha'(t), \beta'(t))$:

$$x'(t)\sin(\alpha(t)) - y'(t)\cos(\alpha(t)) = 0$$
$$(x'(t) - \ell\sin(\alpha(t))\alpha'(t))\sin(\alpha(t) - \beta(t)) - (y'(t) + \ell\cos(\alpha(t))\alpha'(t))\cos(\alpha(t) - \beta(t)) = 0.$$

Any solution is of the form

$$\begin{pmatrix} x'(t) \\ y'(t) \\ \alpha'(t) \\ \beta'(t) \end{pmatrix} = \lambda(t)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \mu(t)\begin{pmatrix} \ell\cos(\alpha(t))\cos(\beta(t)) \\ \ell\sin(\alpha(t))\cos(\beta(t)) \\ -\sin(\beta(t)) \\ 0 \end{pmatrix}.$$

Let us set

$$X := \frac{\partial}{\partial\beta} \quad \text{(steer vector field)}$$
$$Y := \ell\cos\beta\cos\alpha\frac{\partial}{\partial x} + \ell\cos\beta\sin\alpha\frac{\partial}{\partial y} - \sin\beta\frac{\partial}{\partial\alpha} \quad \text{(drive vector field).}$$

Note that the two control vector fields X and Y span a bracket-generating distribution on M, which describes the space of possible velocities. Indeed, check that $X, Y, [X, Y]$ and $[Y, [X, Y]]$ span TM. What does this mean?

CHAPTER 3

The Cotangent Bundle

Constructions in the category of vector spaces can be generalised to the category of vector bundles. In particular, for any vector bundle we can form its dual and wedge products of it. In this case of the tangent bundle this leads to the cotangent bundle and wedge products of it, whose sections are called 1-forms respectively k-forms.

3.1. 1-forms

Suppose M is a manifold of dimension m and $p : E \to M$ a vector bundle of rank k. Given two local trivialisations of E, $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ and $\phi_\beta : p^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^k$, the transition map is of the form:

$$\begin{aligned} \phi_\beta \circ \phi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^k &\to U_\alpha \cap U_\beta \times \mathbb{R}^k \\ (y, v) &\mapsto (y, \phi_{\beta\alpha}(y)v), \end{aligned}$$

for a unique smooth map $\phi_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$.

Remark 3.1.

- The local trivialisations (U_α, ϕ_α) (also called vector bundle charts) of E form a so-called vector bundle atlas of E.
- Any vector bundle over M may be also defined as smooth map $E \to M$ that is equipped with a maximal vector bundle atlas.
- The family of maps $\phi_{\beta\alpha}$ (associated to a vector bundle atlas) satisfy

$$\begin{aligned} \phi_{\alpha\alpha}(y) &= y \\ \phi_{\alpha\beta}(y)\phi_{\alpha\gamma}(y) &= \phi_{\alpha\gamma}(y) \quad \text{(cocycle condition)}. \end{aligned}$$

  The Čech cohomology class of the cocycle of transition functions determines a vector bundle up to isomorphism.

For any $x \in M$ consider the dual vector space $E_x^* = \{\lambda : E_x \to \mathbb{R} : \lambda \text{ is linear}\}$ of E_x. Set

$$E^* := \bigsqcup_{x \in M} E_x^*,$$

and denote by $q : E^* \to M$ the natural projection.

Lemma 3.2. *For any vector bundle $p : E \to M$ of rank k, $q : E^* \to M$ is also naturally a vector bundle of rank k, which is called the* ***dual vector bundle of*** *$E \to M$.*

Proof. By construction, $q : E^* \to M$ is a surjection such that $q^{-1}(x) = E_x^*$ is a k-dimensional vector space for any $x \in M$. Fix $x \in M$ and let

(U_α, u_α) be a chart for M with $x \in U_\alpha$. By possibly shrinking U_α, we can assume that $E \to M$ trivialises over U_α, i.e. there exists a local trivialisation

$$\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

defined over U_α. (Note that $u_\alpha \times id \circ \phi_\alpha : p^{-1}(U_\alpha) \to u_\alpha(U_\alpha) \times \mathbb{R}^k \subset \mathbb{R}^m \times \mathbb{R}^k$ is a chart for the manifold E). Now define a bijection

$$\phi^*_\alpha : q^{-1}(U_\alpha) \to U_\alpha \times (\mathbb{R}^k)^*, \tag{3.1}$$

where $\phi^*_\alpha|_{E_y} := ((\phi_\alpha|_{E_y})^{-1})^* : E^*_y \to \{y\} \times (\mathbb{R}^k)^*$. Then $\mathrm{pr}_1 \circ \phi^*_\alpha = q|_{q^{-1}(U_\alpha)}$. Moreover,

$$(u_\alpha \times \mu) \circ \phi^*_\alpha : q^{-1}(U_\alpha) \to u_\alpha(U_\alpha) \times \mathbb{R}^k \subset \mathbb{R}^m \times \mathbb{R}^k,$$

defines a bijection for any choice of isomorphism $\mu : (\mathbb{R}^k)^* \cong \mathbb{R}^k$. Let $(u_\beta \times \mu) \circ \phi^*_\beta : q^{-1}(U_\beta) \to u_\beta(U_\beta) \times \mathbb{R}^k$ another such bijection for a chart (U_β, u_β) and local trivialisation (U_β, u_β) with $U_\alpha \cap U_\beta \neq \emptyset$. Then

$$\begin{aligned} ((u_\beta \times \mu) \circ \phi^*_\beta) \circ ((u_\alpha \times \mu) \circ \phi^*_\alpha)^{-1} :& u_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^k \to u_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^k \\ & (y, v) \mapsto (u_{\beta\alpha}(y), (\mu \circ (\phi_{\beta\alpha}(u_\alpha^{-1}(y))^*)^{-1} \circ \mu^{-1})(v)) \end{aligned}$$

is smooth, since $u_{\beta\alpha}$, $\phi_{\beta\alpha}$, and inversion and dualising in $\mathrm{GL}(k, \mathbb{R})$ are smooth. Hence, we can use the bijections of the form (3.1) to equip E^* with the structure of a smooth manifold of dimension $m+k$ whose maximal atlas is induced by these bijections (cf. the construction of the smooth structure on TM for a abstract manifold M in Section 2.31). By construction, it follows that $q : E^* \to M$ is a smooth vector bundle for this smooth structure on E^*. □

Definition 3.3.

(a) For any manifold M the dual vector bundle $q : T^*M \to M$ of the tangent bundle $p : TM \to M$ is called the **cotangent bundle** of M.]We write $T^*_xM := q^{-1}(x)$ for its fiber over $x \in M$.
(b) A (smooth) section of $q : T^*M \to M$ is called a (smooth) **1-form** on M. We write $\Gamma(T^*M)$ or $\Omega^1(M)$ for the set of 1-forms, which is a real vector space and a modul over $C^\infty(M, \mathbb{R})$ by Lemma 2.14.

Suppose (U, u) is a chart for M. Then the map

$$\phi^* := u^{-1} \times \mathrm{Id} \circ T^*u : T^*U = q^{-1}(U) \xrightarrow{T^*u} u(U) \times (\mathbb{R}^m)^* \xrightarrow{u^{-1}\times \mathrm{Id}} U \times (\mathbb{R}^m)^*,$$

where

$$T^*_yu := T^*u|_{T^*_yU} := ((T_yu)^{-1})^* = ((T_yu)^*)^{-1},$$

is a local trivialisation of $T^*M \to M$.

Definition 3.4. Let $\{\lambda_1, ..., \lambda_m\}$ be the basis of $(\mathbb{R}^m)^*$ dual to the standard basis of $\{e^1, ..., e^m\}$ of $\mathbb{R}^m$, i.e. $\lambda_i(e^j) = \delta_{ij}$. Then we write du^i for the section of $T^*U \to U$ defined by

$$du^i(y) = (\phi^*)^{-1}(y, \lambda^i) = (T^*u)^{-1}(u(y), \lambda_i) \quad \forall y \in U.$$

Evidently, $du^1(y)$,..., $du^m(y)$ form a basis of $T^*_yU = T^*_yM$ for any $y \in U$, which is dual to the basis $\frac{\partial}{\partial u^1}(y)$,...,$\frac{\partial}{\partial u^m}(y)$ of $T_yU = T_yM$.

For smooth functions $\omega_i : U \to \mathbb{R}$, $i = 1, ..., m$, the sum

$$\sum_{i=1}^{m} \omega_i du^i,$$

is a local 1-form defined on U. In particular, locally there are many 1-forms on a manifold and from the existence of partitions of unity it follows that this is also true globally. Conversely, any $\omega \in \Omega^1(M)$ may be restricted to U, where it can be written as

$$\omega|_U = \sum_{i=1}^{m} \omega_i du^i, \tag{3.2}$$

for unique smooth functions $\omega_i \in C^\infty(U, \mathbb{R})$. Note that smoothness of ω is equivalent to ω_i being smooth for all i.

Definition 3.5. Suppose $\omega \in \Omega^1(M) = \Gamma(T^*M)$ is a 1-form on a manifold M and let (U, u) be a chart for M. Then (3.2) or $(\omega_1, ..., \omega_m)$ is called the **local coordinate expression** of ω with respect to (U, u).

Note that we have a bilinear map:

$$\begin{aligned}\Gamma(T^*M) \times \Gamma(TM) &\to C^\infty(M, \mathbb{R})\\ (\omega, \xi) &\mapsto (\omega(\xi) : x \mapsto \omega_x(\xi_x)).\end{aligned}$$

By construction, $du^i(\frac{\partial}{\partial u^j})(x) = \delta_{ij}$ for all $x \in U$ and $\omega|_U(\frac{\partial}{\partial u^i}) = \omega^i$ and $du^i(\xi|_U) = \xi^i$.

Remark 3.6. Note that for a not necessarily smooth section ω of T^*M, i.e. a map $\omega : M \to T^*M$ such that $q \circ \omega = \mathrm{Id}_M$, the following are equivalent:

- ω is smooth;
- ω has smooth local coordinate expressions for any chart of M;
- $\omega(\xi)$ is smooth for any local smooth vector vector field ξ.

Let us now compute how the local coordinate expression of a 1-form $\omega \in \Omega^1(M)$ changes when we change the chart: suppose (U_α, u_α) and (U_β, u_β) are two charts of M. Recall that

$$\frac{\partial}{\partial u^i_\alpha} = \sum_{i=1}^{m} A^j_i \frac{\partial}{\partial u^j_\beta},$$

where $A^j_i(x) = \frac{\partial u^j_{\beta\alpha}}{\partial x^i}(x)$. Hence,

$$\omega|_{U\alpha \cap U_\beta} = \sum_i \omega^\alpha_i du^i_\alpha = \sum_i \omega^\beta_i du^i_\beta,$$

where

$$\omega^\alpha_i = \omega(\frac{\partial}{\partial u^i_\alpha}) = \sum_j A^j_i \omega(\frac{\partial}{\partial u^j_\beta}) = \sum_j A^j_i \omega^\beta_j,$$

resp.

$$\omega^\beta_i = \sum_j B^j_i \omega^\alpha_j,$$

where B^i_j is the inverse to A^i_j.

DEFINITION 3.7. For $f \in C^\infty(M, \mathbb{R})$ we may define a 1-form $df \in \Omega^1(M)$ by

$$df(x)(\xi_x) = T_x f \xi_x \quad \text{for } x \in M, \xi_x \in T_x M.$$

Indeed, $df : M \to T^*M$ is smooth, since $df(\xi) = \xi \cdot f$ for any $\xi \in \mathfrak{X}(M)$, and $df(x) \in T_x^*M$ for all $x \in M$.

The operator $d : C^\infty(M, \mathbb{R}) \to \Omega^1(M)$ is the easiest special case of the so-called exterior derivative on differential forms as we shall see. In local coordinates (U, u) we have:

$$df|_U = \sum_i df(\frac{\partial}{\partial u^i})du^i = \sum_i \frac{\partial f}{\partial u^i} du^i.$$

Note that for $f = u^i$, we have $du^i = \sum_j du^i(\frac{\partial}{\partial u^j})du^j = du^i$, which explains our notation for the local 1-forms du^i.

3.2. Review: Multi-linear algebra

Suppose V_1,..., V_r are (real) finite-dimensional vector spaces. For a vector space W we write

$$L(V_1, ..., V_r; W)$$

for the vector space of r-linear maps $V_1 \times ... \times V_r \to W$.

DEFINITION 3.8.

