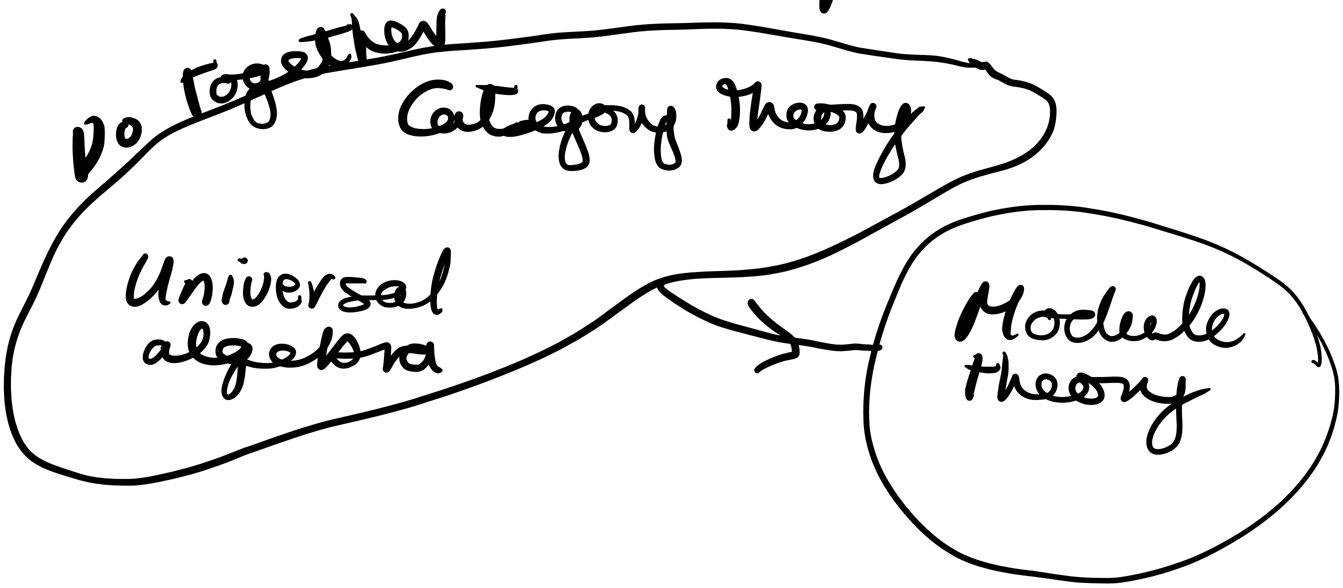
Algebra 3

- John Bourke - bourke@math.muni.cz
- lecture notes + IS each week

- Course has 3 components:

