Algebra 3

- John Bourke - bourke@math.muni.cz
- lecture notes + 1S each week

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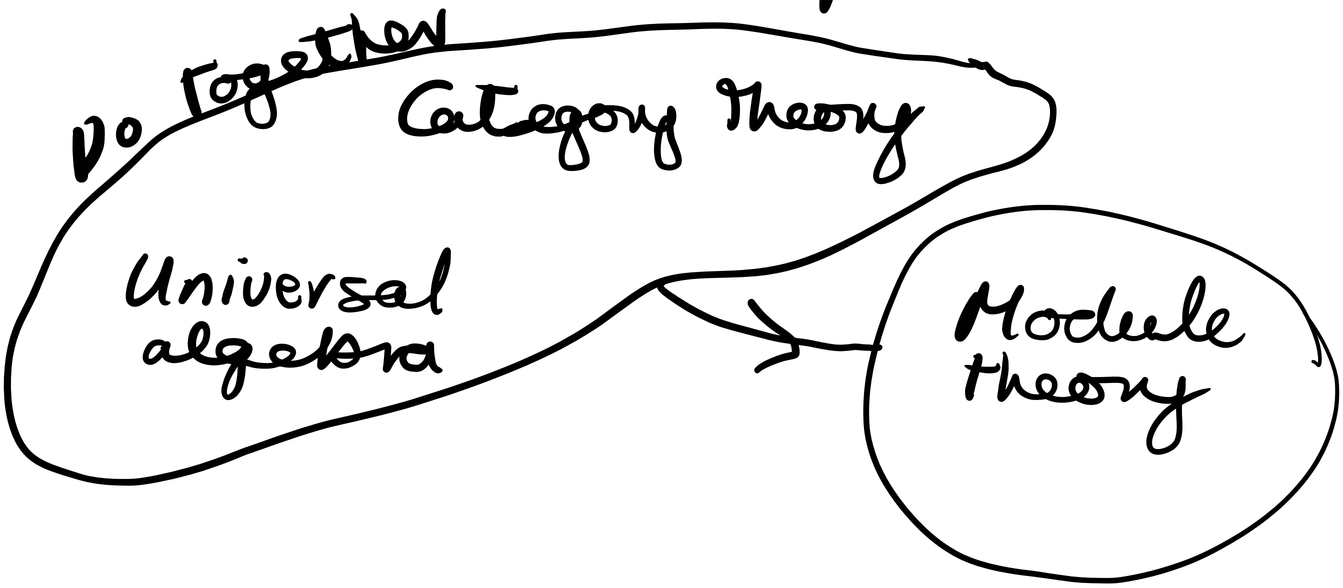

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Do together

Category theory

Universal algebra

Module theory

Today: start with category theory.

What is category theory?

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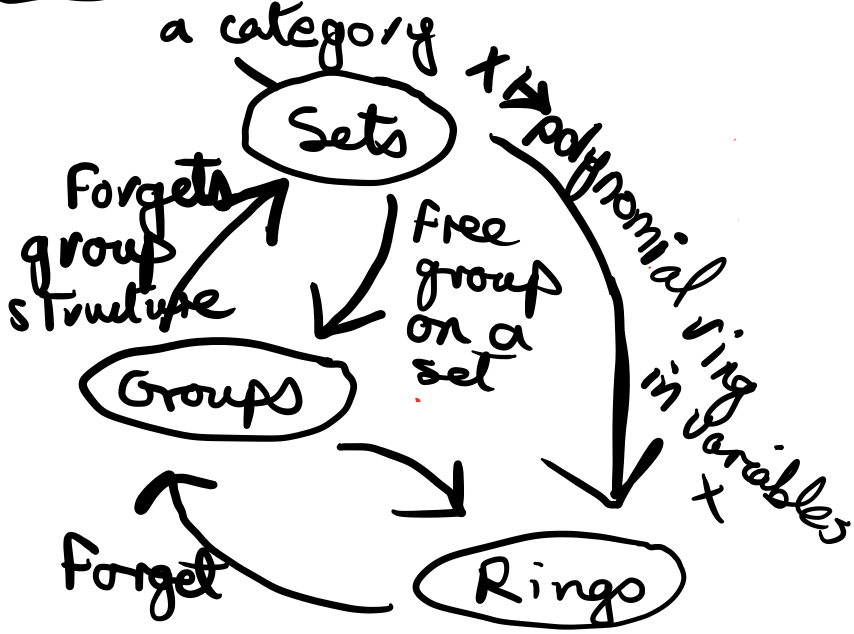