(a) The **tensor product** of V_1,..., V_r is the vector space

$$V_1 \otimes ... \otimes V_r := L(V_1^*, ..., V_r^*; \mathbb{R}).$$

(b) For $(v_1, ..., v_r) \in V_1 \times ... \times V_r$ we write $v_1 \otimes ... \otimes v_r \in V_1 \otimes ... \otimes V_r$ for the map

$$v_1 \otimes ... \otimes v_r : (\lambda_1, ..., \lambda_r) \mapsto \Pi_{i=1}^r \lambda_i(v_i).$$

Note that the map

$$\begin{aligned} \otimes : V_1 \times ... \times V_r &\to V_1 \otimes ... \otimes V_r \\ (v_1, ..., v_r) &\mapsto v_1 \otimes ... \otimes v_r, \end{aligned}$$

is r-linear, i.e. $\otimes \in L(V_1, ..., V_r; V_1 \otimes ... \otimes V_r)$.

Properties of the tensor product:

- **Universal property**: For any r-linear map $f : V_1 \times ... \times V_r \to W$ to a vector space W there exists a unique linear map

$$\tilde{f} : V_1 \otimes ... \otimes V_r \to W$$

  such that $f = \tilde{f} \circ \otimes$. In particular, $f \mapsto \tilde{f}$ defines an isomorphism $L(V_1, ..., V_r; W) \cong L(V_1 \otimes ... \otimes V_r, W)$.
- There are natural isomorphisms:

$$\begin{aligned} (V_1 \otimes V_2) \otimes V_3 &\cong V_1 \otimes V_2 \otimes V_3 \\ (V_1 \oplus V_2) \otimes V_3 &\cong V_1 \otimes V_2 \oplus V_2 \otimes V_3 \end{aligned}$$

- **Basis**: If $\{e_{i,j}\}_{1\leq j\leq n_i}$ is a basis of V_i for $i = 1, ..., r$, then
$$\{e_{1,j_i} \otimes ... \otimes e_{r,j_r}\}_{1\leq j_i\leq n_i, 1\leq i\leq r}$$
is a basis for $V_1 \otimes ... \otimes V_r$. In particular, $\dim(V_1 \otimes ... \otimes V_r) = \Pi_{i=1}^r \dim(V_i) = n_1 \cdot ... \cdot n_r$.
- There exists canonical isomorphisms:
$$\begin{aligned} V_1^* \otimes V_2^* &\cong (V_1 \otimes V_2)^* \\ \lambda_1 \otimes \lambda_2 &\mapsto (v_1 \otimes v_2 \mapsto \lambda_1(v_1)\lambda_2(v_2)) \end{aligned}$$
$$\begin{aligned} V_1^* \otimes V_2 &\cong L(V_1, V_2) \\ \lambda_1 \otimes v_2 &\mapsto (v_1 \mapsto \lambda_1(v_1)v_2). \end{aligned}$$
- If $f_i : V_i \to W_i$ are linear maps for $i = 1, ..., r$, then by the universal property there exists a unique linear map
$$f_1 \otimes ... \otimes f_r : V_1 \otimes ... \otimes V_r \to W_1 \otimes ... \otimes W_r,$$
such that $f_1 \otimes ... \otimes f_r \circ \otimes = \otimes \circ f_1 \times ... \times f_r : V_1 \times ... \times V_r \to W_1 \otimes ... \otimes W_r$.

Definition 3.9. Suppose V is a real finite-dimensional vector space and write $L^r(V, \mathbb{R}) := L(\underbrace{V, ..., V}_{r}; \mathbb{R}) = V^* \otimes ... \otimes V^*$.

(a) A r-linear map $\omega \in L^r(V, \mathbb{R})$ is called **alternating**, if
$$\omega(v_{\sigma(1)}, ..., v_{\sigma(r)}) = \mathrm{sign}(\sigma)\omega(v_1, ..., v_r)$$
for all $v_1, ..., v_r \in V$ and for all $\sigma \in S_r := \{\text{bijections of } \{1, ..., r\}\}$. Note that ω is alternating $\iff$ ω vanishes if one inserts an element twice. We write
$$\bigwedge^r V^* := L^r_{\mathrm{alt}}(V, \mathbb{R}) \subset L^r(V, \mathbb{R})$$
for the subspace of r-linear alternating maps.

(b) There is natural projection $\mathrm{Alt} : L^r(V, \mathbb{R}) \to L^r_{\mathrm{alt}}(V, \mathbb{R})$, called **alternator**, given by
$$\mathrm{Alt}(\omega)(v_1, ..., v_r) = \frac{1}{r!} \sum_{\sigma\in S_r} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, ..., v_{\sigma(r)}).$$
Note that, if $\omega \in L^r_{\mathrm{alt}}(V, \mathbb{R})$, then $\mathrm{Alt}(\omega) = \omega$.

It follows that, if $r > \dim(V)$, then $\bigwedge^r V^* = 0$, since any r-linear map is uniquely determined by its values on elements of a basis of V and vanishes if an element is inserted twice. Moreover, if $r = \dim(V)$, then $\bigwedge^r V^*$ is 1-dimensional: fix a basis $\mathcal{B} = \{e^1, ..., e^r\}$ of V, then $\det = \det_{\mathcal{B}} : V \times ... \times V \to \mathbb{R}$ (the determinant of r vectors with respect to the basis $\mathcal{B}$) is an element of $\bigwedge^r V^*$. If $\omega \in \bigwedge^r V^*$, then
$$\omega(v_1, ..., v_r) = \det(v_1, ..., v_r)\omega(e^1, ..., e^r),$$
by multilinearity and being alternating.

Definition 3.10. For a (finite-dimensional) vector space V we set

$$\bigwedge^* V^* := \bigoplus_{r\geq 0} \bigwedge^r V^*$$

with the convention that $\bigwedge^0 V^* := \mathbb{R}$ and $\bigwedge^1 V^* := V^*$. Then $\bigwedge^* V^*$ is a finite-dimensional vector space.

Note that any linear map $f : V \to W$ between (finite-dimensional) vector spaces V and W induces a linear map $f^* : \bigwedge^r W^* \to \bigwedge^r V^*$ given by

$$f^*\omega(v_1, ..., v_r) = \omega(f(v_1), ..., f(v_r)),$$

which extends naturally to a linear map $f^* : \bigwedge^* W^* \to \bigwedge^* V^*$. One has

$$(g \circ f)^* = f^* \circ g^* \tag{3.3}$$

for any other linear map $g : W \to Z$.

Definition 3.11. For $\omega \in \bigwedge^r V^*$ and $\eta \in \bigwedge^s V^*$ their **wedge product** $\omega \wedge \eta \in \bigwedge^{r+s} V^*$ is given by

$$\begin{aligned}\omega \wedge \eta(v_1, ..., v_{r+s}) &:= \frac{(r+s)!}{r!s!}\mathrm{Alt}(\omega \otimes \eta)(v_1, ..., v_{r+s}) \\ &= \frac{1}{r!s!}\sum_{\sigma\in S_{r+s}} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, ..., v_{\sigma(r)})\eta(v_{\sigma(r+1)}, ..., v_{\sigma(r+s)}).\end{aligned}$$

By bilinearity, we can extend $\wedge$ to $\bigwedge^* V^*$:

$$\sum_r \omega_r \wedge \sum_r \eta_r := \sum_{r,s} \omega_r \wedge \eta_s \quad \omega_r, \eta_r \in \bigwedge^r V^*.$$

Proposition 3.12. *The vector space $\bigwedge^* V^* := \bigoplus_{r\geq 0} \bigwedge^r V^*$ is an associative (unitial) graded-anticummutative algebra, i.e.*

(a) $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$ *for all* $\omega, \eta, \zeta \in \bigwedge^* V^*$ *(associativity);*
(b) $1 \in \mathbb{R} = \bigwedge^0 V^*$ *satisfies* $1\wedge\omega = \omega\wedge 1 = \omega$ *for all* $\omega \in \bigwedge^* V^*$ *(unitial);*
(c) $\bigwedge^r V^* \wedge \bigwedge^s V^* \subset \bigwedge^{r+s} V^*$ *(graded algebra);*
(d) $\omega \wedge \eta = (-1)^{rs}\eta \wedge \omega$ *for* $\omega \in \bigwedge^r V^*$ *and* $\eta \in \bigwedge^s V^*$.

Moreover, for any linear map between vector spaces $f : V \to W$, the linear map $f^* : \bigwedge^* W \to \bigwedge^* V^*$ is a graded algebra morphism (of degree 0), i.e.

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta, f^*1 = 1 \text{ and } f^*\bigwedge^r W^* \subset \bigwedge^r V^*.$$

Proof. See algebra class or tutorial. □

Proposition 3.13. *Suppose V is an n-dimensional vector space.*

(a) *If $\omega_1, ..., \omega_r \in V^*$ and $v_1, ..., v_r \in V$, then*

$$\omega_1 \wedge ... \wedge \omega_r(v_1, ..., v_r) = \det(\omega_i(v_j)_{1\leq i,j\leq r}).$$

In particular, $\omega_1, ..., \omega_r$ are linearly independent if and only if $\omega_1 \wedge ... \wedge \omega_r \neq 0$.

(b) *If $\{\lambda_1, ..., \lambda_n\}$ is a basis of V^*, then*

$$\{\lambda_{i_1} \wedge ... \wedge \lambda_{i_r} : 1 \leq i_1 < ... < i_r \leq n\}$$

is a basis of $\bigwedge^r V^*$.

Proof. See algebra class or tutorial. □

3.3. Tensors

Suppose M is a manifold. For a fixed point $x \in M$ consider

$$\underbrace{T_xM \otimes ... \otimes T_xM}_{p-\text{times}} \otimes \underbrace{T_x^*M \otimes ... \otimes T_x^*M}_{q-\text{times}} = L(T_x^*M, ..., T_x^*M, T_xM, ..., T_xM; \mathbb{R}).$$

and denote by $TM^{\otimes p} \otimes T^*M^{\otimes q}$ the disjoint union over all $x \in M$ of these vector spaces. It is easy to see that the natural projection

$$\pi : TM^{\otimes p} \otimes T^*M^{\otimes q} \to M$$

admits the structure of a (smooth) vector bundle, which is induced from the vector bundle structures on TM and T^*M (see tutorial).

Definition 3.14.

- A (smooth) $\binom{p}{q}$-tensor is a (smooth) section of π.
- We write $\mathcal{T}_q^p(M)$ for the vector space of (smooth) $\binom{p}{q}$-tensors on M, which is also a modul over the ring $C^\infty(M, \mathbb{R})$.
- If $\phi \in \mathcal{T}_q^p(M)$ and $\psi \in \mathcal{T}_s^r(M)$, then $\phi \otimes \psi$, defined by
$$(\phi \otimes \psi)(x) = \phi_x \otimes \psi_x \quad \forall x \in M,$$
is a $\binom{p+r}{q+s}$-tensor on M. (Note that $x \mapsto (\phi_x, \psi_x) \mapsto \phi_x \otimes \psi_x$ is smooth as composition of smooth maps).

Suppose (U, u) is a chart for M, then the local tensors of the form

$$\frac{\partial}{\partial u^{i_1}} \otimes ... \otimes \frac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes ... \otimes du^{j_q} \in \mathcal{T}_q^p(U)$$

form a basis of $\pi^{-1}(x)$ at any point $x \in U$. Hence, any tensor $\phi \in \mathcal{T}_q^p(M)$ can be written on U as:

$$\phi|_U = \sum_{i_1,...,i_p;j_1,...,j_q} \phi_{j_1,...,j_q}^{i_1,...,i_p} \frac{\partial}{\partial u^{i_1}} \otimes ... \otimes \frac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes ... \otimes du^{j_q} \tag{3.4}$$

for real-valued functions $\phi_{j_1,...,j_q}^{i_1,...,i_p}$ functions on U, which are called the **local coordinate expressions** of ϕ with respect to (U, u). Smoothness of ϕ is equivalent to the smoothness of the local coordinate expressions for any chart.

Any $\phi \in \mathcal{T}_q^p(M)$ gives rise to a map, which we will also denote by ϕ, given by

$$\begin{aligned} \phi :& \Gamma(T^*M) \times ... \times \Gamma(T^*M) \times \Gamma(TM) \times ... \times \Gamma(TM) \to C^\infty(M, \mathbb{R}) \qquad (3.5) \\ &(\omega^1, ..., \omega^p, \xi_1, ..., \xi_q) \mapsto (x \mapsto \phi_x(\omega^1(x), ..., \omega^p(x), \xi_1(x), ..., \xi_q(x))) \end{aligned}$$

By construction, (3.5) is $C^\infty(M, \mathbb{R})$-linear in each entry and, by (3.4),

$$\phi(\omega^1, ..., \omega^p, \xi_1, ..., \xi_q)$$

is indeed an element of $C^\infty(M, \mathbb{R})$, since on the domain U of any chart (U, u) it is given by

$$\sum_{i_1,...,i_p;j_1,...,j_q} \phi_{j_1,...,j_q}^{i_1,...,i_p} \omega_{i_1}^1 ... \omega_{i_p}^p \xi_1^{j_1} ... \xi_q^{j_q},$$

which is smooth as a sum of products of smooth functions.