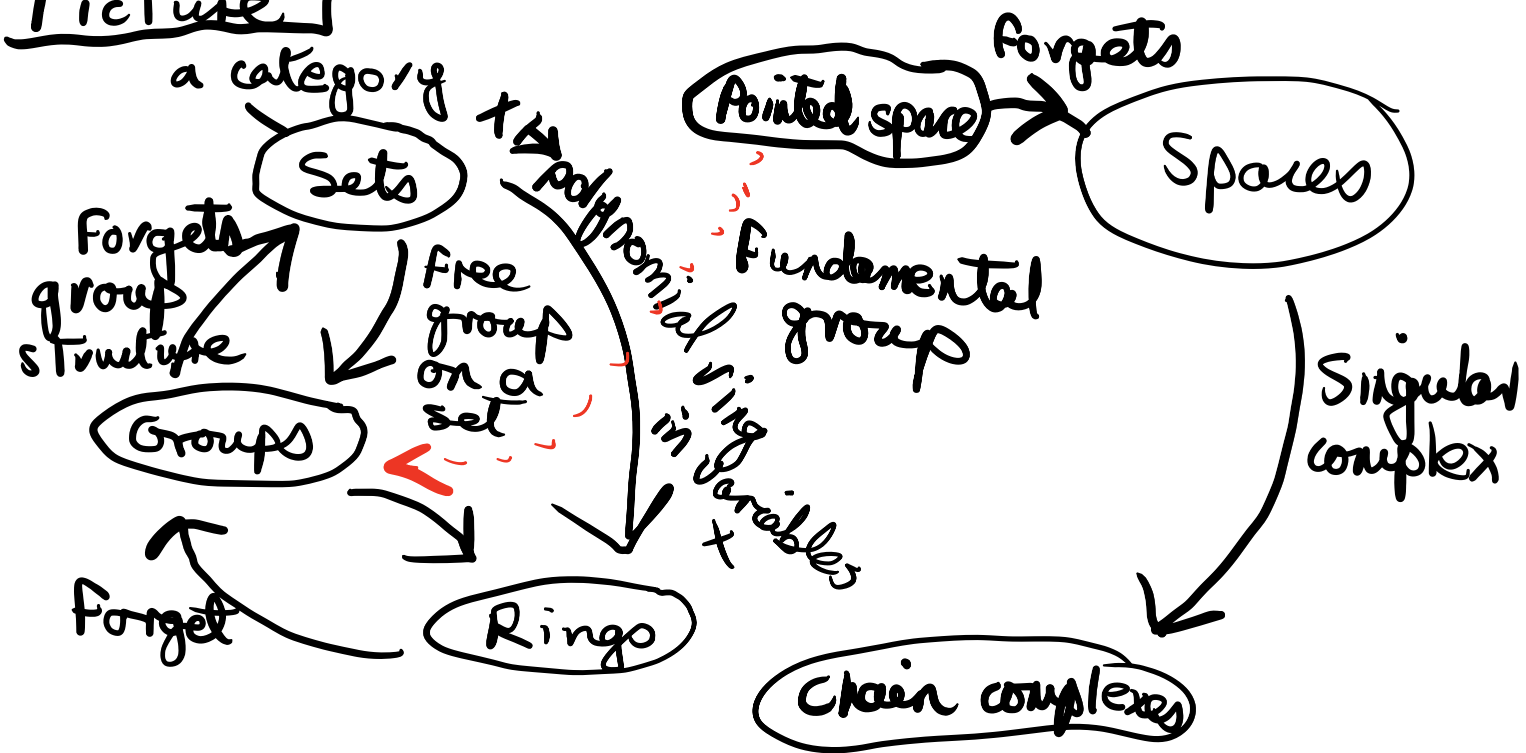


Today: start with category theory.

What is category theory?

- In math, study structures like sets, groups, rings, topological spaces.
- Each forms category of "structures of that type".
- Cat. theory studies the relⁿ between these diff. areas of mathematics (or categories of mathematics)

Picture



- What do these different categories have in common?
- What structure is preserved when we move between them?
- Fundamental notion in category theory is an arrow/morphism:

$$A \longrightarrow B$$

captures relationship between two things.

Def) A category $\mathcal{C}$ consists of :

- a collection $\mathrm{ob}\,\mathcal{C}$ of objects A, B, C,
- & for each pair of objects $A, B \in \mathrm{ob}\,\mathcal{C}$ a collection $\mathcal{C}(A, B)$ of "arrows/morphisms" from A to B,

(depicted $A \xrightarrow{f} B$ or $f : A \to B$ to mean $f \in \mathcal{C}(A, B)$)

- For each $A, B, C \in \mathrm{ob}\,\mathcal{C}$ a function

$$\mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$

$$(B \xrightarrow{g} C, A \xrightarrow{f} B) \longmapsto A \xrightarrow{g \circ f} C$$

called composition.

- For each object $A \in \mathrm{ob}\,\mathcal{C}$ an arrow $1_A : A \longrightarrow A$ called the identity on A.
- These satisfy the following axioms:

Associativity

- Given $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

- Left & right unit laws
- Given $f : A \longrightarrow B$ we have

$$1_B \circ f = f = f \circ 1_A .$$

- Notation: often write $gf : A \to C$ instead of $g \circ f$.
- Associativity & unit laws imply that given

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$$

there is a unique way to compose the arrows $f_n f_{n-1} \cdots f_2 f_1$, indep. of brackets & identities.