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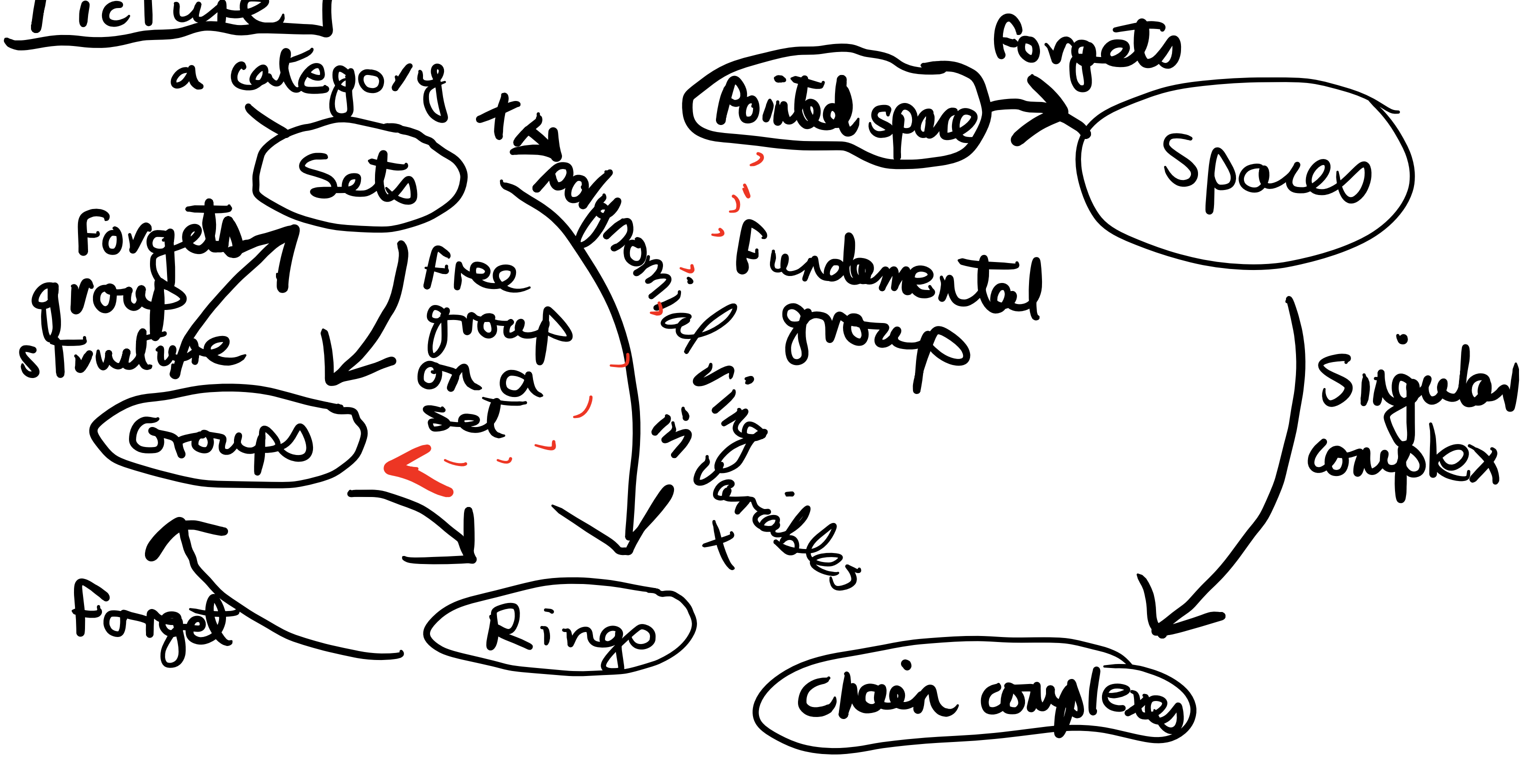