REMARK 3.15. Note that

$$\phi^{i_1,\dots,i_p}_{j_1,\dots,j_q} = \phi|_U(du^{i_1},..,du^{i_p},\frac{\partial}{\partial u^{j_1}},...,\frac{\partial}{\partial u^{j_q}}).$$

Moreover, smoothness of a tensor ϕ is equivalent to $\phi(\omega^1,...,\omega^p,\xi_1,...,\xi_q)$ being smooth for any smooth local 1-forms $\omega^1,...,\omega^p$ and vector fields $\xi_1,...,\xi_q$. Let us consider two special cases:

- $\phi \in \mathcal{T}^0_1(M) = \Gamma(T^*M)$ is a 1-form and we know already that $\phi(\xi) : M \to \mathbb{R}$ is smooth for $\xi \in \Gamma(TM)$.
- $\phi \in \mathcal{T}^1_0(M) = \Gamma(TM)$ is a vector field and $\phi(\omega) = \omega(\phi) : M \to \mathbb{R}$ is smooth for $\omega \in \Gamma(T^*M)$.

REMARK 3.16. Elements of $\mathcal{T}^p_0(M)$ (respectively, $\mathcal{T}^0_q(M)$) are called p-times contra-variant (respectively, q-times covariant) tensor, which refers to the way they change under coordinate transformations.

The following proposition characterises tensors:

PROPOSITION 3.17. *Associating to a tensor $\phi \in \mathcal{T}^p_q(M)$ the map* (3.5) *defines a linear isomorphism between $\mathcal{T}^p_q(M)$ and the vector space*

$$\mathcal{W}^p_q(M) := L_{C^\infty(M,\mathbb{R})}(\Gamma(T^*M),...,\Gamma(T^*M),\Gamma(TM),...,\Gamma(TM);C^\infty(M,\mathbb{R}))$$

of $C^\infty(M,\mathbb{R})$-multilinear maps.

PROOF. We already know that $\phi \in \mathcal{T}^p_q(M)$ gives rise via (3.5) to a map in $\mathcal{W}^p_q(M)$ and evidently that association is linear and injective. Conversely, let

$$\phi : \Gamma(T^*M) \times ... \times \Gamma(T^*M) \times \Gamma(TM) \times ... \times \Gamma(TM) \to C^\infty(M,\mathbb{R})$$

be $C^\infty(M,\mathbb{R})$-multilinear, i.e. an element in $\mathcal{W}^p_q(M)$. Then we have to show that

$$\phi(x)(\omega^1(x),...,\xi_q(x)) := \phi(\omega^1,...,\xi_q)(x)$$

for 1-forms ω^i and vector fields ξ_j just depends on the value of these 1-forms and vector fields at x. If this is the case, $x \mapsto \phi_x$ is an element of $\mathcal{T}^p_q(M)$. It is sufficient to show that if a 1-form or vector field σ vanishes at x, so does $\phi(\omega^1,...,\sigma,...,\xi_q)(x)$.

Suppose first σ vanishes identically on a open neighbourhood $U \subset M$ of $x \in M$ and let $f \in C^\infty(M,\mathbb{R})$ be such that $f|_{M\setminus U} = 1$ and $f(x) = 0$ (which exists by Corollary 1.35). Then $\sigma = f\sigma$ and by $C^\infty(M,\mathbb{R})$-linearity, we have

$$\phi(\omega^1,...,\sigma,...,\xi_q)(x) = \phi(\omega^1,...,f\sigma,...,\xi_q)(x) = f(x)\phi(\omega^1,...,\sigma,...,\xi_q)(x) = 0.$$

This shows that for a chart (U,u) with $x \in U$, $\phi(\omega^1,...,\xi_q)|_U$ (and in particular its value at x) just depend on the restrictions of the the 1-forms ω^i and vector fields ξ_j to U. Hence, we have

$$\phi(\omega^1,...,\xi_q)|_U = \sum_{i_1,...,i_p;j_1,...,j_q} \phi^{i_1,...,i_p}_{j_1,...,j_q}\omega^1_{i_1}...\omega^p_{i_p}\xi^{j_1}_1...\xi^{j_q}_q. \qquad (3.6)$$

Therefore, if $\sigma(x) = 0$, then its local coordinate expressions vanish at x, and $\phi(\omega^1,...,\sigma,...,\xi_q)(x) = 0$ by (3.6). □

EXAMPLE 3.1. A tensor $g \in \mathcal{T}_2^0(M)$ on a manifold M of dimension m is a called a **(pseudo-)Riemannian metric** on M, if for any $x \in M$ the bilinear form

$$g(x) : T_xM \times T_xM \to \mathbb{R}$$

is symmetric and non-degenerate. If M is connected, the signature (p, q) of $g(x)$ does not depend on x and is referred to as the signature of g $(p+q = m)$. In particular, if g is positive definite, i.e. the signature is $(m, 0)$, then g is called a **Riemannian metric**, and if g has signature $(m-1, 1)$ or $(1, m-1)$, then it is called a **Lorentzian metric**.

If $M = \mathbb{R}^m$, the standard inner product gives rise to the Euclidean (or standard) metric on $\mathbb{R}^m$ given by

$$g = dx^1 \otimes dx^1 + ... + dx^m \otimes dx^m.$$

Similarly, the standard Lorentzian inner product on $\mathbb{R}^m$ gives rise to the standard Lorentzian (or Minkowski) metric given by

$$g = -dx^1 \otimes dx^1 + ... + dx^m \otimes dx^m.$$

3.4. Differential forms

Suppose M is a manifold of dimension m.

DEFINITION 3.18.

(a) A **(differential)** k**-form** on M is a $\binom{0}{k}$-tensor $\omega \in \mathcal{T}_k^0(M)$ such that $\omega(x) \in \wedge^k T_x^*M$ for all $x \in M$.

(b) We write $\Omega^k(M) \subset \mathcal{T}_k^0(M)$ for the subspace of k-forms on M, which is also a mudul over $C^\infty(M, \mathbb{R})$. We use the convention that $\Omega^0(M) = \mathcal{T}_0^0(M) = C^\infty(M, \mathbb{R})$.

Note that for $k > m$ one has $\Omega^k(M) = \{0\}$.

REMARK 3.19.

- $\wedge^k T^*M := \sqcup_{x\in M} \wedge^k T_x^*M \subset \underbrace{T^*M \otimes ... \otimes T^*M}_{k-\text{times}}$ is a subbundle.
- $\Omega^k(M) = \Gamma(\wedge^k T^*M) =$ space of sections of $\wedge^k T^*M$.

By Proposition 3.17, we can consider a k-form $\omega \in \Omega^k(M)$ also as a k-linear, alternating map

$$\omega : \Gamma(TM) \times ... \times \Gamma(TM) \to C^\infty(M, \mathbb{R}),$$

that is linear over $C^\infty(M, \mathbb{R})$ in each entry.

DEFINITION 3.20. Suppose $f : M \to N$ is a smooth map between manifolds. If $\omega \in \Omega^k(N)$, then $f^*\omega$, called **the pull-back of** ω via f, is a k-form on M given by:

$$f^*\omega(x)(\xi_1, ..., \xi_k) = \omega(f(x))(T_xf\xi_1, ..., T_xf\xi_k) \quad \xi_i \in T_xM.$$

If $\xi_1, .., \xi_k \in \Gamma(TM)$, then

$$f^*\omega(\xi_1, ..., \xi_k) = (\omega \circ f)(Tf \circ \xi_1, ..., Tf \circ \xi_k),$$

which shows that $f^*\omega$ is indeed a smooth tensor field on M.

REMARK 3.21. More generally, one can pull-back $\begin{pmatrix}0\\k\end{pmatrix}$-tensor via smooth maps.

We have a natural map

$$\begin{aligned}\mathrm{Alt}: \mathcal{T}^0_k(M) &\to \Omega^k(M)\\ \mathrm{Alt}(\phi)(x) &:= \mathrm{Alt}(\phi_x),\end{aligned}$$

where we have $\mathrm{Alt}(\omega) = \omega$ for $\omega \in \Omega^k(M)$.

DEFINITION 3.22. For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ the **wedge product** $\omega \wedge \eta \in \Omega^{k+\ell}(M)$ of ω and η is given by

$$(\omega \wedge \eta)(x) = \omega_x \wedge \eta_x = \frac{(\ell+k)!}{k!\ell!}\mathrm{Alt}(\omega_x \otimes \eta_x).$$

For $f \in \Omega^0(M) = C^\infty(M,\mathbb{R})$ and $\omega \in \Omega^k(M)$: $f \wedge \omega = f\omega$.

Extending $\wedge$ by bilinearity to a map

$$\wedge : \Omega^*(M) \times \Omega^*(M) \to \Omega^*(M),$$

where $\Omega^*(M) := \bigoplus_{k\geq 0}\Omega^k(M)$, Proposition 3.12 implies:

PROPOSITION 3.23. *The vector space $\Omega^*(M)$ is an associative (unitial) graded-anticommutative algebra over $C^\infty(M,\mathbb{R})$ (in particular, over $\mathbb{R}$), i.e. satisfies (a)-(d) of Proposition 3.12, since it does so pointwise.*

Moreover, we have:

PROPOSITION 3.24. *Let $f : M \to N$ be a smooth map between manifolds. Then $f^* : \Omega^k(N) \to \Omega^k(M)$ extends to a morphism*

$$f^* : \Omega^*(N) \to \Omega^*(M)$$

of (unitial) graded algebras, i.e. f^* is $\mathbb{R}$-linear, $f^*1 = 1$, $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$ and $f^*\Omega^k(N) \subset \Omega^k(M)$. Moreover, if $g : N \to P$ is another smooth map between manifolds, then

$$(g \circ f)^* = f^* \circ g^*.$$

PROOF. Since $(f^*\omega)(x) = (T_xf)^*\omega(f(x))$ for all $x \in M$, the first claim follows from Proposition 3.12 and the second claim from (3.3) and $T(g \circ f) = Tg \circ Tf$. □

If (U, u) is a chart for M, then $\{du^{i_1} \wedge ... \wedge du^{i_k} : 1 \leq i_1 < ... < i_k \leq m\}$ form a basis of $\wedge^k T_xM$ for any $x \in U$. For $\omega \in \Omega^k(M)$ we therefore have

$$\omega|_U = \sum_{1\leq i_1<...<i_k\leq m} \omega_{i_1...i_k} du^{i_1} \wedge ... \wedge du^{i_k},$$

where $\omega_{i_1...i_k} = \omega(\frac{\partial}{\partial u^{i_1}}, ..., \frac{\partial}{\partial u^{i_k}}) \in C^\infty(U,\mathbb{R})$.

Recall that we have an operator

$$\begin{aligned}d : \Omega^0(M) = C^\infty(M,\mathbb{R}) &\to \Omega^1(M)\\ f &\mapsto df\end{aligned}$$

We can extend this operator to an operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for any $k \geq 0$ in a natural way.

DEFINITION 3.25. Suppose $\omega \in \Omega^k(M)$. Then we define

$$d\omega : \underbrace{\Gamma(TM) \times ... \times \Gamma(TM)}_{k+1} \to C^\infty(M, \mathbb{R})$$

by

$$\begin{aligned} d\omega(\xi_0, \xi_1, ..., \xi_k) = & \sum_{i=0}^{k} (-1)^i \xi_i \cdot \omega(\xi_0, ..., \hat{\xi}_i, ..., \xi_k) \\ & + \sum_{i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_k), \end{aligned}$$

where $\hat{\xi}_i$ means that this element is omitted.

LEMMA 3.26. *For $\omega \in \Omega^k(M)$ one has $d\omega \in \Omega^{k+1}(M)$. Moreover, by linearity, we can extend d to a map*

$$d : \Omega^*(M) \to \Omega^{*+1}(M),$$

called ***the exterior derivative*** *on differential forms.*

PROOF.