Eg: When $n = 4$,

$$((f_4 f_3) f_2) f_1 = (f_4 1_{A_3})((f_3 f_2) f_1).$$

- **Commutative diagrams**

Given a diagram such as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h"'] & & B \arrow[d, "g"] \\ C \arrow[r, "i"] & D \arrow[r, "j"] & E \end{tikzcd}$$

we say the diagram commutes if the paths agree: in this case, $gf = jih$.

- In a larger diagram, eg.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h"'] \arrow[drr, phantom, "="] & & B \arrow[r, "k"] \arrow[d, "g"] \arrow[dr, phantom, "="] & F \arrow[d, "l"] \\ C \arrow[r, "i"] & D \arrow[r, "j"] & E \arrow[r, "m"'] & G \end{tikzcd}$$

diag. commutes

the commutativity of subdiagrams implies the commutativity of the outer diagram.

This is diagram chasing.

Examples

- **Set** : objects are sets, morphisms $f: A \to B$ are functions.
- Given $A \xrightarrow{f} B$ & $B \xrightarrow{g} C$ the function $gf: A \to C$ is def. by $gf(x) = g(f(x))$.
  We have $1_A(x) = x$.

Categories of sets with structure

- **Mon**, the cat of monoids & monoid homomorphisms:
  $f: (A, m_A, e_A) \longrightarrow (B, m_B, e_B)$
  $f(e_A) = e_B$,
  $f(m_A(x,y)) = m_B(fx, fy)$
- **Grp**, the category of groups & group homomorphisms.
- **Rng** ~ rings, ring hom...
  k-**Vect** ~ k-vector spaces, lin. transf.
- These are examples of algebraic categories.

- More generally, given

signature $\underbrace{(\Omega}$, $\underbrace{E)}$ set of equations

we can consider the cat. (Ω, E)-Alg. This captures all of the above examples. Return to this example later.

- Top is the category of topological spaces and continuous functions.

Def) A morphism $f : A \to B \in \mathcal{C}$ is an <u>isomorphism</u> if $\exists\, g : B \to A$ such that $gf = 1_A$ & $fg = 1_B$.

Remark) Can express this via the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "1_A"'] & B \arrow[d, "g"] \arrow[dr, "1_B"] \arrow[dl, phantom, "="] & \\
& A \arrow[r, "f"'] \arrow[ur, phantom, "="] & B
\end{tikzcd}$$

We say that g is the inverse of f and write $g = f^{-1}$.
This is justified by:

Proposition) If $f : A \to B$ is an iso, then its inverse is unique.

Proof) Suppose $g, h : B \to A$ are inverses to f.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "1_B"'] \arrow[drr, bend left=60, "g"] \arrow[drr, bend right=60, "h"'] & A \arrow[d, "f"] \arrow[dr, "1_A"] \arrow[dl, phantom, "="] & \\
& B \arrow[r, "h"'] \arrow[ur, phantom, "="] & A
\end{tikzcd}$$

Therefore $g = h$.

In alg. notation, the corresponding proof is

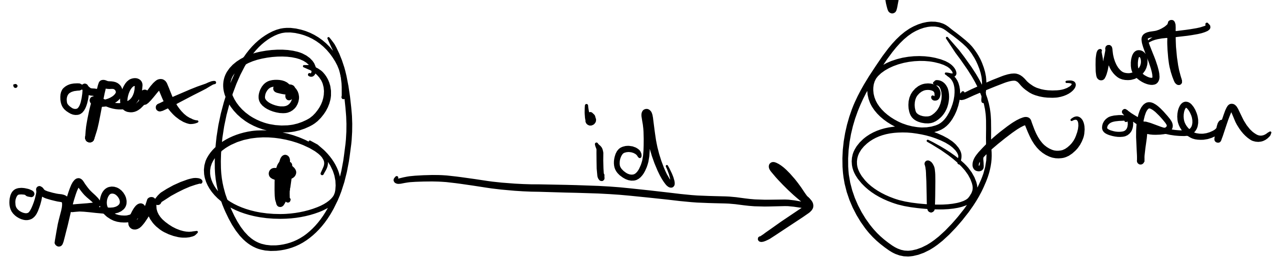
$$g = 1_A \circ g = (h \circ f) \circ g = h \circ (f \circ g)$$
$$= h \circ 1_B = h.$$

