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$$\begin{array}{ccc} & \xrightarrow{p_i} & D_i \\ \langle & \downarrow \alpha & \text{for } i \\ p_j & \searrow & D_j \end{array}$$

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$$\begin{array}{ccc} & p_i \nearrow & D_i \\ L & \xrightarrow{\quad \Downarrow \rho_\alpha \quad \text{for}} & i \\ & p_j \searrow & D_j \\ & & j \end{array}$$

- it is a limit cone (or just that L) if given a cone $(k_i : A \rightarrow D_i)_{i \in J}$ $\exists ! \ k : A \rightarrow L$ such that $p_i \circ k = k_i$ for all i .

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$$\begin{array}{ccc} & \xrightarrow{p_i} & D_i \\ L & \xrightarrow{\quad \Downarrow \rho \alpha \quad} & \downarrow \alpha \\ & \xrightarrow{p_j} & D_j \end{array}$$
- It is a limit cone (or just that L) if given a cone $(k_i: A \rightarrow D_i)_{i \in J}$
 $\exists! \ k: A \rightarrow L$ such that $p_i \circ k = k_i$ for all i .
- If $U: A \rightarrow B$ is a functor it takes the cone $(p_i: L \rightarrow D_i)_{i \in J}$ to a cone $(U p_i: UL \rightarrow UD_i)_{i \in J}$ for UD .

Def) We say that u preserves
the limit L of D if the
cone $(u_L \xrightarrow{u_{pi}} uD_i)_{i \in J}$
is a limit cone.

- Similarly, we can speak of
a functor preserving
colimits.

Theorem) Right adjoints
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Proof) Consider

$$J \xrightarrow{D} A \begin{array}{c} \xleftarrow{\perp} \\[-1ex] \xrightarrow{u} \end{array} B \text{ & a}$$

limit cone $(\text{L}^F_i \rightarrow D_i)_{i \in J}$.

We must show that $(\text{UL}^F_i \rightarrow UD_i)_{i \in J}$
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Proof) Consider

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limit cone $(\lim^F_i D_i)_{i \in J}$.

We must show that $(\lim^{uf}_i UD_i)_{i \in J}$
is a limit cone.

- consider cone $(x \xrightarrow{\kappa_i} UD_i)_{i \in J}$.

- using bijections $A(Fx, D_i) \xrightarrow{\cong} B(x, UD_i)$
we obtain maps

$Fx \xrightarrow{\varphi^{-1}\kappa_i} D_i$ & claim These form a cone
to D:

we must show

$$F_X \xrightarrow{\varphi^{-1}k_i} D_i$$
$$\Downarrow D\alpha$$
$$\varphi^{-1}k_j \xrightarrow{} D_j$$

we must show

$$F_x \xrightarrow{\varphi^{-1}k_i} D_i \quad \text{but this is equivalent to}$$
$$\Downarrow D\alpha$$

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showing images of these maps
 $F_x \xrightarrow{} D_j$ under
 $A(F_x, D_j) \xrightarrow{\varphi} B(X, UD_j)$
are equal.

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Well $\varphi\varphi^{-1}k_j = k_j$.

we must show

$$\begin{array}{ccc} \varphi^{-1}k_i & \rightarrow & D_i \\ Fx & \xrightarrow{\quad \cong \quad} & UD\alpha \\ \varphi^{-1}k_j & \rightarrow & D_j \end{array}$$

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- Well $\varphi \varphi^{-1}k_j = k_j$.