- $d\omega$ is alternating: Suppose $\xi_j = \xi_{j+1}$. Then the fact that ω is alternating and $[\xi, \xi] = 0$ implies that

$$\begin{aligned} d\omega(\xi_0, .., \xi_j, \xi_{j+1}, .., \xi_k) = & (-1)^j \xi_j \cdot \omega(\xi_0, ..., \hat{\xi}_j, \xi_{j+1} ..., \xi_k) \\ & + (-1)^{j+1} \xi_{j+1} \cdot \omega(\xi_0, ..., \xi_j, \hat{\xi}_{j+1} ..., \xi_k) \\ & + \sum_{i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, \xi_{j+1}, ..., \xi_k) \\ & + \sum_{i<j} (-1)^{i+j+1} \omega([\xi_i, \xi_j], \xi_0, ..., \hat{\xi}_i, ...\xi_j, \hat{\xi}_{j+1}, ..., \xi_k) \\ & + \sum_{j+1<i} (-1)^{j+i} \omega([\xi_j, \xi_i], \xi_0, ..., \hat{\xi}_j, \xi_{j+1}, ..., \hat{\xi}_i, ..., \xi_k) \\ & + \sum_{j+1<i} (-1)^{j+1+i} \omega([\xi_{j+1}, \xi_i], \xi_0, ..., \xi_j, \hat{\xi}_{j+1}, ..., \hat{\xi}_i, ..., \xi_k) \end{aligned}$$

  Note that the the first and the second, the third and the fourth, and the fifth and the sixth term cancel each other, since they are the same but with a different sign.
- $d\omega$ is $C^\infty(M, \mathbb{R})$-linear in each entry: by being alternating, it is enough to show this for one entry. For $f \in C^\infty(M, \mathbb{R})$ one has

$$\begin{aligned} d\omega(f\xi_0, \xi_1, ..., \xi_k) = & (f\xi_0) \cdot \omega(\xi_1, ..., \xi_k) + \sum_{i>0} (-1)^i \xi_i \cdot \omega(f\xi_0, ..., \hat{\xi}_i, ..., \xi_k) \\ & + \sum_{1 \le i} (-1)^i \omega([f\xi_0, \xi_i], \xi_1, ..., \hat{\xi}_i, ..., \xi_k) \\ & + \sum_{1 \le i<j} (-1)^{i+j} f\omega([\xi_i, \xi_j], \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_k) \\ & = f d\omega(\xi_0, \xi_1, ..., \xi_k), \end{aligned}$$

since $[f\xi_0, \xi_i] = f[\xi_0, \xi_i] - (\xi_i \cdot f)\xi_0$ and

$$\xi_i \cdot \omega(f\xi_0, ..., \hat{\xi}_i, ..., \xi_k) = (\xi_i \cdot f)\omega(\xi_0, ..., \hat{\xi}_i, ..., \xi_k) + f(\xi_i \cdot \omega(\xi_0, ..., \hat{\xi}_i, ..., \xi_k)).$$

□

THEOREM 3.27. *The operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfies the following properties.*

(a) *For $f \in C^\infty(M, \mathbb{R})$, $df(\xi) = \xi \cdot f$ for all $\xi \in \Gamma(T^*M)$.*

(b) *For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ we have*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d^2 = d \circ d = 0$.

(d) *d is a local operator: if $\omega \in \Omega^k(M)$ vanishes identically on an open subset $U \subset M$, then $d\omega$ also vanishes identically on U. In particular, for $\omega \in \Omega^k(M)$, $d\omega|_U$ just depends on $\omega|_U$ for an open subset $U \subset M$.*

(e) *If (U, u) is a chart for M and $\omega \in \Omega^k(M)$, then*

$$d\omega|_U = \sum_{i_1 < ... < i_k} d(\omega_{i_1 ... i_k}) \wedge du^{i_1} \wedge ... \wedge du^{i_k} = \sum_{i_1 < ... < i_k; i_0} \frac{\partial \omega_{i_1 ... i_k}}{\partial u^{i_0}} du^{i_0} \wedge du^{i_1} ... \wedge du^{i_k},$$

where $\omega|_U = \sum_{i_1 < ... < i_k} \omega_{i_1 ... i_k} du^{i_1} \wedge ... \wedge du^{i_k}$.

(f) *d is a natural operator: If $f : M \to N$ is a smooth map between manifolds, then*

$$d(f^*\omega) = f^* d\omega.$$

PROOF.

(a) Clear from the definition.

(d) Suppose $\omega|_U = 0$. Then for arbitrary vector fields $\xi_1, ..., \xi_k$ one has $\omega(\xi_1, ..., \xi_k)|_U = 0$ and also $\xi_0 \cdot \omega(\xi_1, ..., \xi_k)|_U = 0$ for any vector field ξ_0. Hence, $d\omega|_U = 0$. In particular, if $\omega|_U = \eta|_U$, then

$$0 = d(\omega - \eta)|_U = d\omega|_U - d\eta|_U,$$

which implies $d\omega|_U = d\eta|_U$.

(e) We first prove a special case of (b). Suppose $\omega \in \Omega^k(M)$ and $f \in \Omega^0(M) = C^\infty(M, \mathbb{R})$. Then

$$\begin{aligned} d(f\omega)(\xi_0, ..., \xi_k) &= \sum_{i=0}^{k} (-1)^i \xi_i \cdot (f\omega(\xi_0, ..., \hat{\xi}_i, ..., \xi_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} f\omega([\xi_i, \xi_j], \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_k) \\ &= (fd\omega)(\xi_0, ..., \xi_k) + \sum_{i=0}^{k} (-1)^i (\xi_i \cdot f)\omega(\xi_0, ..., \hat{\xi}_i, ..., \xi_k), \end{aligned}$$

which shows that

$$d(f\omega) = fd\omega + df \wedge \omega. \tag{3.7}$$

Now we claim that

$$d(du^{i_1} \wedge ... \wedge du^{i_k}) = 0.$$

Indeed, $d(du^{i_1} \wedge ... \wedge du^{i_k})$ vanishes upon insertion of any $k+1$ coordinate vector fields, since $du^{i_1} \wedge ... \wedge du^{i_k}$ is a constant upon insertion of k coordinate vector fields and $[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] = 0$. Hence, (d) and (3.7) imply

$$\begin{aligned}(d\omega)|_U &= d(\sum_{i_1<...<i_k} \omega_{i_1...i_k} du^{i_1} \wedge ... \wedge du^{i_k}) \\ &= \sum_{i_1<...<i_k} d\omega_{i_1...i_k} \wedge du^{i_1} \wedge ... \wedge du^{i_k} \\ &= \sum_{i_1<...<i_k;i_0} \frac{\partial \omega_{i_1...i_k}}{\partial u^{i_0}} du^{i_0} \wedge du^{i_1} ... \wedge du^{i_k}.\end{aligned}$$

(b) By (d) we can prove this in local coordinates. Suppose

$$\omega|_U = \sum_I \omega_I du^I \text{ and } \eta|_U = \sum_J \eta_J du^J,$$

where $I = (i_1 < i... < i_k)$ and $J = (j_1 < i... < j_\ell)$ are multi-indices and $du^I = du^{i_1} \wedge ... \wedge du^{i_k}$. Then one has

$$\omega \wedge \eta|_U = \sum_{I,J} \omega_I \eta_J du^I \wedge du^J.$$

Hence, one gets

$$\begin{aligned}d(\omega \wedge \eta)|_U &= \sum_{I,J} d(\omega_I \eta_J) du^I \wedge du^J \\ &= \sum_{I,J} d(\omega_I)\eta_J du^I \wedge du^J + \sum_{I,J} \omega_I d(\eta_J) du^I \wedge du^J \\ &= \sum_I d(\omega_I) du^I \wedge \sum_J \eta_J du^J + (-1)^k \sum_I \omega_I du^I \wedge \sum_J d(\eta_J) du^J \\ &= (d\omega \wedge \eta + (-1)^k \omega \wedge d\eta)|_U.\end{aligned}$$

(c) We can prove this again in local coordinates. We have already seen that $d(du^{i_1} \wedge ... \wedge du^{i_k)} = 0$. Hence, (b) and (e) imply that

$$(d^2\omega)|_U = \sum_I d^2(\omega_I) \wedge du^I.$$

It remains to show that $d^2 f = 0$ for any $f \in C^\infty(M, \mathbb{R})$. Indeed, by definition of the Lie bracket one has

$$\begin{aligned}d(df)(\xi_0, \xi_1) &= \xi_0 \cdot df(\xi_1) - \xi_1 \cdot df(\xi_0) - df([\xi_0, \xi_1]) \\ &= \xi_0 \cdot (\xi_1 \cdot f) - \xi_1 \cdot (\xi_0 \cdot f) - [\xi_0, \xi_1] \cdot f = 0.\end{aligned}$$

(f) Suppose $g \in \Omega^0(N) = C^\infty(N, \mathbb{R})$. Then $f^* g = g \circ f$. Hence, for any vector field $\xi \in \mathfrak{X}(M)$ one has

$$d(f^*g)(\xi) = \xi \cdot (g \circ f) = (Tf\xi) \cdot g = dg(Tf\xi) = f^* dg(\xi). \qquad (3.8)$$

Suppose now (U, u) is a chart for N with $f(N) \cap U \neq \emptyset$. For $\omega_U = \sum_I \omega_I du^I$ we have by Proposition 3.24 that

$$f^*\omega|_{f^{-1}(U)} = \sum_I f^*\omega_I f^* du^I = \sum_{i_1<...<i_k} f^*\omega_{i_1...i_k} f^* du^{i_1} \wedge .... f^* du^{i_k}.$$

By (3.8), $d(f^*\omega_I) = f^*d(\omega_I)$ and $d(f^*u^i) = f^*du^i$, and therefore (c) implies

$$d(f^*\omega|_{f^{-1}(U)}) = \sum_I d(f^*\omega_I) f^*du^I = f^*(\sum_I d\omega_I \wedge du^I) = f^*(d\omega|_U).$$

□

3.5. Lie derivatives

Suppose $f : M \to N$ is a local diffeomorphism between manifolds. Then at any point $x \in M$

$$T_x f : T_x M \to T_{f(x)} N$$
$$T_x^* f = (T_x f)^* : T_{f(x)}^* N \to T_x^* M$$

are linear isomorphism. Recall that $((T_x f)^{-1})^* = (T_x^* f)^{-1}$.

Definition 3.28. Suppose $f : M \to N$ is a local diffeomorphism and let $\phi \in \mathcal{T}_q^p(N)$ be a $\binom{p}{q}$-tensor. Then the **pull-back of ϕ via f** is the $\binom{p}{q}$-tensor $f^*\phi \in \mathcal{T}_q^p(M)$ on M given by:

$$f^*\phi(x)(\omega^1, ..., \omega^p, \xi_1, ..., \xi_q) = \phi(f(x))((T_x^* f)^{-1}\omega^1, ..., T_x^* f)^{-1}\omega^p, T_x f\xi_1, ..., T_x f\xi_q),$$

for $\omega^i \in T_x^* M$ and $\xi_j \in T_x M$.

Applied to local flows of vector fields we get:

Definition 3.29. Suppose M is a manifold, $\xi \in \mathfrak{X}(M)$, and $\phi \in \mathcal{T}_q^p(M)$. Then the **Lie derivative** $\mathcal{L}_\xi \phi \in \mathcal{T}_p^q(M)$ of ϕ along ξ is the $\binom{p}{q}$-tensor given by

$$(\mathcal{L}_\xi \phi)(x) := \frac{d}{dt}|_{t=0}((\mathrm{Fl}_t^\xi)^*\phi(x)) \text{ for all } x \in M.$$

Note that $t \mapsto (\mathrm{Fl}_t^\xi)^*\phi(x)$ is a smooth curve (defined for small t, depending on x) in the vector space $T_x M^{\otimes p} \otimes T_x^* M^{\otimes q}$ and hence its derivative at $t = 0$ is again an element in this space. One checks that $\mathcal{L}_\xi \phi : M \to TM^{\otimes p} \otimes T^*M^{\otimes q}$ is a smooth section (see below).

Proposition 3.30. *Suppose $\xi \in (M)$ is a vector field on a manifold M.*

(1) $\mathcal{L}_\xi f = df(\xi)$ *for* $f \in \Omega^0(M) = \mathcal{T}_0^0(M) = C^\infty(M, \mathbb{R})$.
(2) $\mathcal{L}_\xi \eta = [\xi, \eta]$ *for* $\eta \in \mathfrak{X}(M)$.
(3) *For* $\phi \in \mathcal{T}_q^p(M)$ *and* $\psi \in \mathcal{T}_s^r(M)$ *one has*

$$\mathcal{L}_\xi(\phi \otimes \psi) = \mathcal{L}_\xi \phi \otimes \psi + \phi \otimes \mathcal{L}_\xi \psi.$$

In particular, $\mathcal{L}_\xi(\omega \wedge \mu) = \mathcal{L}_\xi \omega \wedge \mu + \omega \wedge \mathcal{L}_\xi \mu$ *for* $\omega \in \Omega^k(M)$ *and* $\mu \in \Omega^\ell(M)$.

