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- It is a limit cone (or just that L is the limit) if given a cone $(k_i : A \longrightarrow D_i)_{i \in J}$ $\exists!\ k : A \longrightarrow L$ such that $p_i \circ k = k_i$ for all i.

What do adjoint functors preserve?

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- If $U : A \to \mathcal{B}$ is a functor it takes the cone $(p_i : L \to D_i)_{i \in J}$ to a cone $(Up_i : UL \to UD_i)_{i \in J}$ for UD.

Def) We say that U <u>preserves the limit L</u> of D if the cone $(UL \xrightarrow{Up_i} UD_i)_{i\in J}$ is a limit cone.

- Similarly, we can speak of a functor preserving colimits.

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We must show that $(UL \xrightarrow{Up_i} UDi)_{i \in J}$ is a limit cone.

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Proof) Consider

$$\begin{tikzcd} J \arrow[r, "D"] & A \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & B \arrow[l, shift right, "F"'] \end{tikzcd}$$

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We must show that $(UL \xrightarrow{Up_i} UDi)_{i \in J}$ is a limit cone.

- Consider cone $(x \xrightarrow{k_i} UDi)_{i \in J}$.

- Using bijections $A(Fx, Di) \xrightarrow{\varphi} B(x, UDi)$ we obtain maps

$Fx \xrightarrow{\varphi^{-1} k_i} Di$ & claim these form a cone to D:

we must show

$$\begin{tikzcd}
 & D_i \arrow[dd, "D\alpha"] \\
FX \arrow[ur, "\varphi^{-1}k_i"] \arrow[dr, "\varphi^{-1}k_j"'] \arrow[r, phantom, "="] & \\
 & D_j
\end{tikzcd}$$

we must show

$$\begin{tikzcd} & D_i \arrow[d, "D_\alpha"] \\ FX \arrow[ur, "\varphi^{-1}k_i"] \arrow[dr, "\varphi^{-1}k_j"'] \arrow[r, phantom, "="] & D_j \\ & \end{tikzcd}$$

but this is equivalent to showing images of these maps $Fx \rightrightarrows D_j$ under $A(FX, D_j) \xrightarrow{\varphi} B(X, UD_j)$ are equal.

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- $\varphi(D\alpha \circ \varphi^{-1}k_i) \underset{\text{by naturality of } \varphi}{=} UD\alpha \circ \varphi\varphi^{-1}k_i = UD\alpha \circ k_i$

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by naturality of φ

so their images are the two paths of

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which agree, since the k_i are a cone

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- Since the maps $\varphi^{-1}k_i : FX \longrightarrow D_i$ form a cone we obtain a unique $\ell : FX \longrightarrow L$ such that

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for all i.

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such that $*$

$$\begin{tikzcd} FX \arrow[r, "l"] \arrow[dr, "\varphi^{-1}k_i"'] \arrow[dr, phantom, bend left=10, "="] & L \arrow[d, "p_i"] \\ & D_i \end{tikzcd}$$

for all i.

- Using the bijection $\mathcal{A}(FX, L) \xrightarrow{\varphi} \mathcal{B}(X, UL)$ this corresponds to a map

$X \xrightarrow{\varphi\ell} UL$ & the equations ⊛

$X \xrightarrow{\eta_L} UL$ & the equations $\circledast$ corresp. to the equations

$(\ast\ast)$
$$\begin{tikzcd} X \arrow[r, "\eta_L"] & UL \arrow[d] & Up_i \\ K_i \arrow[rr] & & UD_i \end{tikzcd}$$
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- In partic, $\varphi_\ell : X \to UL$ is the unique map st. (**) commutes; therefore $(UL \xrightarrow{Up_i} UD_i)_{i \in J}$ is a <u>limit cone</u>. $\square$

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- For example, $U : \mathbf{Grp} \to \mathbf{Set}$ preserves products, equalisers, terminal object etc. More generally, forgetful functors from algebraic cats to **Set** preserve all limits.

Dually

Theorem

Left adjoints preserve colimits.

Exercise

Prove that forgetful functor
$U : \mathrm{Grp} \longrightarrow \mathrm{Set}$ does
not have a right adjoint
(ie. is not a left adjoint.)

Example

- Let **Field** = cat. of fields & homomorphisms : preserve addition, mult., 0 & 1.

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- Let **Field** = cat. of fields & homomorphisms : preserve addition, mult., 0 & 1.
- Fields are commutative rings s.t. $0 \neq 1$ & $x \neq 0 \Rightarrow \exists y : xy = 1$

These equations involve negation

$\Rightarrow$ not universal algebra

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So it suffices to show Field does not have an initial object.

- Firstly, let $f: R \to S$ be a field homomorphism. We claim f is injective.

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- Firstly, let $f: R \to S$ be a field homomorphism. We claim f is injective. Indeed, suppose $fx = 0$ for $x \neq 0$. Then $\ker f \hookrightarrow R$ is an ideal of R, non-zero, so as R is a field, $\ker f = R$. Therefore $f1 = 0$ so $1 = f1 = 0$ which is a contradiction; hence f is injective.

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for p, q coprime.

- Since $F \hookrightarrow \mathbb{Z}_p, \mathbb{Z}_q$ are injective they reflect equations $p.1 = 0$ & $q.1 = 0$, so these equations hold in F.

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Hence in F

$$1 = 1.1 = (np + mq).1$$
$$= n(p1) + m(q1) =$$
$$n0 + m0 = 0.$$

Hence $\underline{1 = 0}$ in F, so <u>F not a field</u>. $\square$

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<u>Proof</u> Let $\mathcal{U} : \mathcal{A} \longrightarrow \mathcal{B}$ have left adj. F, and consider <u>mono</u> $a \xrightarrow{f} b \in \mathcal{A}$.