- $\varphi(D\alpha \circ \varphi^{-1}k_i) = UD\alpha \circ \varphi \varphi^{-1}k_i = UD\alpha \circ k_i$
by naturality
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 $Fx \rightarrow D_j$ $A(Fx, D_j) \xrightarrow{\cong} B(x, UD_j)$
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so their images are the two paths
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 $k_j \rightarrow UD_j$ the k_i are a cone

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$$F_x \xrightarrow{\varphi^{-1}k_i} D_i \quad \text{and} \quad F_x \xrightarrow{\varphi^{-1}k_j} D_j$$

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 $F_x \xrightarrow{\varphi} D_j$ under
 $A(F_x, D_j) \xrightarrow{\varphi} B(X, UD_j)$
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- Well $\varphi\varphi^{-1}k_j = k_j$.
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$$\begin{array}{c} K_i \xrightarrow{\quad} UD_i \\ X \xrightarrow{\varphi} UD_x \\ K_j \xrightarrow{\quad} UD_j \end{array} \quad \text{which agree, since the } k_i \text{ are a cone}$$

- Since the maps $\varphi^{-1}k_i : F_x \rightarrow D_i$ form a cone we obtain a unique

$$l : F_x \rightarrow L$$

such that  $F_x \xrightarrow{l} L$ $\varphi^{-1}k_i \xrightarrow{\quad} D_i$ for all i .

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- Since the maps $\varphi^{-1}k_i : Fx \rightarrow D_i$ form a cone we obtain a unique
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such that

$$Fx \xrightarrow{l} L \xrightarrow{p_i} D_i \quad \text{for all } i.$$

$\varphi^{-1}k_i$

- Using the bijection
 $A(Fx, L) \xrightarrow{\cong} B(x, UL)$ this corresponds to a map

$x \xrightarrow{\cong} UL$ & the equations 

$x \xrightarrow{\psi_L} u_L$ & the equations ~~not~~
corresp. to the equations

** $x \xrightarrow{\psi_L} u_L$ up; using naturality
 $\kappa_i \downarrow_{u_D i}$ of ψ ,

$x \xrightarrow{\epsilon_L} u_L$ & the equations (*)

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$\text{(*)} \quad x \xrightarrow{\epsilon_L} u_L$ up; using naturality
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- In partic, $\varphi_L : x \rightarrow u_L$ is the unique map st. (*) commutes;
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- For example, $U : \text{Gp} \rightarrow \text{Set}$ preserves products, equalisers, terminal object etc. More generally, Forgetful Functors from algebraic cats to Set preserve all limits.

Dually Theorem

Left adjoints preserve colimits.

Exercise
Prove that forgetful functor
 $U : \text{Grp} \longrightarrow \text{Set}$ does
not have a right adjoint
(i.e. is not a left adjoint.)

Example

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These equations involve negation
 \Rightarrow not universal algebra

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Functor $U: \text{Field} \rightarrow \text{Set}$
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So it suffices
to show Field does not
have an initial object.

• Firstly, let $F: R \rightarrow S$
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$$\begin{array}{ccc}
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 F & \downarrow & \\
 & \xrightarrow{\text{inj}} & \mathbb{Z}_q
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↓

$$F \xrightarrow{\text{inj}} \mathbb{Z}_q$$

- Since $F \hookrightarrow \mathbb{Z}_p, \mathbb{Z}_q$ are injective they reflect equations $p \cdot 1 = 0$ & $q \cdot 1 = 0$, so these equations hold in F .

But as p, q coprime,
 $\exists n, m \in \mathbb{Z}$ such that

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Hence in F

$$\begin{aligned} l &= l \cdot l = (np + mq) \cdot l \\ &= n(p \cdot l) + m(q \cdot l) = \\ &n \cdot 0 + m \cdot 0 = 0. \end{aligned}$$

Hence $l = 0$ in F ,

so F not a Field.



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Consider $x \xrightarrow{\begin{matrix} u \\ v \end{matrix}} Ua \xrightarrow{UF} Ub$

satisfying $UF.u = UF.v$. We
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Since F is mono, therefore
 $\ell^{-1}u = \ell^{-1}v$.

Therefore $u = v$ so that
 uF is mono, as claimed. \square

Dually

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Defⁿ - A subcategory A of B is a subcollection of objects and arrows in B closed under identities and composition.