(4) *For* $\phi \in \mathcal{T}^p_q(M)$,

$$(\mathcal{L}_\xi\phi)(\omega^1, ..., \omega^p, \eta_1, ..., \eta_q) = \mathcal{L}_\eta(\phi(\omega^1, ..., \eta_q)) - \sum_{i=1}^{p} \phi(\omega^1, ..., \mathcal{L}_\xi\omega^i, ..., \omega^p, \eta_1, ..., \eta_q) - \sum_{j=1}^{q} \phi(\omega^1, ..., \omega^p, ..., \eta_1, ..., \mathcal{L}_\xi\eta_j, ..., \eta_q),$$

for all 1-forms $\omega^1, ..., \omega^p$ *and all vector fields* $\eta_1, ..., \eta_q$ *on* M*. In particular, for* $\mu \in \Omega^q(M)$,

$$\begin{aligned}(\mathcal{L}_\xi\mu)(\eta_1, ..., \eta_q) &= \mathcal{L}_\xi(\mu(\eta_1, ..., \eta_q)) - \sum_{i=1}^{k} \mu(\eta_1, ..., \mathcal{L}_\xi\eta_i, ..., \eta_q) \\ &= \xi \cdot (\mu(\eta_1, ..., \eta_q)) - \sum_{i=1}^{k} \mu(\eta_1, ..., [\xi, \eta_i], ..., \eta_q),\end{aligned}$$

for all vector fields $\eta_1, ..., \eta_q$ *on* M*.*

PROOF.

(1) $\mathcal{L}_\xi f(x) = \frac{d}{dt}|_{t=0}(\mathrm{Fl}^\xi_t)^* f)(x) = \frac{d}{dt}|_{t=0} f(\mathrm{Fl}^\xi_t(x)) = T_x f \xi(x) = df(\xi)(x)$.
(2) See tutorial.
(3) This follows from the fact that $f^*(\phi \otimes \psi) = f^*\phi \otimes f^*\psi$ for any local diffeomorphism $f : M \to M$ (which in turn follows from the definition of the tensor product) and the bilinearity of $\otimes : (\phi, \psi) \mapsto \phi \otimes \psi$.
(4) Note that the full contraction map

$$C : \phi \otimes \omega^1 \otimes ...\omega^p \otimes \eta_1 \otimes ... \otimes \eta_q \mapsto \phi(\omega^1, ..., \omega^p, \eta_1, ..., \eta_q)$$

is linear and commutes with the pull-back of local diffeomorphisms $f : M \to M$, that is,

$$\begin{aligned}C(f^*(\phi \otimes \omega^1 \otimes ... \otimes \eta_q)) &= C(f^*\phi \otimes f^*\omega^1 \otimes ... \otimes f^*\eta_q)) \\ &= \phi(\omega^1, ..., \eta_q) \circ f = f^*C(\phi \otimes \omega^1 \otimes ... \otimes \eta_q).\end{aligned}$$

Using this the result follows immediately from (3).

□

Note that the formulae for $\mathcal{L}_\xi\phi$ and $\mathcal{L}_\xi\omega$ in (4) imply in particular that $\mathcal{L}_\xi\phi$ is again a smooth tensor field as the right-hand side is smooth upon insertion of smooth vector fields and 1-forms.

On differential forms we have the following operators:

- $d : \Omega^k(M) \to \Omega^{k+1}(M)$
- $\mathcal{L}_\xi : \Omega^k(M) \to \Omega^k(M)$ for $\xi \in \mathfrak{X}(M)$
- $$\begin{aligned}\iota_\xi : \Omega^k(M) &\to \Omega^{k-1}(M) \\ \omega &\mapsto \iota_\xi\omega = \omega(\xi, \_\,, ..., \_\,)\end{aligned}$$

  for $\xi \in \mathfrak{X}(M)$.

DEFINITION 3.31. A **graded derivation of the algebra** $(\omega^*(M), \wedge)$ **of degree** r is a linear map $D : \Omega^*(M) \to \Omega^*(M)$ such that

- D maps $\Omega^k(M)$ to $\Omega^{k+r}(M)$, and
- for any $\omega \in \Omega^k(M)$ and any $\eta \in \Omega^\ell(M)$,

$$D(\omega \wedge \eta) = D(\omega) \wedge \eta + (-1)^{rk} \omega \wedge D(\eta).$$

PROPOSITION 3.32. *Suppose M is a manifold and $\xi \in \mathfrak{X}(M)$.*

(1) *d is a graded derivation of degree 1.*
(2) *$\mathcal{L}_\xi$ is a graded derivation of degree 0.*
(3) *ι_ξ is a graded derivation of degree -1.*

Moreover, if D_1 and D_2 are two graded derivations of degree r_1 respectively r_2, then

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1 \tag{3.9}$$

is a graded derivation of degree $r_1 + r_2$.

PROOF. (1) and (2) we have already seen. For (3) and the last statement see tutorial. □

REMARK 3.33. Denoting by $\mathrm{Der}_r(\Omega(M))$ the vector space of derivations of degree r, then the vector space $\bigoplus_{r\in\mathbb{Z}} \mathrm{Der}_r(\Omega(M))$ is a graded Lie algebra with respect to the graded commutator (3.9).

PROPOSITION 3.34. *Suppose D is a graded derivation of degree r of $(\omega^*(M), \wedge)$.*

(1) *D is a local operator: if $\omega \in \Omega^k(M)$ vanishes identically on an open subset $U \subset M$, then so does $D\omega$. In particular, if two differential forms agree on some open set U, so do their images under D.*
(2) *If $\widetilde{D}$ is anotehr graded derivation of degree r such that for any $f \in C^\infty(M, \mathbb{R})$*

$$\widetilde{D}(f) = D(f) \text{ and } \widetilde{D}(df) = D(df),$$

then $\widetilde{D} = D$.

PROOF. See tutorial. □

REMARK 3.35. The previous proposition implies in particular that d is the unique graded derivation of degree 1 such that $Df = df$ and $D(df) = 0$.

PROPOSITION 3.36. *Suppose M is manifold and $\xi, \eta \in \mathfrak{X}(M)$ vector fields.*

(1) $[d, \mathcal{L}_\xi] = d \circ \mathcal{L}_\xi - \mathcal{L}_\xi \circ d = 0$
(2) $[d, \iota_\xi] = d \circ \iota_\xi + \iota_\xi \circ d = \mathcal{L}_\xi$
(3) $[d, d] = 2\, d \circ d = 0$
(4) $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_\xi \circ \mathcal{L}_\eta - \mathcal{L}_\eta \circ \mathcal{L}_\xi = \mathcal{L}_{[\xi\, \eta]}$
(5) $[\mathcal{L}_\xi, \iota_\eta] = \mathcal{L}_\xi \circ \iota_\eta - \iota_\eta \circ \mathcal{L}_\xi = \iota_{[\xi\, \eta]}$
(6) $[\iota_\xi, \iota_\eta] = \iota_\xi \circ \iota_\eta + \iota_\eta \circ \iota_\xi = 0$

PROOF. See tutorial. □

CHAPTER 4

Integration on Manifolds

Recall that the transformation formula (or coordinate change formula) for multiple integrals: Suppose $U \subset \mathbb{R}^m$ is an open subset and $\phi : U \to \phi(U)$ a diffeomorphism between open subsets of $\mathbb{R}^m$. Let $f : \phi(U) \to \mathbb{R}$ be a smooth function with compact support. Then

$$\int_{\phi(U)} f = \int_U (f \circ \phi)|\det(D\phi)|. \tag{4.1}$$

This looks like the transformation of m-forms on manifolds of dimension m under coordinate changes. Indeed, suppose M is a smooth manifold of dimension m, $\omega \in \Omega^m(M)$ an m-form and (U_α, u_α) a chart of M. Then we know that

$$\omega|_{U_\alpha} = \omega^\alpha_{1...m} du^1_\alpha \wedge ... \wedge du^m_\alpha,$$

where $\omega^\alpha_{1...m} = \omega(\frac{\partial}{\partial u^1_\alpha}, ..., \frac{\partial}{\partial u^m_\alpha}) : U_\alpha \to \mathbb{R}$. Suppose now (U_β, u_β) is another chart with $U = U_\alpha = U_\beta$ and set $\phi := u_\beta \circ u_\alpha^{-1} : u_\alpha(U) \to u_\beta(U)$. Let us now compare the the local coordinate expression $\omega^\alpha_{1...m} \circ u_\alpha^{-1}$ and $\omega^\beta_{1...m} \circ u_\beta^{-1}$ of the functions $\omega^\alpha_{1...m}$ and $\omega^\beta_{1...m}$:

$$\begin{aligned}
\omega^\alpha_{1...m}(u_\alpha^{-1}(y)) &= \omega(u_\alpha^{-1}(y))(T_y u_\alpha^{-1} e^1, ..., T_y u_\alpha^{-1} e^m) \\
&= \omega(u_\alpha^{-1}(y))(T_{\phi(y)} u_\beta^{-1} \circ T_y \phi\, e^1, ..., T_{\phi(y)} u_\beta^{-1} \circ T_y \phi\, e^m) \\
&= \det(D_y\phi)\omega(u_\alpha^{-1}(y))(T_{\phi(y)} u_\beta^{-1} e^1, ..., T_{\phi(y)} u_\beta^{-1} e^m) \qquad (4.2)\\
&= \underbrace{\det(D_y\phi)}_{\neq 0} \omega^\beta_{1...m}(u_\beta^{-1}(\phi(y))).
\end{aligned}$$

If we assume U is connected and hence so is $u_\alpha(U)$, then the sign of $\det(D_y\phi)$ is either always positive or negative on $u_\alpha(U)$. Then (4.1) says that the integral over the local coordinate expression of the function $\omega^\alpha_{1...m}$ is up to a sign independent of the choice of charts. To fix the issue with the sign, we need now to talk about orientation.

4.1. Orientation

4.1.1. Orientation of vector spaces. Suppose $\mathbb{V}$ is a real vector space of dimension m. Then two ordered bases $(v^1, ..., v^m)$ and $(w^1, ..., w^m)$ of $\mathbb{V}$ have the same orientation (respectively, opposite orientations), if the linear map $T : \mathbb{V} \to \mathbb{V}$, mapping one to the other, i.e. $T(v^i) = w^i$ for all i, has positive (respectively, negative) determinant. It is easy to see that "having the same orientation" defines an equivalence relation on the set of ordered bases of $\mathbb{V}$ and there are exactly two equivalence classes.

Definition 4.1. Suppose $\mathbb{V}$ is a real vector space of dimension $m \geq 1$.

(1) An **orientation** for $\mathbb{V}$ is a choice of one of the two equivalence classes of ordered bases of $\mathbb{V}$.
(2) Having fixed an orientation for $\mathbb{V}$, any basis in the chosen equivalence class is called **positively oriented** and any one in the opposite equivalence class is called **negatively oriented**.
(3) For a zero-dimensional vector space, we define an orientation to be a choice of one of the numbers 1 or -1.
(4) A vector space with a choice of **orientation** is called an **oriented vector space**.

Example 4.1. The **standard orientation** on $\mathbb{R}^m$ is the one induced by the standard basis $(e^1, ..., e^m)$. Note that with respect to the standard orientation a basis $(v^1, ..., v^m)$ of $\mathbb{R}^m$ is positively oriented, if $\det((v^1, ..., v^m)) > 0$.

Proposition 4.2. *Suppose $\mathbb{V}$ is a vector space of dimension m. Then any nonzero element $\omega \in \bigwedge^m \mathbb{V}^*$ induces an orientation on $\mathbb{V}$ as follows:*

(1) *If $m \geq 1$, then the orientation is given by all ordered bases $(v^1, ..., v^m)$ of $\mathbb{V}$ such that $\omega(v^1, ..., v^m) > 0$.*
(2) *If $m = 0$, the induced orientation is defined to be 1 for $\omega > 0$ and -1 for $\omega < 0$.*

Moreover, two nonzero elements in $\bigwedge^m \mathbb{V}^*$ define the same orientation if and only if one is a positive multiple of the other.

Proof. See tutorial. □

The previous proposition shows that the choice of an orientation for an m-dimensional vector space $\mathbb{V}$ is equivalent to the choice of one of the two connected components of $\bigwedge^m \mathbb{V}^* \setminus \{0\}$.

Example 4.2. Suppose that $\lambda_1, ..., \lambda_m$ is the basis of $(\mathbb{R}^m)^*$ dual to the standard basis in $\mathbb{R}^m$. Then $\lambda_1 \wedge ... \wedge \lambda_m \in \bigwedge^m (\mathbb{R}^m)^*$ induces the standard orientation on $\mathbb{R}^m$.

Definition 4.3. Suppose $\mathbb{V}$ and $\mathbb{W}$ are oriented vector spaces of dimension m. A linear isomorphism $F : \mathbb{V} \to \mathbb{W}$ is called **orientation preserving**, if F maps any positively oriented basis of $\mathbb{V}$ to a positively oriented basis of $\mathbb{W}$.