Consider $x \underset{v}{\overset{u}{\rightrightarrows}} Ua \xrightarrow{Uf} Ub$

satisfying $Uf.u = Uf.v$. We must show $u = v$.

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$$Fx \underset{\varphi^{-1}v}{\overset{\varphi^{-1}u}{\rightrightarrows}} a \xrightarrow{f} b$$
& the diagram commutes by naturality of φ.

Since f is mono, therefore $\varphi^{-1}u = \varphi^{-1}v$.

Therefore $u = v$ so that uF is mono, as claimed. $\square$

Dually

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Fully & Faithful Functors

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$\boxed{\text{Def}^n}$ - A subcategory A of B is a subcollection of objects and arrows in B closed under identities and composition.
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Examples

- $U : \mathbf{Grp}, \mathbf{Mon}, \mathbf{Ring} \longrightarrow \mathbf{Set}$ are all faithful.

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for each $b \in B$, an object $Rb \in A$ & morphism $\eta_b : b \longrightarrow Rb$ such that, given $f : b \to c$ with $c \in A$,
$\exists!\ Rb \xrightarrow{\bar{f}} c$ such that

$$\begin{tikzcd} Rb \arrow[dr, "\bar{f}"] & \\ b \arrow[u, "\eta_b"] \arrow[r, "f"'] \arrow[ur, phantom, "="] & c \end{tikzcd}$$
.

Additional material: not compulsory.

Horizontal composition

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define $\begin{tikzcd} A \arrow[r, bend left, "HF"] \arrow[r, bend right, "HG"'] \arrow[r, phantom, "H\eta \Downarrow"] & C \end{tikzcd}$ to be the

nat. transf. with components:
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$$\eta_{FX} : GFX \longrightarrow HFX \text{ at } X \in A.$$

Given
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{B} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "I"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{C} \end{tikzcd}$$

we have two ways of defining a composite, as either path in

$$\begin{tikzcd} HF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "H\alpha"'] & IF \arrow[d, Rightarrow, "I\alpha"] \\ HG \arrow[r, Rightarrow, "\beta G"'] & IG \end{tikzcd}$$

(i.e. $I\alpha \cdot \beta F$) ;

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(ie. $I\alpha . \beta F$) ;

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at the morphism $\alpha_X : FX \to GX$.

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we have two ways of defining a composite, as either path in

$$\begin{tikzcd} HF \arrow[r, Rightarrow, "\beta F"] \arrow[d, Rightarrow, "H\alpha"'] & IF \arrow[d, Rightarrow, "I\alpha"] \\ HG \arrow[r, Rightarrow, "\beta G"'] & IG \end{tikzcd}$$ (i.e. $I\alpha . \beta F$) ;

these agree by naturality of β :

$$\begin{tikzcd} HFX \arrow[r, "\beta FX"] \arrow[d, "H\alpha_X"'] & IFX \arrow[d, "I\alpha_X"] \\ HGX \arrow[r, "\beta GX"'] & IGX \end{tikzcd}$$

at the morphism $\alpha_X : FX \to GX$.

The resulting nat t.
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "HF"] \arrow[r, bend right=50, "IG"'] \arrow[r, phantom, "\beta * \alpha \Downarrow"] & \mathcal{C} \end{tikzcd}$$

is called horizontal composite.

Given $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{C} \end{tikzcd}$

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$\boxed{\text{Remark}}$ Categories, Functors & natural transformations form a 2-category!

① The Triangle equations

Theorem

An adjunction $\begin{tikzcd} A \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "F"'] \end{tikzcd}$

is specified by a pair of functors as above + natural transformations

$$\begin{tikzcd} B \arrow[rr, "1"] \arrow[dr, "F"'] & \arrow[d, phantom, "\eta \Downarrow"] & B \\ & A \arrow[ur, "U"'] & \end{tikzcd} \quad \& \quad \begin{tikzcd} & B \arrow[dr, "F"] \arrow[d, phantom, "\varepsilon \Downarrow"] & \\ A \arrow[ur, "U"] \arrow[rr, "1"'] & {} & A \end{tikzcd}$$

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i.e.

$$\begin{tikzcd} & FUFb \arrow[d, "\varepsilon_{Fb}"] \\ Fb \arrow[ur, "F\eta_b"] \arrow[r, "1"', "="] & Fb \end{tikzcd} \quad \& \quad \begin{tikzcd} & UFUa \arrow[d, "U\varepsilon_a"] \\ Ua \arrow[ur, "\eta_{Ua}"] \arrow[r, "1"', "="] & Ua \end{tikzcd}.$$

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$$\begin{tikzcd} a \arrow[rr, "\varphi\beta"] \arrow[dr, "\eta_a"'] & {''} & Ub \\ & UFa \arrow[ur, "U\beta"'] & \end{tikzcd}$$

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$$1_{Ua} \longmapsto \varepsilon_a : FUa \to a \longmapsto Ua \xrightarrow{\eta_{Ua}} UFUa \xrightarrow{U\varepsilon_a} Ua$$

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~~Proof~~ (Sketch)

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- We defined the unit $1 \Rightarrow UF$ last time.
- Similarly

$$\begin{array}{ccc} \mathcal{B}(Ua, Ua) & \xrightarrow{\varphi^{-1}} & \mathcal{A}(FUa, a) \\ 1 & \longmapsto & FUa \xrightarrow{\varepsilon_a} a \end{array}$$

determines the counit.

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$\square$

Remark) This def^n of adjunction via unit & counit + triangle equations captures sense in which adjunction is a weaker form of iso/equivalence.