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Examples

- \mathcal{U} : Grp, Mon, Ring \longrightarrow Set
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Remark

If $A \xrightarrow{i} B$ is a full subcategory,
observe that a left adjoint to i
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If $A \xrightarrow{i} B$ is a full subcategory, observe that a left adjoint to i is specified by: For each $b \in B$, an object $R_b \in A$ & morphism $\pi_b: b \rightarrow R_b$ such that,

Examples

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Remark

If $A \xrightarrow{i} B$ is a full subcategory, observe that a left adjoint to i is specified by:

for each $b \in B$, an object $Rb \in A$ & morphism $\pi_b: b \rightarrow Rb$ such that, given $f: b \rightarrow c$ with $c \in A$,

$\exists! Rb \xrightarrow{\bar{F}} c$ such that

$$\begin{array}{ccc} Rb & \searrow \bar{F} \\ \pi_b \uparrow & \parallel & \downarrow \\ b & \xrightarrow{f} & c \end{array}$$

Additional material : net compulsory.

Horizontal composition

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Given $A \xrightarrow{\alpha} B \xrightarrow{H} C$ we can define $A \xrightarrow{H\alpha} C$ to be the nat. transf. with components : at $x \in A$, $H\alpha_x \xrightarrow{H\alpha x} H\alpha x$.

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Horizontal composition

- We have already seen vertical composition of natural transformations.
- We now turn to a second kind of composition, called horizontal composition.

- Given $A \xrightarrow{F} B \xrightarrow{H} C$ we can

define $A \xrightarrow{HF} C$ to be the nat. transf. with components :

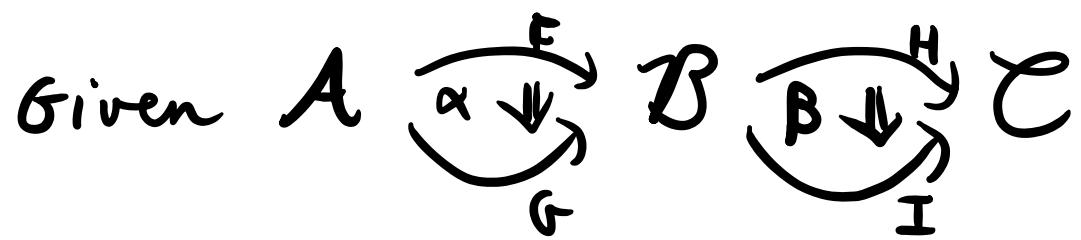
$$\text{at } x \in A, \quad HFX \xrightarrow[G]{H\alpha_x} HGX.$$

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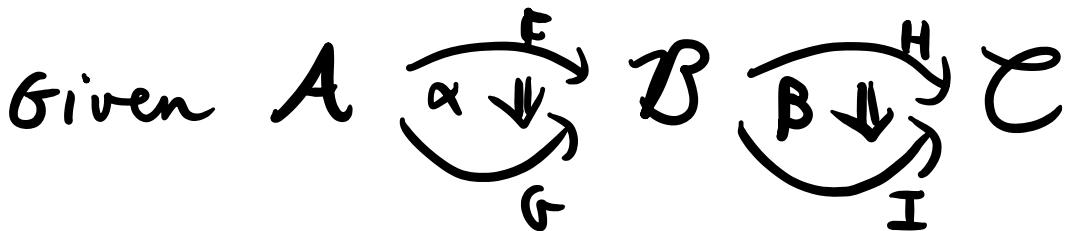
define $A \xrightarrow{HF} C$ as the

natural transf. with component

$$\alpha_F x : GFx \longrightarrow HFX \text{ at } x \in A.$$



we have two ways of defining
a composite, as either path
in $HG \xrightarrow{\beta_F} IF \xleftarrow{I\alpha}$ (i.e. $I\alpha \cdot \beta_F$)
 $H\alpha \downarrow \quad \downarrow I\alpha$
 $HG \xrightarrow{\beta_G} IG$;



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in $HFX \xrightarrow{\beta_F} IFX \xleftarrow{I\alpha} (i.e. I\alpha \circ \beta_F)$

