

Exercise 4

Core exercises

① Consider the adjoint functors

$$\mathbf{Vect} \xrightleftharpoons[F]{U} \mathbf{Set}$$

Describe a natural Transformation
 $\epsilon: FU \Rightarrow \text{Id}$.

② Consider the Forgetful Functor

$U: \mathbf{Rng} \longrightarrow \mathbf{Set}$
It has a left adjoint. Can you describe it explicitly?

Further exercises

③ (Initial & terminal objects as adjoints)

- let I be the terminal cat
(one object, one identity morphism)
& consider $A \xrightarrow{!} I$ the unique
map.

Show that A has :

- an initial object $\Leftrightarrow !$ has a left adj.
- a terminal object $\Leftrightarrow !$ has a right adj.

④ Let $\text{Set} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{U} \end{matrix} \text{Mon}$ with $W = UF$.

Describe a natural transformation

$$\mu: WW \longrightarrow W.$$

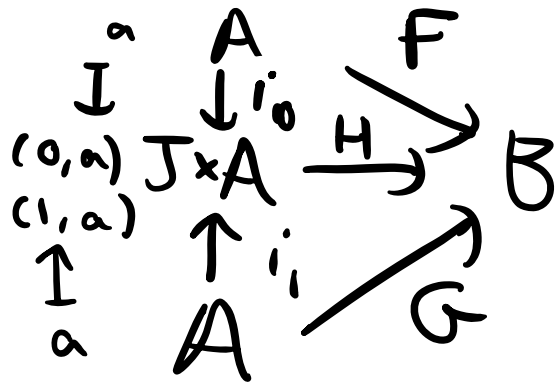
④ (Natural transformations as homotopies.)

- Let A, B be categories. The product category $A \times B$ has objects (a, b) where $a \in A, b \in B$ & morphisms $(a, b) \xrightarrow{(f, g)} (c, d)$ where $f: a \rightarrow c$ & $g: b \rightarrow d$.
- Let $F, G: A \Rightarrow B$ be functors, and $J = \{0 \rightarrow 1\}$.

Show that natural transformations $F \Rightarrow G$ are the same thing as functors

$$H: J \times A \rightarrow B$$

such that
the diagram



commutes.