Proposition 4.4. *Suppose $\mathbb{V}$ and $\mathbb{W}$ are vector space of dimension m equipped with orientations induced by $\omega \in \bigwedge^m \mathbb{V}^* \setminus \{0\}$ respectively $\nu \in \bigwedge^m \mathbb{W}^* \setminus \{0\}$. A linear isomorphism $F : \mathbb{V} \to \mathbb{W}$ is orientation preserving if and only if the induced map $F^* : \bigwedge^m \mathbb{W}^* \to \bigwedge^m \mathbb{V}^*$ maps ν to $c\omega$ for some positive real number c.*

Proof. Exercise, see tutorial. □

4.1.2. Orientations on manifolds.

Definition 4.5. Suppose M is a manifold.

(1) M is called **orientable**, if for any $x \in M$ one can choose an orientation of T_xM such that the following holds: for any point $x \in M$ there exists an open neighbourhood $U \subset M$ and a local frame

$\xi_1, ..., \xi_m$ of TM defined on U such that $(\xi_1(x), ..., \xi_m(x))$ forms a positively oriented basis of T_xM for all $x \in U$.

(2) If M is orientable, a choice of orientation on each tangent space T_xM for $x \in M$ satisfying the property in (1) is called an **orientation** on M.

(3) An orientable manifold with a choice of orientation is called an **oriented manifold**.

Remark 4.6. If M is a zero-dimensional manifold, then the condition in (1) is vacuous and an orientation is simply the choice of ± 1 attached to any of its points, i.e. a map $\epsilon : M \to \{\pm 1\}$.

Remark 4.7. Note that being orientable is equivalent to the possibility of choosing an orientation on each tangent space such that for any local frame local frame $\xi_1, ..., \xi_m$ of TM over a connected subset U, the basis $(\xi_1(x), ..., \xi_m(x))$ of T_xM is either positively or negatively oriented for all $x \in U$.

Note an on a connected orientable manifold, there are exactly two orientations, which coincide if they coincide at a single tangent space. Also, evidently, an orientation on a manifold induces an orientation on any of its open subsets, making them oriented manifolds in a natural way.

Definition 4.8. A local diffeomorphism $F : M \to N$ between oriented manifolds is called **orientation preserving**, if at each point $x \in M$ the tangent map $T_xF : T_xM \to T_{F(x)}N$ is orientation preserving.

Proposition 4.9. *Suppose M is an m-dimensional manifold.*

(1) *If there exists nowhere vanishing m-form $\omega \in \Gamma(\wedge^m TM)$ (i.e. $\omega(x) \neq 0$ for all $x \in M$), then the orientation on T_xM defined by $\omega(x)$ as in Proposition 4.2 defines an orientation on M. In particular, M is orientable.*

(2) *Conversely, if M is orientable and oriented, there exists a nowhere vanishing m-form $\omega \in \Gamma(\wedge^m TM)$ inducing (as in (1)) the given orientation. Moreover, ω is unique up to multiplication by a positive smooth function on M.*

Proof.

(1) Suppose $\omega \in \Gamma(\wedge^m TM)$ is nowhere vanishing. It remains to check that the orientations on T_xM induced by $\omega(x) \in \wedge^m T_xM$ satisfy the property in (1) of Definition 4.5. If $m = 0$ this condition is vacuous, so assume $m \geq 1$. Let $x \in M$ be any point and $(\xi_1, ..., \xi_m)$ a local frame of TM defined on a connected open neighbourhood U of x. Then $\omega(\xi_1, ..., \xi_m) : U \to \mathbb{R}$ is smooth, in particular continuous, and nowhere vanishing on U. Hence, either $\omega(\xi_1, ..., \xi_m)$ is either always positive or always negative on U. Hence, $\xi_1(x), ..., \xi_m(x)$ is a positive oriented basis of T_xM for all $x \in U$ in the first case and in the second case simply replace ξ_1 by $-\xi_1$ to obtain a local frame with that property.

(2) Fix an orientation on M. Let $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ be an atlas for M such that for all $\alpha \in I$, U_α is connected and $\frac{\partial}{\partial u_\alpha^1}, ..., \frac{\partial}{\partial u_\alpha^m}$

defines a positively oriented basis for all $x \in U_\alpha$. Such an atlas exists, since by Remark 4.7 for any chart (U, u) for M defined on a connected open subset U the corresponding coordinate vector fields $\frac{\partial}{\partial u^1}, ..., \frac{\partial}{\partial u^m}$ define either a positively oriented basis or a negatively oriented basis for all $x \in U$ and in the latter case replacing u^1 by $-u^1$ leads to a chart whose coordinate vector fields define a positively oriented basis at all $x \in U$. Now let $\mathcal{F} := \{f_k : M \to \mathbb{R} : k \in \mathbb{N}\}$ be a partition of unity subordinate to the cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. For each $k \in \mathbb{N}$ choose $\alpha_k \in I$ such that $\text{supp}(f_k) \subset U_{\alpha_k}$. Then $\omega^k := f_k du^1_{\alpha_k} \wedge ... \wedge du^m_{\alpha_k}$ can be extended by zero to an m-form defined on all of M and local finiteness implies that

$$\omega := \sum_{k \in \mathbb{N}} \omega^k$$

defines a (smooth) m-form on M. Fix $x \in M$. Since $\sum_k f_k(x) = 1$, there exists $k \in \mathbb{N}$ such that

$$\omega^k(x)(\frac{\partial}{\partial u^1_{\alpha_k}}(x), ..., \frac{\partial}{\partial u^m_{\alpha_k}}(x)) = f_k(x) > 0.$$

Since any chart (U_α, u_α) in $\mathcal{A}$ is by construction orientation preserving, so are all the transition maps for elements in $\mathcal{A}$. Since the transition maps of $\mathcal{A}$ are orientation preserving and any element in $\mathcal{F}$ has non-negative values on M, for any $\ell \in \mathbb{N}$

$$\omega^\ell(x)(\frac{\partial}{\partial u^1_{\alpha_k}}(x), ..., \frac{\partial}{\partial u^m_{\alpha_k}}(x)) \geq 0.$$

Hence, $\omega(x) \neq 0$. Therefore, there exists a nowhere vanishing m-form on M as claimed. The last statement is obvious.

□

The proof of the previous Proposition suggest the following definition:

Definition 4.10. Suppose M is an m-dimensional manifold, $m \geq 1$.

(1) An **oriented atlas** for M is an atlas $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ for M (i.e. a subatlas of the maximal atlas of M) such that for any $\alpha, \beta \in I$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ the transition map

$$u_\beta \circ u_\alpha^{-1} : u_\alpha(U_{\alpha\beta}) \to u_\beta(U_{\alpha\beta})$$

is an orientation preserving diffeomorphism between open subsets of $\mathbb{R}^m$, where $\mathbb{R}^m$ is equipped with its standard orientation.

(2) Two oriented atlases are equivalent, if their union is again an oriented atlas. Any oriented atlas is contained in a maximal oriented atlas.

Proposition 4.11. *Suppose M is a manifold of dimension $m \geq 1$. Then the following are equivalent:*

(1) *M is orientable.*
(2) *M admits an oriented atlas.*
(3) *There exists an m-form $\omega \in \Gamma(\wedge^m TM)$ such that $\omega(x) \neq 0$ for all $x \in M$.*

Moreover, if M is orientable, fixing an orientation is equivalent to fixing a maximal oriented atlas or a nowhere vanishing m-form up to multiplication by a positive smooth function.

Proof. By Proposition 4.9, we already know that (1) if and only if (3). Also, the proof of (2) of Proposition 4.9 shows that (1) implies (2) and that (2) implies (3). The remaining statements are obvious. □

Having fixed an orientation on an orientable manifold M, we call a chart (U, u) of M **oriented**, if it is contained in the corresponding maximal oriented atlas.

Example 4.3. $M = \mathbb{R}^m$ is orientable with standard orientation induced by the nowhere vanishing m-form $dx^1 \wedge ... \wedge dx^m$.

Example 4.4. The m-sphere $S^m \subset \mathbb{R}^{m+1}$ is an oriented manifold, which can be seen as follows. Consider the m-form on $\mathbb{R}^{m+1}$ given by

$$\omega = \sum_{i=1}^{m+1} (-1)^{i-1} x^i dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^{m+1}.$$

If $x \in S^m$, then $T_x S^m = \{v \in T_x\mathbb{R}^{m+1} \cong \mathbb{R}^{m+1} : \langle x, v\rangle = 0\} = x^\perp$. Now, if $(v^1, ..., v^m)$ is basis of $x^\perp = T_x S^m$, then

$$\omega_x(v^1, ..., v^m) = \det(x, v^1, ..., v^m) \neq 0.$$

Hence, ω restricts to a nowhere vanishing m-form on S^m, inducing the standard orientation on S^m. Otherwise put, the vector field

$$\nu = \sum_{i=1}^{m+1} x^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^{m+1})$$

is at every point $x \in S^m$ orthogonal to $T_x S^m$ and for $\Omega = dx^1 \wedge ... \wedge dx^{m+1}$ one has

$$i_\nu \Omega = \omega.$$

Example 4.5. The Möbius band $M \subset \mathbb{R}^3$, which can be parametrized by

$$\phi(r, \alpha) = \begin{pmatrix} \cos\alpha(1 + \frac{r}{2}\cos\frac{\alpha}{2}) \\ \sin\alpha(1 + \frac{r}{2}\cos\frac{\alpha}{2}) \\ \frac{r}{2\sin\frac{\alpha}{2}} \end{pmatrix}$$

for $0 \leq \alpha < 2\pi$ and $r \in [-1, 1]$, is not orientable.

Example 4.6. Real projective space $\mathbb{R}P^m$ is orientable if and only if m is odd; see tutorial.

Example 4.7. For any manifold M its tangent space TM is an oriented manifold. Recall that any atlas for M induces an atlas for TM in a natural way, which turns out to be oriented, since its transition maps (2.3) are orientation preserving; check that (tutorial).

4.2. Integration

For a k-form $\omega \in \Omega^k(M)$ on a manifold M let us write

$$\operatorname{supp}(\omega) = \overline{\{x \in M : \omega(x) \neq 0\}}$$

for its support and denote by $\Omega_c^k(M)$ the k-forms with compact support.

Proposition 4.12. *Suppose M is a (smooth) oriented manifold of dimension m. Then there exists a unique linear map*

$$\int_M : \Omega_c^m(M) \to \mathbb{R},$$

such that for any oriented chart (U, u) of M and any $\omega \in \Omega_c^m(U)$ one has

$$\int_M \omega = \int_{u(U)} (u^{-1})^* \omega := \int_{u(U)} (u^{-1})^* \omega(e^1, ..., e^m).$$

Proof. Suppose $\mathcal{A} = \{(U_i, u_i)\}_{i \in I}$ is an oriented atlas for M and let $\mathcal{F} = \{f_i\}_{i \in I}$ be a partition of unity subordinate to the cover $\mathcal{U} = \{U_i\}_{i \in I}$ such that $\operatorname{supp}(f_i) \subset U_i$. Note that for any $\omega \in \Omega_c^m(M)$ one has

$$\omega = \sum_{i \in I} f_i \omega,$$

where $f_i \omega \in \Omega_c^m(U_i)$. Hence, by linearity, one must have

$$\int_M \omega = \sum_{i \in I} \int_{u_i(U_i)} (u_i^{-1})^* (f_i \omega)(e^1, ..., e^m). \tag{4.3}$$

This shows uniqueness. The formula also allows to define a linear functional $\int_M$ and that it satisfies the requested property follows from the transformation rules (4.1) and (4.2). Indeed, if $\omega \in \Omega_c^m(U)$, then by (4.1) and (4.2), one gets

$$\begin{aligned}
\int_M \omega &= \sum_{i \in I} \int_{u_i(U_i \cap U)} (u_i^{-1})^* (f_i \omega)(e^1, ..., e^m) \\
&= \sum_{i \in I} \int_{u(U_i \cap U)} (u^{-1})^* (f_i \omega)(e^1, ..., e^m) \\
&= \int_{u(U)} (u^{-1})^* (\sum_{i \in I} f_i \omega)(e^1, ..., e^m) = \int_{u(U)} (u^{-1})^* \omega(e^1, ..., e^m).
\end{aligned}$$

□

Note that the functional $\int_M$ in Proposition 4.12 depends on the chosen orientation on M. If M is connected and $-M$, the manifold M equipped with the opposite orientation, then

$$\int_{-M} = -\int_M .$$

Also, if $U \subset M$ is an open subset of an oriented manifold M, equipped with the induced orientation, then for any $\omega \in \Omega_c^m(M)$ with $\operatorname{supp}(\omega) \subset U$, one has

$$\int_M \omega = \int_U \omega|_U .$$

An immediate corollary from Proposition 4.12, following from the uniqueness of the functional, is:

COROLLARY 4.13. *Suppose $\Phi : M \to N$ is an orientation preserving diffeomorphism between oriented m-dimensional manifolds M and N. Then for any $\omega \in \Omega_c^m(N)$ one has*

$$\int_M \Phi^*\omega = \int_N \omega.$$

REMARK 4.14. For a zero-dimensional manifold M with an orientation $\epsilon : M \to \{\pm 1\}$ and $\omega \in \Omega_c^0(M)$ one defines

$$\int_M \omega = \sum_{x \in M} \epsilon(x)\omega(x).$$

REMARK 4.15.