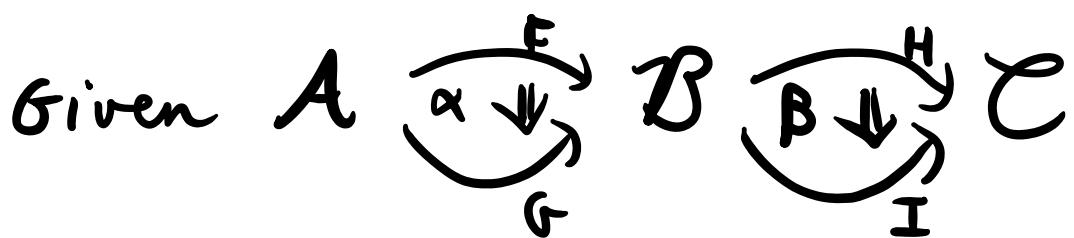
$$\begin{array}{ccc} H\alpha & \downarrow & \\ HG & \xrightarrow{\beta_G} & IG \end{array};$$

these agree by naturality of

$\beta : HFX \xrightarrow{\beta_{FX}} IFX$

$$\begin{array}{ccc} H\alpha_x & \downarrow & \\ HG_x & \xrightarrow{\beta_{GX}} & IG_x \end{array}$$

at the morphism $\alpha_x : FX \rightarrow GX$.



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at the morphism $\alpha_x : Fx \rightarrow Gx$.

The resulting nat t.

$$A \xrightarrow[\text{Bnat}]{{\overset{HF}{\curvearrowright}}} C$$

is called horizontal composite.

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Remark

Categories, Functions &
natural Transformations

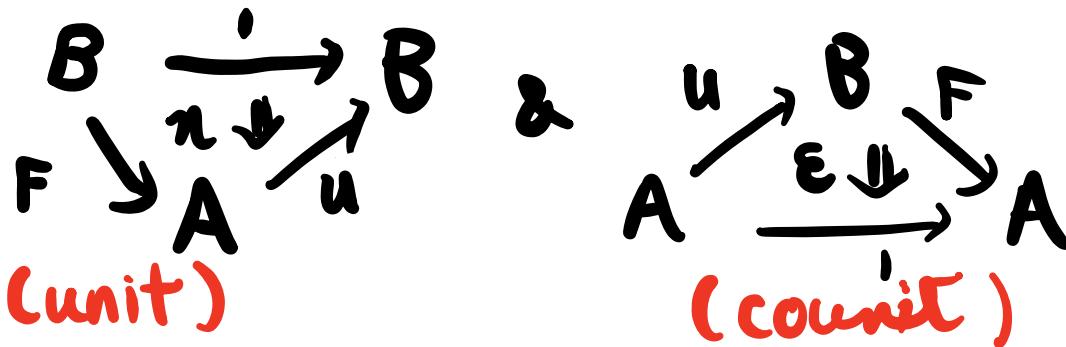
form a 2-category!

① The Triangle equations

Theorem

An adjunction $A \xrightleftharpoons[u]{F} B$

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as above + natural transformations



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$$\begin{array}{ccc} B & \xrightarrow{i} & B \\ F \downarrow & \Downarrow \eta & \downarrow u \\ A & \xrightarrow{u} & A \end{array} \quad \& \quad \begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow \varepsilon & \Downarrow F & \downarrow \\ A & \xrightarrow{i} & A \end{array}$$

(unit) (countit)

such that

$$\begin{array}{ccc} B & \xrightarrow{i} & B \\ F \downarrow & \Downarrow \eta & \downarrow u \\ A & \xrightarrow{u} & A \end{array} \quad \& \quad \begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow \varepsilon & \Downarrow F & \downarrow \\ A & \xrightarrow{i} & A \end{array} = id_F \quad \& \quad \begin{array}{ccc} B & \xrightarrow{i} & B \\ \downarrow u & \Downarrow \varepsilon & \downarrow \\ A & \xrightarrow{i} & A \end{array} = id_u.$$