Examples

- In **Set**, the isomorphisms $f: A \to B$ are the bijections.
- In other algebraic cats such as **Grp**, an isomorphism is sometimes defined as a bijective homomorphism: this coincides with the categorical notion, since if a homomorphism is bijective its inverse (as a function) is also a homomorphism.
- For **Top**, an isomorphism is a homeomorphism: a cts function with a cts inverse.

In Top, $\exists$ bijective cts maps which are not homeomorphisms

eg.



not a homeomorphism.

- In summary, the categorical def^n of isomorphism captures the correct notion in all of our examples.

Def^n) - A category $\mathcal{C}$ is said to be locally small if for all $A, B \in \mathcal{C}$ the collection $\mathcal{C}(A,B)$ is a set.

- If, furthermore, the coll^n ob $\mathcal{C}$ is a set, we say that it is small.

Examples of locally small cats

- Set is not small: one cannot form a "set of all sets". However, it is locally small as the collection Set(A,B) of functions from A to B forms a set.
- All of the examples considered so far are locally small, but not small.

Examples of small categories

Ex 1) - $\emptyset$ empty cat

- $\mathbb{1} = (\bullet)$ 1 object, 1 identity morph
- plus identities

$$\boxed{0 \longrightarrow 1}\ ,\quad \boxed{\begin{tikzcd} a \arrow[r, "f"] \arrow[rr, bend right, "h"'] & b \arrow[r, "g"] & c \end{tikzcd}}\ , \ldots$$

$\boxed{\text{Ex. 2}}$ Preorders & posets

- Preorder $(X, \leq)$
  satisfying $x \leq x$
  $x \leq y$ & $y \leq z \Rightarrow x \leq z$
  Poset : also $x \leq y$ & $y \leq x \Rightarrow x = y$.
- If $(X, \leq)$ a preorder,
  can form a category X^*
  whose objects are elements of X
  & such that there exists a single
  morph. $x \to y \iff x \leq y$ &
  no morphisms from x to y
  otherwise.
- In fact,
  Preorders $\overset{\text{same as}}{\equiv}$ small cats with at most 1 morph. between any two objects.
  hopefully later in course
  (equivalence of cats)

Example 3

mult, unit

If $(M, \times, e)$ is a monoid we can form a category ΣM w/ one object $\bullet$ & whose morphisms $\bullet \xrightarrow{m} \bullet$ are the elements of M.

We compose by
$$\begin{tikzcd} \bullet \arrow[r, "m"] \arrow[dr, "n+m"'] & \bullet \arrow[d, "n"] \\ & \bullet \end{tikzcd} \qquad \bullet \xrightarrow[1]{e} \bullet$$

unit is $1_\bullet$

& assoc. & unit laws for monoid $\Rightarrow$ axioms for a category.

- If $(M, \times, e)$ is a group (all elts. have inverse) then ΣM is a groupoid : a category in which all morphisms are isomorphisms

- Monoids $\equiv$ 1-object small cats
  $\cup$ $\qquad\qquad$ $\cup$
  Groups $\equiv$ 1-ob. small groupoids

- Natural : e.g. symmetric group S_n consists of the bijections $f \circlearrowright \bar{n} = \{1, \ldots, n\}$.

Groups of transformations $\sim$ symmetries of an object.

Functors

Def^n) Let $\mathcal{A}, \mathcal{B}$ be categories. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

- a function $\mathrm{ob}\,\mathcal{A} \longrightarrow \mathrm{ob}\,\mathcal{B}$ which we write as $X \longmapsto FX$,
- for each pair $X, Y \in \mathcal{A}$ a function $\mathcal{A}(X,Y) \longrightarrow \mathcal{B}(FX, FY)$ written as $X \xrightarrow{\alpha} Y \longmapsto FX \xrightarrow{F\alpha} FY$.