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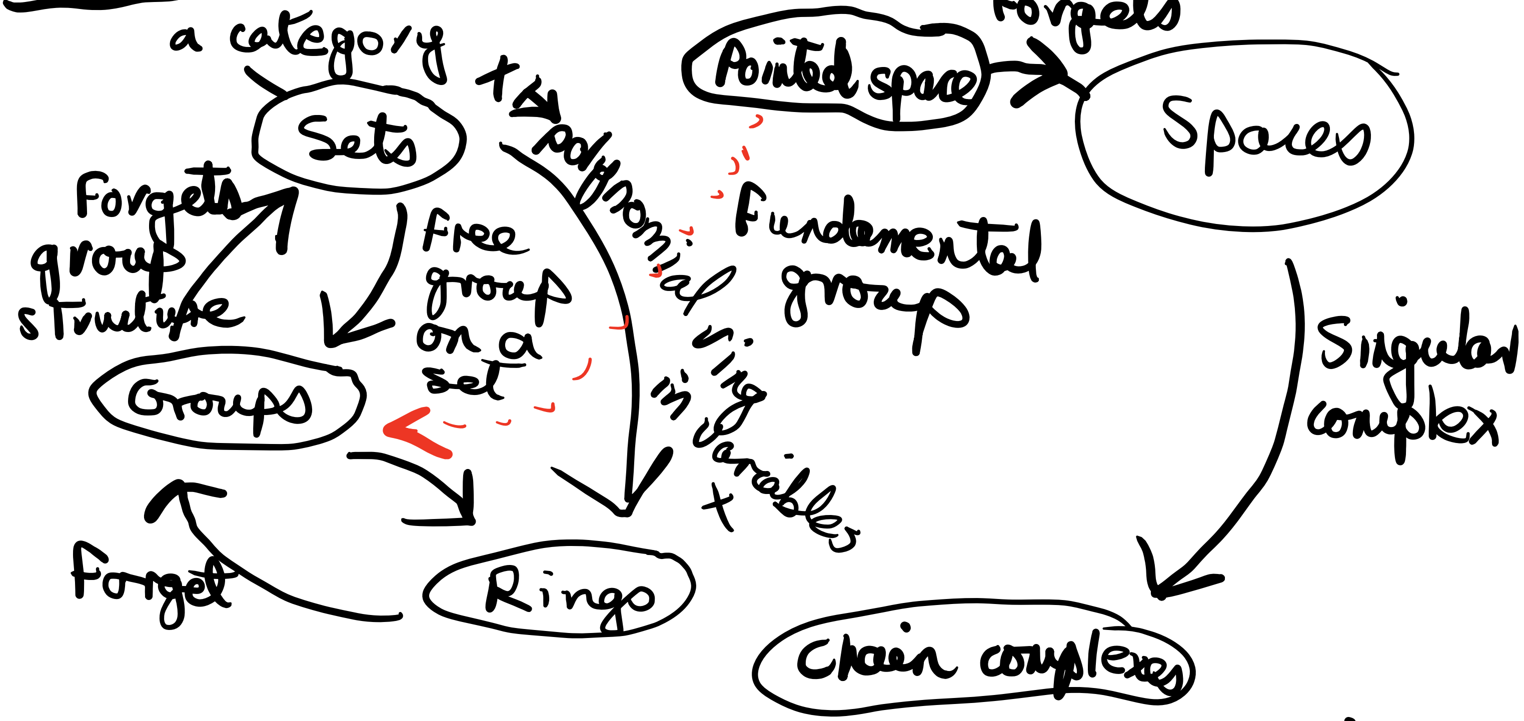