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Theorem

An adjunction $A \xrightleftharpoons[u]{F} B$

is specified by a pair of functors
as above + natural transformation

$$\begin{array}{ccc} B & \xrightarrow{\quad i \quad} & B \\ F \downarrow & \nearrow u & \downarrow \varepsilon \\ A & & A \end{array}$$

(unit)

$$\begin{array}{ccc} & u \nearrow & B \\ A & \xrightarrow{\quad \varepsilon \quad} & A \\ & F \downarrow & \end{array}$$

(counit)

such that

$$\begin{array}{ccc} B & \xrightarrow{\quad i \quad} & B \\ F \downarrow & \nearrow u & \downarrow \varepsilon \\ A & \xrightarrow{\quad u \quad} & A \end{array}$$

$$\begin{array}{ccc} & u \nearrow & B \\ A & \xrightarrow{\quad \varepsilon \quad} & A \\ & F \downarrow & \end{array} \quad \begin{array}{ccc} & i \quad \nearrow & B \\ A & \xrightarrow{\quad u \quad} & A \end{array} = id_u.$$

i.e.

$$Fb \xrightarrow{\quad i \quad} FUFB \xrightarrow{\quad \varepsilon_{FB} \quad} FB$$

$$Ua \xrightarrow{\quad u \quad} UFUa \xrightarrow{\quad \varepsilon_a \quad} Ua$$

~~Proof~~ (Sketch)

- Consider an adjunction $A(Fb, a) \xrightarrow{\epsilon} B(b, Fa)$.
- We defined the unit $1 \Rightarrow UF$ last time.

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- Consider an adjunction $A(Fb, a) \xrightarrow{\epsilon} B(b, Fa)$.
- We defined the unit $1 \Rightarrow UF$ has fine.
- Similarly $B(Ua, Fa) \xrightarrow{\epsilon^{-1}} A(FUa, a)$
 $1 \quad \longleftarrow \quad FUa \xrightarrow{\epsilon_a} a$
determines the counit.

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- Consider an adjunction $A(Fb, a) \xrightarrow{\epsilon} B(b, Fa)$.
- We defined the unit $1 \Rightarrow UF$ has time.
- Similarly $B(Ua, Fa) \xrightarrow{\varphi^{-1}} A(FUa, a)$

$$1 \quad \downarrow \qquad \qquad \qquad FUa \xrightarrow{\epsilon_a} a$$

determines the counit.

By naturality of φ^{-1} , have

$$@ \quad b \xrightarrow{\alpha} Ua$$

$$Fb \xrightarrow{\varphi^{-1}\alpha} a$$

$$F\alpha \downarrow \qquad \qquad \qquad \nearrow \epsilon_a \\ FUa$$

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 - We defined the unit $1 \Rightarrow UF$ has fine.
 - Similarly $B(Ua, Fa) \xrightarrow{\epsilon^{-1}} A(FUa, a)$
- $\begin{matrix} & \epsilon^{-1} \\ I & \longleftarrow \\ & Fa \xrightarrow{\epsilon_a} a \end{matrix}$
- determines the counit.

By naturality of ϵ^{-1} , have

$$@ \begin{array}{c} b \xrightarrow{\alpha} Ua \\ Fb \xrightarrow{\epsilon^{-1}\alpha} a \\ F\alpha \searrow \quad \nearrow Fa \\ FUa \end{array}$$

+

$$@ \begin{array}{c} \beta : Fa \rightarrow b \\ a \xrightarrow{\epsilon\beta} Ub \\ \text{no } " \xrightarrow{UFa} UF \\ \text{(last week)} \end{array}$$