(1) If $M = \mathbb{R}$ is equipped with its standard notation, then for any real numbers $a < b$ and any $\omega = f dt \in \Omega^1(\mathbb{R})$ we have

$$\int_{[a,b]} \omega = \int_a^b f(t)dt.$$

(2) Line integrals: Suppose $V \subset \mathbb{R}^m$ is an open subset, $\omega = \sum_{i=1}^m \omega_i dx^i \in \Omega^1(V)$ and $\gamma : I \to V$ a smooth curve defined on an open interval $I \subset \mathbb{R}$. Then, $\gamma^*\omega \in \Omega^1(I)$ and for $a, b \in I$ one has

$$\int_{[a,b]} \gamma^*\omega = \sum_{i=1}^m \int_{[a,b]} (\omega_i \circ \gamma)(\gamma_i'(t))dt.$$

This is the line integral of ω along $\gamma|_{[a,b]} = \alpha$, also denoted by $\int_\alpha \omega$.

4.3. Manifolds with boundary

It is useful and natural to extend the notion of manifolds to the notion of manifolds with boundary, in particular in the context of integration.

Consider the m-dimensional half-space $\mathrm{H}^m \subset \mathbb{R}^m$ given by

$$\mathrm{H}^m := \{(x^1, ..., x^m) \in \mathbb{R}^m : x^1 \leq 0\}.$$

and denote by

$$\mathrm{int}(\mathrm{H}^m) := \{(x^1, ..., x^m) \in \mathbb{R}^m : x^1 < 0\}$$

$$\partial(\mathrm{H}^m) := \mathrm{H}^m \setminus \mathrm{int}(\mathrm{H}^m) = \{(x^1, ..., x^m) \in \mathbb{R}^m : x^1 = 0\}$$

the set of interior respectively boundary points of the closed subset $\mathrm{H}^m \subset \mathbb{R}^m$. Recall that for an open subset $U \subset \mathrm{H}^m$ a map $F : U \to \mathbb{R}^d$ is smooth at $x \in U$, if there exists an open neighbourhood $\tilde{U} \subset \mathbb{R}^m$ of x in $\mathbb{R}^m$ and a smooth map $\tilde{F} : \tilde{U} \to \mathbb{R}^d$ such that $\tilde{F}|_{\tilde{U} \cap U} = F|_{\tilde{U} \cap U}$. For a smooth map $F : U \to \mathbb{R}^d$ its restriction to $U \cap \mathrm{int}(\mathrm{H}^m)$ is smooth in the usual sense.

DEFINITION 4.16. Suppose M is a topological space.

(1) A **chart with values in** H^m for M is a homeomorphism $u : U \to u(U)$ between open subsets $U \subset M$ and $u(U) \subset \mathrm{H}^m$.

(2) A **smooth atlas with values in** H^m for M is a collection $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ of charts with values in H^m such that
- $M = \bigcup_{\alpha \in I} U_\alpha$, and
- for any $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$ the transition map
$$u_{\beta\alpha} := u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to u_\beta(U_\alpha \cap U_\beta)$$
is smooth.

Analogously as for atlases with values in $\mathbb{R}^m$ one has a natural notion of compatibility for atlases with values in H^m and for any atlas with values in H^m there is a unique maximal atlas with values in H^m containing it.

Definition 4.17. An m**-dimensional (smooth) manifold with boundary** is a Hausdorff second countable topological space M equipped with a maximal smooth atlas of charts with values in H^m.

Proposition 4.18. *Suppose $(M, \mathcal{A})$ is (smooth) m-dimensional manifold with boundary and let $x \in M$ be point. If there exists $(U, u) \in \mathcal{A}$ such that $x \in U$ and $u(x) \in \partial H^m$, then for every chart $(V, v) \in \mathcal{A}$ with $x \in V$ one has $v(x) \in \partial H^m$.*

Proof. Suppose, by contrary, that there exist chart $(U, u), (V, v) \in \mathcal{A}$ such that $x \in U \cap V$ and $u(x) \in \mathrm{int}(\mathrm{H}^m)$ and $v(x) \in \partial\mathrm{H}^m$. Since the transition map $\phi := v \circ u^{-1} : u(U \cap V) \to v(U \cap V)$ and its inverse $\phi^{-1} = u \circ v^{-1}$ are smooth, there exists in particular an open neighbourhood W of $v(x)$ in $\mathbb{R}^m$ and a smooth map $\psi : W \to \mathbb{R}^m$ that agrees with ϕ^{-1} on $W \cap v(U \cap V)$. Since $u(x) \in \mathrm{int}(\mathrm{H}^m)$, the set $u(U \cap V)$ contains an open neighbourhood $\widetilde{W}$ of $u(x)$ in $\mathbb{R}^m$ (note that $u(U \cap V)$ as an open subset of H^m must be of the form $W' \cap \mathrm{H}^m$ for an open subset W' of $\mathbb{R}^m$, hence $\widetilde{W} = W' \cap \mathrm{int}(\mathrm{H}^m)$ is an open neighbourhood of $u(x)$ as requested). By possibly shrinking $\widetilde{W}$ we may assume that $\phi(\widetilde{W}) \subset W$. By construction,
$$\psi \circ \phi|_{\widetilde{W}} = \phi^{-1} \circ \phi|_{\widetilde{W}} = \mathrm{Id}_{\widetilde{W}}.$$
Therefore, for all $y \in \widetilde{W}$, $D_{\phi(y)}\psi \circ D_y\phi = \mathrm{Id}_{T_y\mathbb{R}^m}$, which implies that $D_y\phi : T_y\mathbb{R}^m \to T_{\phi(y)}\mathbb{R}^m$ is injective and hence an isomorphism for all $y \in \widetilde{W}$. By the Inverse Function Theorem, it follows in particular that ϕ is an open map. So, in particular, $\phi(\widetilde{W})$ is an open subset of $\mathbb{R}^m$ that contains $v(x)$ and is contained in $v(V) \subset \mathrm{H}^m$. This contradicts the fact that $v(x) \in \partial\mathrm{H}^m$. □

Definition 4.19. Suppose $(M, \mathcal{A})$ is (smooth) m-dimensional manifold with boundary.

(1) A point $x \in M$ is called a **boundary point**, if for a (hence, every) chart $(U, u) \in \mathcal{A}$ with $x \in U$ one has $u(x) \in \partial\mathrm{H}^m$. We denote the set of boundary points by
$$\partial M = \{x \in M : x \text{ is a boundary point}\}.$$
(2) A point $x \in M$ is called an **interior point**, if $x \in \mathrm{int}(M) := M \setminus \partial M$.

Since any open subset of $\mathbb{R}^m$ can be mapped diffeomorphically onto an open subset of H^m that consist of interior points only, any m-dimensional manifold is an m-dimensional manifold with boundary whose boundary ∂M is the empty set.

EXAMPLE 4.8. Any open subset $U \subset \mathrm{H}^m$ is an m-dimensional manifold with boundary $\partial U = \partial \mathrm{H}^m \cap U$.

EXAMPLE 4.9. The unit ball in $\mathbb{R}^{m+1}$, given by

$$\mathrm{B}^m := \{x \in \mathbb{R}^{m+1} : ||x|| \leq 1\},$$

is an m-dimensional manifold with boundary $\partial \mathrm{B}^m = S^{m-1}$.

Analogously as for manifolds without boundary, using charts, one defines smoothness for functions on manifolds with boundary and for maps between manifolds with boundary. The latter restrict on manifolds with boundary $\partial M = \emptyset$ to the notion of smoothness defined in Definition 1.21. Hence, the category of manifolds is a subcategory of the category of manifolds with boundary. Also, the theorem about the existence of partitions of unity extends to manifolds with boundary.

PROPOSITION 4.20. *Suppose $(M, \mathcal{A})$ is (smooth) manifold with boundary $\partial M \neq \emptyset$ of dimension $m \geq 2$. Then ∂M, equipped with the subspace topology, inherits from $\mathcal{A}$ the structure of a smooth manifold without boundary of dimension $m-1$ with the property that the natural inclusion $i : \partial M \hookrightarrow M$ is a smooth immersion. In particular, any smooth function on M restricts to a smooth function on ∂M.*

PROOF. For any chart $(U_\alpha, u_\alpha) \in \mathcal{A}$ with $U_\alpha \cap \partial M \neq \emptyset$, the restriction

$$u_\alpha|_{U_\alpha \cap \partial M} : U_\alpha \cap \partial M \to u_\alpha(U_\alpha) \cap (\{0\} \times \mathbb{R}^{m-1})$$

defines a homeomorphism between an open subset of ∂M and an open subset of $\mathbb{R}^{m-1} \cong \{0\} \times \mathbb{R}^{m-1} \subset \mathbb{R}^m$. Setting

$$\mathcal{A}|_{\partial M} := \{(U_\alpha \cap \partial M, u_\alpha|_{U_\alpha \cap \partial M}) : (U_\alpha, u_\alpha) \in \mathcal{A})\},$$

therefore defines a smooth atlas for ∂M with values in $\mathbb{R}^{m-1}$, since the transitions maps as restrictions of smooth maps are smooth. By construction, i is smooth with respect to $\mathcal{A}|_{\partial M}$ and $\mathcal{A}$. □

As for manifolds without boundary, the tangent space of a manifold with boundary M at some point $x \in M$ is defined to be the space of derivations at x, that is,

$$T_x M = \mathrm{Der}_x(C^\infty(M, \mathbb{R}), \mathbb{R})$$

and for a smooth map $f : M \to N$ between manifolds with boundary one defines the tangent map $T_x f : T_x M \to T_{f(x)} N$ at a point $x \in M$ by the same formula as in Section 2.31. One may show then again that for any point $x \in M$ (for an interior point that is clear), the tangent space $T_x M$ is an m-dimensional vector space spanned by $\frac{\partial}{\partial u^1}(x), ..., \frac{\partial}{\partial u^m}(x)$ for any chart $u : U \to u(U)$ with $x \in U$. Moreover, at any boundary point $x \in \partial M$ the tangent space $T_x \partial M$ at x to the boundary can be identified, via the natural inclusion $T_x i : T_x \partial M \hookrightarrow T_x M$, with the $m-1$-dimensional subspace of $T_x M$ spanned by $\frac{\partial}{\partial u^2}(x), ..., \frac{\partial}{\partial u^m}(x)$. Therefore, at a boundary point $x \in \partial M$ elements in $T_x M$ fall into three classes: either a vector $\xi_x \in T_x M$ is tangent to ∂M, i.e. contained in $T_x \partial M$, or if it is an element of $T_x M \setminus T_x \partial M$, then it is either inward pointing or outward pointing. According to our choice of half-space, $\xi_x \in T_xM \setminus T_x\partial M$ is inward pointing, if the coefficient of $\frac{\partial}{\partial u^1}(x)$ when writing ξ_x as linear combination of $\frac{\partial}{\partial u^1}(x), \dots, \frac{\partial}{\partial u^m}(x)$ is < 0 and outward pointing, if it is > 0. Note that this is independent of the choice of the chart. Similarly, as for manifolds without boundary, one can form the disjoint union TM of all tangent spaces and equip it with the structure of a smooth manifold with boundary $p^{-1}(\partial M)$ of dimension $2\dim(M)$ such that the natural projection $p : TM \to M$ is smooth. Then one can define vector fields, differential forms and general types of tensors in the same way as for manifolds without boundary. Also, the exterior derivative of differential forms extend to manifolds with boundary. Finally, also the notions of orientability and orientation extend without problem to manifolds with boundary.

Proposition 4.21. *Suppose M is (smooth) manifold with boundary $\partial M \neq \emptyset$ of dimension $m \geq 2$ and assume it is orientable.*

(1) *Any oriented atlas for M induces, via restriction, an oriented atlas on ∂M.*
(2) *An orientation on M induces an orientation on ∂M that has the property that for any $x \in \partial M$ a basis of $T_x\partial M$ is positively oriented if and only if adding an outward pointing vector as the first element to it defines a positively oriented basis of T_xM. In particular, ∂M is orientable.*

Proof.