This satisfies the axioms:

- given $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \in \mathcal{A}$, we have $F(\beta \circ \alpha) = F\beta \circ F\alpha$.
- given $X \in \mathcal{A}$ we have $F(1_X) = 1_{FX}$.

In other words, a functor sends obs to obs, arrows to arrows (respecting their domain & codomain) & preserves identities and composition.

Examples

- Forgetful functors:

Eg. $U : \mathrm{Grp} \longrightarrow \mathrm{Set}$

$$(X, \underset{\text{mult.}}{m}, \underset{\text{unit}}{e}) \longmapsto X$$

which sends a group to its underlying set, ie. forgets the group structure.

Sim. it sends a group homomorphism to its underlying function:

$$f : (X, m_X, e_X) \longrightarrow (Y, m_Y, e_Y) \longmapsto f : X \to Y$$

Clearly preserves composition & identities.

- Similarly, there are forgetful functors from cats of "sets with structure"

to Set, eg
$U : \mathbf{Top} \xrightarrow{U} \mathbf{Set}$, $U : k\text{-}\mathbf{Vect} \longrightarrow \mathbf{Set}$
or Rng, Mon etc...
Similarly, $U : \mathbf{Rng} \longrightarrow \mathbf{Mon}$
$(R, +, \times, 0, 1) \longmapsto (R, \times, 1)$.

There is a functor $F : \mathbf{Set} \longrightarrow \mathbf{Mon}$
where $FX =$ "list/free" monoid on X
- its elements are (possibly empty) lists $[x_1, x_2, \dots, x_n]$ of elements of X with unit $[-]$ empty list.
- Composition of lists is "concatenation":
$[x_1, \dots, x_n].[y_1, \dots, y_m] = [x_1, \dots, x_n, y_1, \dots, y_m]$.
- This makes FX a monoid.
- Given a function $f : X \longrightarrow Y$ obtain a monoid homomorphism
$$Ff : FX \longrightarrow FY$$
$$[x_1, \dots, x_n] \longmapsto [fx_1, \dots, fx_n]$$
& easy to check that this satisfies axioms for a functor.

Example) Let G be a group & consider its suspension ΣG as a groupoid with one object $\bullet$ & $\Sigma G(\bullet, \bullet) = G$.
Then a functor $\Sigma G \longrightarrow \mathbf{Set}$ is what?

$$\begin{array}{c} \bullet \\ g \downarrow \\ \bullet \end{array} \longmapsto \begin{array}{c} X \\ \downarrow g.- \\ X \end{array} \quad \begin{array}{c} x \\ \downarrow \\ g.x \end{array}$$

The functor axioms say $e.x = x$ & $h.(g.x) = (h.g).x$.

Therefore a functor $\Sigma G \longrightarrow \mathrm{Set}$ is exactly a <u>G-set</u>: a set X with an action of the group G.

What is a functor $\Sigma G \longrightarrow \mathrm{Vect}$?

Example Let Top_* be category of pointed spaces (X, a)

top space — $a \in X$ basepoint

& continuous maps which preserve basepoint.

There is a functor $\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$ where $\pi_1(X, a)$ = fundamental group of loops around a (up to homotopy)

eg.

$\pi_1(S_1, *) = \mathbb{Z}$.

There are also functors $H_n : \mathrm{Top} \longrightarrow \mathrm{Ab}$ ~ abelian group

sending a top. space to its n'th homological group.

Example : CAT

- Given functors $A \xrightarrow{F} B \xrightarrow{G} C$ we can compose them in obvious way:

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & B \arrow[r, "G"] & C \end{tikzcd}$$

$$x \longmapsto G(F(x)) = GFX$$

$$X \xrightarrow{\alpha} Y \longmapsto GFX \xrightarrow{GF\alpha} GFY$$

- Likewise, we have identity functor
$1_A : A \to A : x \mapsto x$
- In this way, we obtain a (large) category CAT of categories & functors. In fact, this is a 2-category!