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- We defined the unit $1 \Rightarrow UF$ last time.
- Similarly $B(Ua, Fa) \xrightarrow{\epsilon^{-1}} A(FUa, a)$
 $\downarrow \quad \downarrow$
 $FUa \xrightarrow{\epsilon_a} a$
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By naturality of ϵ^{-1} , have

$$\begin{array}{ccc}
 @ b & \xrightarrow{\alpha} & Ua \\
 Fb & \xrightarrow{\epsilon^{-1}\alpha} & a \\
 F\alpha \downarrow & & \nearrow \epsilon_a \\
 FUa & &
 \end{array}
 \quad +
 \quad
 \begin{array}{ccc}
 @ \beta : Fa \rightarrow b \\
 a & \xrightarrow{\epsilon\beta} & Ub \\
 \nearrow \text{new } UFa & \nearrow \text{new } UF & \\
 & &
 \end{array}$$

(last week)

- Now consider $B(Ua, Fa) \xrightarrow{\epsilon^{-1}} A(FUa, a) \xrightarrow{\epsilon} B(Ua, Fa)$
 sends
 $1_{Fa} \mapsto \epsilon_a : FUa \rightarrow a \mapsto Ua \xrightarrow{\pi_{Ua}} UFUa \xrightarrow{U\epsilon_a} Ua$
 so this equals the identity

~~Proof~~ (Sketch)

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$$1 \quad \xrightarrow{\epsilon^{-1}} \quad FUa \xrightarrow{\epsilon_a} a$$

determines the counit.

By naturality of ϵ^{-1} , have

$$\begin{array}{c} @ b \xrightarrow{\alpha} Ua \\ Fb \xrightarrow{\epsilon^{-1}\alpha} a \\ F\alpha \downarrow \quad \nearrow \epsilon_a \\ FUa \end{array} + @ \beta : Fa \rightarrow b \\ a \xrightarrow{\epsilon\beta} Ub \\ \pi_a \downarrow \quad \nearrow UFa \quad \uparrow \eta_F \\ (last \ week) \end{array}$$

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sends
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 so this equals the identity
- Similarly the other triangle equation holds.

- Conversely consider ϵ & π satisfying triangle equations.

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- We have $A(Fb, a) \xrightarrow{\epsilon} B(b, ua)$
 $Fb \xrightarrow{\alpha} a \mapsto a \xrightarrow{\pi a} uFa \xrightarrow{u\alpha} ubs$
natural in a, b .

- Conversely consider ϵ & π satisfying triangle equations.
- We have $A(Fb, a) \xrightarrow{\epsilon} B(b, ua)$
 $Fb \xrightarrow{\alpha} a \mapsto a \xrightarrow{\pi_a} uFa \xrightarrow{u\alpha} ub$
 natural in a, b .
- So enough to show ϵ a bijection.

Conversely consider ϵ & π satisfying triangle equations.

We have $A(Fb, a) \xrightarrow{\varphi} B(b, ua)$
 $Fb \xrightarrow{\alpha} a \mapsto a \xrightarrow{\pi_a} uFa \xrightarrow{u\alpha} ua$
natural in a, b .

So enough to show φ a bijection.

Inverse given by

$$B(b, ua) \xrightarrow{\Theta} A(Fb, a)$$
$$b \xrightarrow{\alpha} ua \quad \downarrow \quad Fb \xrightarrow{F\alpha} Fa \xrightarrow{\epsilon_a} a$$

& $\Theta\varphi = 1$, $\varphi\Theta = 1$ follow from triangle equations.

□

- Conversely consider ϵ & η satisfying triangle equations.
 - We have $A(Fb, a) \xrightarrow{\epsilon} B(b, Ua)$
 $Fb \xrightarrow{\alpha} a \mapsto a \xrightarrow{\eta a} UFa \xrightarrow{U\alpha} Ua$
 natural in a, b .
 - So enough to show φ a bijection.
 - Inverse given by
 $B(b, Ua) \xrightarrow{\Theta} A(Fb, a)$
 $b \xrightarrow{\alpha} Ua \mapsto Fb \xrightarrow{F\alpha} FUa \xrightarrow{\epsilon a} a$
- & $\Theta\varphi = 1$, $\varphi\Theta = 1$ follow from triangle equations.

□

Remark) This defⁿ of adjunction via unit & counit + triangle equations captures sense in which adjunction is a weaker form of iso / equivalence.