(1) Suppose $\mathcal{A}$ is an oriented atlas for M. Then in the proof of Proposition 4.20 we have seen that $\mathcal{A}|_{\partial M}$ is an atlas for ∂M (defining the natural smooth structure on ∂M induced by the one on M). For two chart $(U_\alpha, u_\alpha), (U_\beta, u_\beta) \in \mathcal{A}$ such that $U_\alpha \cap U_\beta \cap \partial M \neq \emptyset$ consider the transition map:

$$u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1} : u_\beta(U_\alpha \cap U_\beta) \to u_\alpha(U_\alpha \cap U_\beta)$$

Since the transition maps sends boundary points to boundary points, for $x \in U_\alpha \cap U_\beta \cap \partial M$ its derivative $Du_{\alpha\beta}(u_\beta(x))$ at $u_\beta(x)$ has the block matrix form

$$\begin{pmatrix} \lambda & 0 \\ v & A \end{pmatrix},$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, $v \in \mathbb{R}^{m-1}$ and $A \in \mathrm{GL}(m-1, \mathbb{R})$. Also, the transition map has to sends interior points to interior point, which implies that $\lambda > 0$. Since $\mathcal{A}$ is oriented, $\det(Du_{\alpha\beta}(u_\beta(x))) = \lambda \det(A) > 0$, which hence implies $\det(A) > 0$. Since A is the derivative at $u_\beta(x)$ of the transition map of the charts on ∂M induced by u_α and u_β, the result follows.
(2) The first statement follows immediately from (1) and Proposition 4.11 and the second from the construction of the induced orientation.

□

4.4. Theorem of Stokes

Suppose M is an m-dimensional manifold with boundary ∂M and let $\omega \in \Omega_c^{m-1}(M)$ be an $m-1$-form with compact support. By definition ω vanishes on the open set $M \setminus \mathrm{supp}(\omega)$ and so does $d\omega$ by (d) of Theorem 3.27, which implies that $\mathrm{supp}(d\omega) \subset \mathrm{supp}(\omega)$. Hence, in particular, $d\omega$ is again of compact support.

Theorem 4.22. *Suppose M is an oriented m-dimensional manifold with boundary ∂M and let $\omega \in \Omega_c^{m-1}(M)$. Then*

$$\int_M d\omega = \int_{\partial M} i^*\omega = \int_{\partial M} \omega, \tag{4.4}$$

where $i : \partial M \hookrightarrow M$ is the natural inclusion. In particular, if $\partial M = \emptyset$, then $\int_M d\omega = 0$.

Proof. Since $\mathrm{supp}(\omega)$ is compact, there exist finitely many charts (U_j, u_j), $j = 1, ..., \ell$, of an oriented atlas of M such that $\mathrm{supp}(\omega) \subset U_1 \cup ... \cup U_\ell$ and smooth functions $f_j : M \to [0,1]$ for $j = 1, ..., \ell$ such that $\mathrm{supp}(f_j) \subset U_j$ and $\sum_{j=1}^{\ell} f_j|_{\mathrm{supp}(\omega)} \equiv 1$. Then

$$\omega = \sum_{j=1}^{\ell} f_j \omega \quad \text{and} \quad d\omega = \sum_{j=1}^{\ell} d(f_j\omega),$$

and $\mathrm{supp}(d(f_j\omega)) \subset \mathrm{supp}(f_j\omega) \subset U_j$. Therefore,

$$\int_M d\omega = \sum_{j=1}^{\ell} \int_{U_j} d(f_j\omega). \tag{4.5}$$

In contrast, the right-hand side of (4.4) may be computed using the charts $(U_j \cap \partial M, u_j|_{U_j \cap \partial M})$ and the functions $f_j|_{\partial M}$ for $j = 1, ..., \ell$ as

$$\int_{\partial M} \omega = \sum_{j=1}^{\ell} \int_{U_j \cap \partial M} f_j\omega. \tag{4.6}$$

In order to prove that (4.5) equals (4.6) it therefore suffices to prove that

$$\int_{U_j} d(f_j\omega) = \int_{U_j \cap \partial M} f_j\omega \quad \forall j.$$

Without loss of generality we hence may assume that $\mathrm{supp}(\omega)$ is contained in the domain of a single chart (U, u). Then one has

$$\omega = \sum_{i=1}^{m} \omega_i \, du^1 \wedge ... \wedge \widehat{du^i} \wedge ... \wedge du^m,$$

for smooth functions $\omega_i : M \to \mathbb{R}$ with compact support contained in U. Since the tangent space $T_x\partial M$ for a point $x \in \partial M$ is spanned by $\frac{\partial}{\partial u^i}(x)$ for $i \geq 2$, $du^1|_{\partial M} = 0$ and therefore

$$\omega|_{\partial M} = \omega_1 \, du^2 \wedge ... \wedge du^m.$$

This implies that

$$\int_{\partial M} \omega = \int_{\partial u(U)} \omega^1 \circ u^{-1} = \int_{\{0\} \times \mathbb{R}^{m-1}} \omega^1 \circ u^{-1}, \tag{4.7}$$

where the last equality follows from the fact that ω^1 is of compact support contained in U. On the other hand,

$$\begin{aligned} d\omega &= \sum_{i=1}^{\ell} \frac{\partial \omega_i}{\partial u^i}\, du^i \wedge du^1 \wedge ... \wedge \widehat{du^i} \wedge ... \wedge du^m \\ &= (\sum_{i=1}^{m} (-1)^{i-1} \frac{\partial \omega_i}{\partial u^i})\, du^1 \wedge ... \wedge du^m \end{aligned}$$

implies that

$$\begin{aligned} \int_M d\omega &= \sum_{i=1}^{m} (-1)^{i-1} \int_{u(U)} \frac{\partial(\omega_i \circ u^{-1})}{\partial x^i} \\ &= \sum_{i=1}^{m} (-1)^{i-1} \int_{(-\infty,0]\times\mathbb{R}^{m-1}} \frac{\partial(\omega_i \circ u^{-1})}{\partial x^i}, \end{aligned}$$

where the second equality follows from the facts that the functions ω_i have compact support contained in U. Fubini's Theorem allows to decompose any component in that sum into integrals over the individual coordinates, where it does not matter in which order one integrates over the individual components. So for each i we may integrate $\frac{\partial(\omega_i \circ u^{-1})}{\partial x^i}$ first along the i-th coordinate, which by the Fundamental Theorem of calculus and the fact that the functions ω_i have compact support gives

$$\begin{aligned} \int_M d\omega = &\int_{\mathbb{R}^{m-1}} \underbrace{(\int_{-\infty}^{0} \frac{\partial(\omega_1 \circ u^{-1})}{\partial x^1} dx^1)}_{=(\omega_1 \circ u^{-1})(0,x^2,...,x^m)}\, dx^2....dx^m \\ &+ \sum_{i=2}^{m} (-1)^{i-1} \int_{(-\infty,0]\times\mathbb{R}^{n-2}} \underbrace{(\int_{-\infty}^{\infty} \frac{\partial(\omega_i \circ u^{-1})}{\partial x^i} dx^i)}_{=0}\, dx^1....\widehat{dx^i}...dx^m \\ = &\int_{\mathbb{R}^{m-1}} (\omega_1 \circ u^{-1})(0, x^2, ..., x^m)\, dx^2....dx^m, \end{aligned}$$

which equals (4.7).

□

4.5. Excursion: de Rham Cohomology

We know that $(\Omega^*(M), \wedge)$ is an (unitial) associative graded-anticommutative algebra over $\mathbb{R}$ (even over $C^\infty(M, \mathbb{R})$):

- $\Omega^*(M) = \bigoplus_{k\geq 0} \Omega^k(M)$, where $\Omega^k(M) = \{0\}$ for $k > m = \dim(M)$, is a graded vector space
- $\Omega^k(M) \wedge \Omega^\ell(M) \subset \Omega^{k+\ell}(M)$ and for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ one has
$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

Moreover, we have a linear map $d : \Omega^*(M) \to \Omega^*(M)$, which is a graded derivation of degree 1, that is,

- $d(\Omega^k(M)) \subset \Omega^k(M)$
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for any $\omega \in \Omega^k(M)$ and $\eta \in \Omega^*(M)$

DEFINITION 4.23. Suppose $\omega \in \Omega^*(M)$ is a differential form.

- ω is called **closed**, if $d\omega = 0$.
- ω is called **exact**, if there exists $\eta \in \Omega^*(M)$ such that $d\eta = \omega$.

By linearity of d, the set of closed and the set of exact forms define vector subspaces of $\Omega^*(M)$, which we denote as follows:

$$Z^*(M) := \ker(d) = \{\omega \in \Omega^*(M) : d\omega = 0\}$$
$$B^*(M) := \mathrm{im}(d) = \{\omega \in \Omega^*(M) : \exists \eta \text{ s.t. } d\eta = \omega\}.$$

We also set $Z^k(M) := Z^*(M) \cap \Omega^k(M)$ and $B^k(M) := B^*(M) \cap \Omega^k(M)$.

By Theorem, $d^2 = d \circ d = 0$, so $B^*(M) \subset Z^*(M) \subset \Omega^*(M)$. Moreover, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge d\eta$ implies that $B^*(M)$ and $Z^*(M)$ are subalgebras of $\Omega^*(M)$ and $B^*(M)$ even a two-sided ideal in $Z^*(M)$. Therefore,

$$H^*(M) := Z^*(M)/B^*(M) = \bigoplus_{k \geq 0} Z^k(M)/B^k(M)$$

is an (unitial) graded-anticommutative algebra over $\mathbb{R}$. It is called the **de Rham cohomology algebra** of M and

$$H^k(M) := Z^k(M)/B^k(M)$$

the **k-th de Rham cohomology space or group** of M. For $\omega \in Z^k(M)$ we write $[\omega] \in H^k(M)$ for its cohomology class.

REMARK 4.24. By construction, $[\omega]+[\eta] := [\omega+\eta]$ and $[\omega]\wedge[\eta] := [\omega\wedge\eta]$.

One can show that, if M is compact, $H^*(M)$ is finite-dimensional. Also, true for many non-compact spaces, but not always. The cohomology spaces of degree 0 and $\dim(M)$ are easy to compute:

LEMMA 4.25. *Suppose M is a (smooth) manifold of dimension m.*

(1) *$H^0(M) \cong \mathbb{R}^\ell$, where ℓ is the number of connected components of M.*
(2) *If M is compact, connected and oriented, then $H^m(M) \cong \mathbb{R}$.*

PROOF.

(1) $df = 0$ if and only if $f : M \to$ is constant on each connected component of M.
(2) $\int_M : \Omega^m(M) \to \mathbb{R}$ induces by Stokes' Theorem, a linear map
$$H^m(M) \to \mathbb{R},$$
which can be shown to be an isomorphism.

□

Recall that any $C^\infty-$map $f : M \to N$ between manifolds induces an algebra morphism $f^* : \Omega^*(N) \to \Omega^*(M)$. Since $f^* \circ d = d \circ f^*$ by Theorem 3.27, we have $f^*(Z^*(N)) \subset Z^*(M)$ and $f^*(B^*(N)) \subset B^*(M)$, which implies that f^* induces an algebra morphism

$$f^\# : H^*(N) \to H^*(M)$$
$$[\omega] \mapsto [f^*\omega],$$

preserving the degree $f^{\#}(H^k(N)) \subset H^k(M)$. Also, if $g : N \to P$ is another smooth map between manifold, then

$$(g \circ f)^{\#} = f^{\#} \circ g^{\#},$$

since $(g \circ f)^* = f^* \circ g^*$.

PROPOSITION 4.26. *If $f : M \to N$ is a diffemorphism between manifolds, then*

$$f^{\#} : H^*(N) \to H^*(M)$$

is an isomorphism with inverse $(f^{\#})^{-1} = (f^{-1})^{\#}$.

PROOF. The identities $f \circ f^{-1} = \mathrm{Id}_N$ and $f^{-1} \circ f = \mathrm{Id}_M$ imply that

$$(f^{-1})^{\#} \circ f^{\#} = (f \circ f^{-1})^{\#} = \mathrm{Id}_N^{\#} = \mathrm{Id}_{H^*(N)}$$

$$f^{\#} \circ (f^{-1})^{\#} = (f^{-1} \circ f)^{\#} = \mathrm{Id}_M^{\#} = \mathrm{Id}_{H^*(M)}.$$

□

By the previous proposition, diffeomorphic manifolds have isomorphic de Rham cohomology groups. In fact, even more is true: The de Rham cohomology of smooth manifold is a topological invariant. It can be identified with the singular cohomology of M with real coefficients. This shows in particular that homotopic equivalent (hence, in particular homeomorphic), smooth manifolds have isomorphic de Rham cohomology. So tools from topology can be used to compute the de Rham cohomology.

CHAPTER 5

Riemannian Manifolds

5.1. Basic definitions

5.2. Hypersurfaces in $\mathbb{R}^n$

5.3. Riemannian manifolds

5.3.1. Affine connections.

5.3.2. The Levi-Civita connection of a Riemannian manifold.

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