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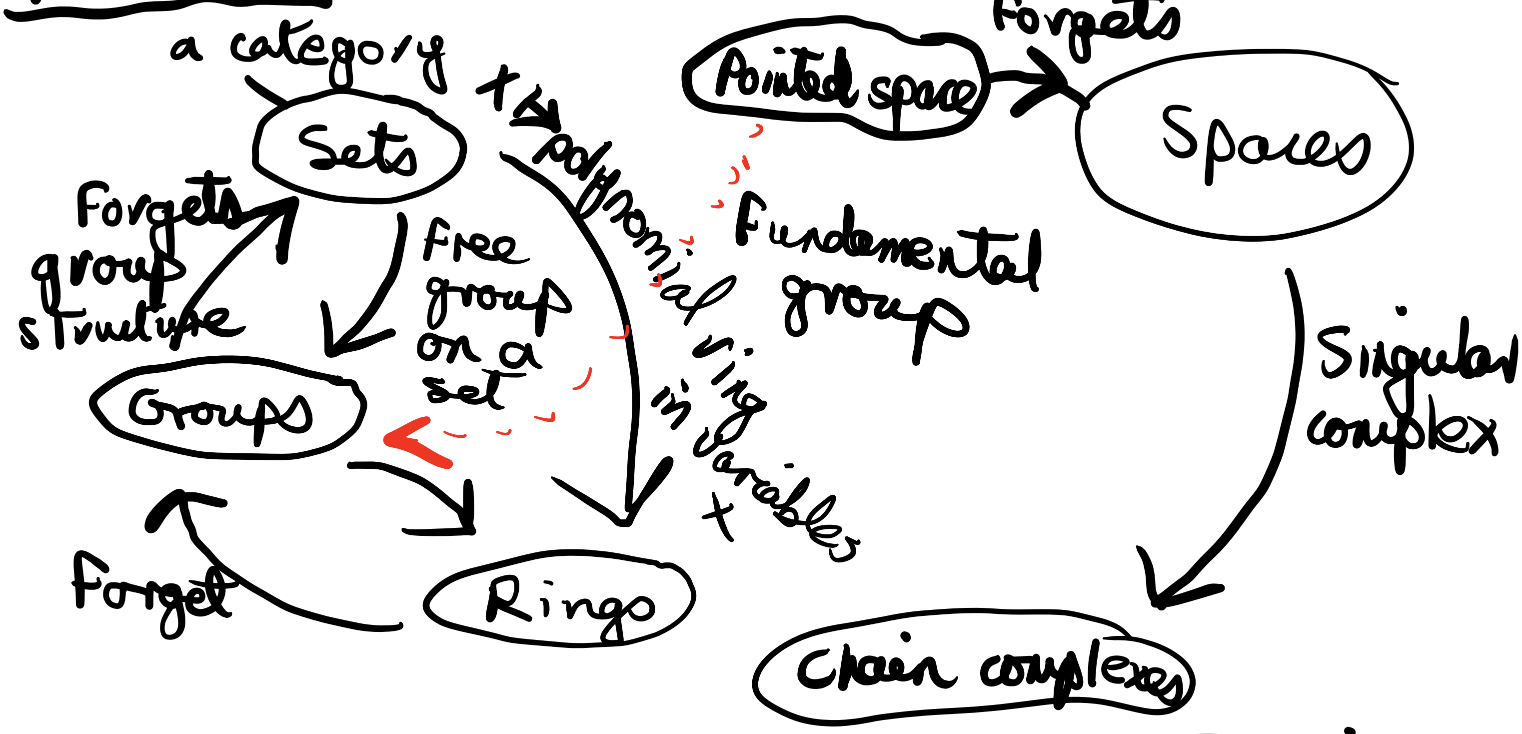

- What do these different categories have in common?
- What structure is preserved when we move between them?
- Fundamental notion in category theory is an arrow/morphism:

$$A \longrightarrow B$$

captures relationship between two things

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- Left & right unit laws
- Given $f : A \longrightarrow B$ we have

