Seminar 3—Global Analysis

1. For a topological space M denote by $C^0(M)$ the vector space of continuous real-valued functions $f : M \to \mathbb{R}$. Any continuous map $F : M \to N$ between topological spaces M and N induces a map $F^* : C^0(N) \to C^0(M)$ given by $F^*(f) := f \circ F : M \to \mathbb{R}$.

   (a) Show that F^* is linear.

   (b) If M and N are (smooth) manifolds, show that $F : M \to N$ is smooth $\iff$ $F^*(C^\infty(N)) \subset C^\infty(M)$.

   (c) If F is a homeomorphism between (smooth) manifolds, show that F is a diffeomorphism $\iff$ $F^*(C^\infty(N)) \subset C^\infty(M)$ and $F^* : C^\infty(N) \to C^\infty(M)$ is an isomorphism.

2. We have seen in the first tutorial that $\mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ of dimension $r(n+m-r)$ in. For $X \in \mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ compute the tangent space

$$T_X\mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m) \subset T_X\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m).$$

3. We have seen in the first tutorial that the Grassmannian manifold $\mathrm{Gr}(r, n)$ can be realized as a submanifold of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of dimension $r(n-r)$. For $E \in \mathrm{Gr}(r, n)$ compute the tangent space

$$T_E\mathrm{Gr}(r, n) \subset T_E\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n).$$

4. Consider the general linear group $\mathrm{GL}(n, \mathbb{R})$ and the special linear group $\mathrm{SL}(n, \mathbb{R})$. We have seen that they are submanifolds of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ (even so called Lie groups) and that $T_{\mathrm{Id}}\mathrm{GL}(n, \mathbb{R}) \cong M_n(\mathbb{R}) = \mathbb{R}^{n^2}$.

   (a) Compute the tangent space $T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R})$ of $\mathrm{SL}(n, \mathbb{R})$ at the identity Id.

   (b) Fix $A \in \mathrm{SL}(n, \mathbb{R})$ and consider the conjugation $\mathrm{conj}_A : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$ by A given by $\mathrm{conj}_A(B) = ABA^{-1}$. Show that conj_A is smooth and compute the derivative $T_{\mathrm{Id}}\mathrm{conj}_A : T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R}) \to T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R})$.

   (c) Consider the map $\mathrm{Ad} : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{Hom}(T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R}), T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R}))$ given by $\mathrm{Ad}(A) := T_{\mathrm{Id}}\mathrm{conj}_A$. Show that Ad is smooth and compute $T_{\mathrm{Id}}\mathrm{Ad}$.

5. Consider $\mathbb{R}^n$ equipped with the standard inner product of signature (p, q) (where $p + q = n$) given by

$$\langle x, y \rangle := \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_i y_i$$

and the group of linear orthogonal transformation of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ given by

$$\mathrm{O}(p, q) := \{A \in \mathrm{GL}(n, \mathbb{R}) : \langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^n\}.$$

(a) Show that

$$\mathrm{O}(p, q) = \{A \in \mathrm{GL}(n, \mathbb{R}) : A^{-1} = I_{p,q} A^t I_{p,q}\},$$

where $I_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$, and that $\mathrm{O}(p, q)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?

(b) Show that $\mathrm{O}(p, q)$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$ with respect to matrix multiplication μ and that $\mu : \mathrm{O}(p, q) \times \mathrm{O}(p, q) \to \mathrm{O}(p, q)$ is smooth (i.e. that $\mathrm{O}(p, q)$ is a Lie group.)

(c) Compute the tangent space $T_{\mathrm{Id}}\mathrm{O}(p, q)$ of $\mathrm{O}(p, q)$ at the identity Id.