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Eg: When $n=4$,

$$((f_4 f_3) f_2) f_1 = (f_4 1_{A_3})((f_3 f_2) f_1).$$

- Commutative diagrams

Given a diagram such as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "i"] & D \arrow[r, "j"] & E \end{tikzcd}$$

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- In a larger diagram, eg.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h"'] \arrow[drr, phantom, "="] & & B \arrow[r, "k"] \arrow[d, "g"] \arrow[dr, phantom, "="] & F \arrow[d, "l"] \\ C \arrow[r, "i"] & D \arrow[r, "j"] & E \arrow[r, "m"'] & G \end{tikzcd}$$

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This is <u>diagram chasing</u>.

Examples

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- Given $A \xrightarrow{f} B$ & $B \xrightarrow{g} C$ the function

  $gf : A \to C$ is def. by

  $gf(x) = g(f(x))$.

  We have $1_A(x) = x$.

Categories of sets with structure

- **Mon**, the cat of monoids & monoid homorphisms:

  $f : (A, m_A, e_A) \longrightarrow (B, m_B, e_B)$

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- More generally, given

$$\underbrace{\Omega}_{\text{signature}}, \underbrace{E}_{\text{set of equations}}$$

$$(\Omega, E)$$

we can consider the cat (Ω, E)-Alg. This captures all of the above examples. Return to this ex later.

- <u>Top</u> is the category of topological spaces and continuous functions.

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Proof) Suppose $g, h: B \to A$ are inverses to f.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "1_B"'] \arrow[drr, bend left=60, "g"] \arrow[drr, bend right=70, "h"'] & A \arrow[d, "f"] \arrow[dr, "1_A"] \arrow[dl, phantom, "="] & \\
& B \arrow[r, "h"'] \arrow[ur, phantom, "="] & A
\end{tikzcd}$$

Therefore $g = h$.

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- In other algebraic cats such as Grp, an isomorphism is sometimes defined as a bijective homomorphism: this coincides with the categorical notion, since if a homomorphism is bijective its inverse (as a function) is also a homomorphism.

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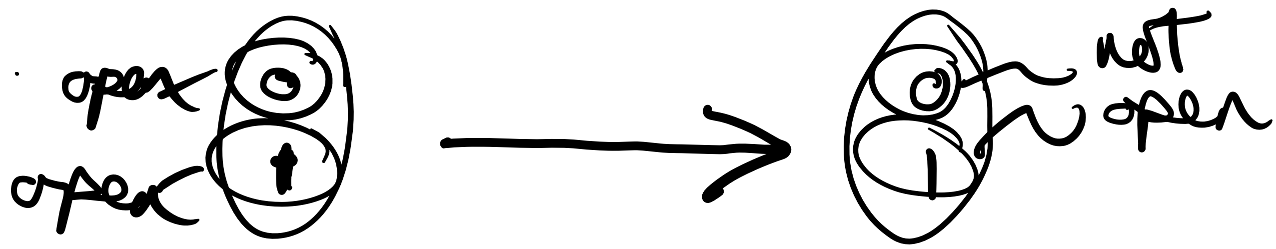

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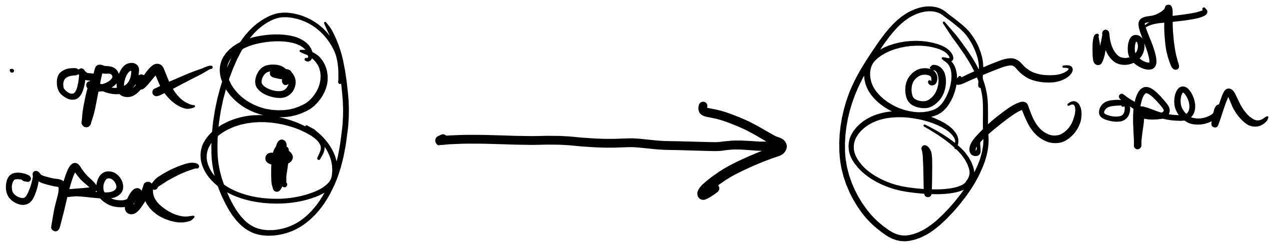

- In summary, the categorical def^n of isomorphism captures the correct notion in all of our examples.

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- Set is not small: one cannot form a "set of all sets". However, it is locally small as the collection Set(A,B) of functions from A to B forms a set.

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- All of the examples considered so far are locally small, but not small.

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- If $(X, \leq)$ a preorder, can form a category X^* whose objects are elements of X & such that there exists a single morph. $x \to y \iff x \leq y$ & no morphisms from x to y otherwise.

$\boxed{\text{Example 3}}$

If $(M, \underset{\text{mult}}{\times}, \underset{\text{unit}}{e})$ is a monoid we can form a category ΣM w/ one object $\bullet$ & whose morphisms $\bullet \xrightarrow{m} \bullet$ are the elements of M.

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- Monoids $\equiv$ 1-object small cats
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  Groups $\equiv$ 1-ob. small groupoids

- Natural : e.g. symmetric group S_n consists of the bijections $f \circlearrowright \bar{n} = \{1, \dots, n\}$.

  Groups of transformations $\sim$ symmetries of an object.

Functors

Def^n) Let $\mathcal{A}, \mathcal{B}$ be categories. A <u>functor</u> $F : \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

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- for each pair $X, Y \in \mathcal{A}$ a function $\mathcal{A}(X, Y) \longrightarrow \mathcal{B}(FX, FY)$ written as $X \xrightarrow{\alpha} Y \longmapsto FX \xrightarrow{F\alpha} FY$.

Functors

Def^n) Let $\mathcal{A}, \mathcal{B}$ be categories. A <u>functor</u> $F : \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

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This satisfies the axioms:

- given $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \in \mathcal{A}$, we have $F(\beta \circ \alpha) = F\beta \circ F\alpha$.
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In other words, a functor sends obs to obs, arrows to arrows (respecting their domain & codomain) & preserves identities and composition.

Examples

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  Eg. $U : \mathrm{Grp} \longrightarrow \mathrm{Set}$

  $\underset{}{(X, \underbrace{m}_{\text{mult.}}, \underbrace{e}_{\text{unit}})} \longmapsto X$

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Similarly, $U: \mathrm{Rng} \longrightarrow \mathrm{Mon}$

$(R, +, \times, 0, 1) \longmapsto (R, \times, 1)$.

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- Given a function $f : X \longrightarrow Y$ obtain a monoid homomorphism

$$Ff : FX \longrightarrow FY$$
$$[x_1, \ldots, x_n] \longmapsto [fx_1, \ldots, fx_n]$$

& easy to check that this satisfies axioms for a functor.

Example) Let G be a group & consider its suspension ΣG as a groupoid with one object $\bullet$ & $\Sigma G(\bullet,\bullet) = G$.

Then a functor $\Sigma G \longrightarrow \mathrm{Set}$ is what?

$$\begin{array}{ccc} \bullet & & X \\ g\downarrow & \longmapsto & \downarrow g\cdot - \\ \bullet & & X \end{array} \qquad \begin{array}{c} x \\ \downarrow \\ g\cdot x \end{array}$$

The functor axioms say $e.x = x$ & $h.(g.x) = (h.g).x$

Therefore a functor $\Sigma G \longrightarrow \mathrm{Set}$ is exactly a G-set: a set X with an action of the group G.

What is a functor $\Sigma G \longrightarrow \mathrm{Vect}$?

$\boxed{\text{Example}}$ Let Top_* be category of pointed spaces (X, a)

top space ... $a \in X$ basepoint

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There are also functors $H_n : Top \longrightarrow Ab$ (abelian group) sending a top. space to its n'th homological group.

Example : CAT

- Given functors $A \xrightarrow{F} B \xrightarrow{G} C$ we can compose them in obvious way :

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[rr, bend right, "G\circ F"'] & B \arrow[r, "G"] & C \end{tikzcd}$$

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- In this way, we obtain a (large) category CAT of categories & functors. In fact, this is a 2-category!