FREE CR DISTRIBUTIONS

GERD SCHMALZ AND JAN SLOVÁK

Abstract. There are only some exceptional CR dimensions and codimensions such that the geometries enjoy a discrete classification of the pointwise types of the homogeneous models. The cases of CR dimensions n and codimensions n^2 are among the very few possibilities of the so called parabolic geometries. Indeed, the homogeneous model turns out to be $\mathrm{PSU}(n+1,n)/P$ with a suitable parabolic subgroup P. We study the geometric properties of such real $(2n+n^2)$-dimensional submanifolds in $\mathbb{C}^{n+n^2}$ for all $n>1$. In particular we show that the fundamental invariant is of torsion type, we provide its explicit computation, and we discuss an analogy to the Fefferman construction of a circle bundle in the hypersurface type CR geometry.

1. Introduction

There is a vast amount of literature on analytical and geometrical aspects of real submanifolds of complex spaces $\mathbb{C}^N$. The generic hypersurfaces in $\mathbb{C}^{n+1}$, i.e. real $(2n+1)$-dimensional contact manifolds equipped with complex structure on the contact distribution, represent the best known example studied in detail for more than hundred years already. From the geometrical point of view, the main reason for their nice and rich structural behaviour lies in the algebraic properties of the Klein's homogeneous model which is represented by the quadric $Q=\mathrm{PSU}(p+1,q+1)/P$ obtained from the standard action of $\mathrm{PSU}(p+1,q+1)$ on $\mathbb{C}^{p+q+2}$. The space Q itself coincides with the space of isotropic lines with respect to the Hermitian form h of signature (p,q) and P is the isotropic subgroup of one such line.

The general case of CR geometries of CR dimension n and codimension k, i.e. real $2n+k$ surfaces in $\mathbb{C}^{2n+k}$ does not permit a similar approach in general, but there are some exceptional dimensions and codimensions which are very similar to the hypersurface case. These exceptional dimensions are $k=n^2$, $k=n^2-1$ for arbitrary $n>1$, $n=k=2$, $n=3,k=2$ and $n=3,k=7$. The case $n=k=2$ was studied in [10] and provides a beautiful way of viewing real 6-dimensional surfaces in $\mathbb{C}^4$. Nowadays, there is the general theory of parabolic geometries and the originally surprising first example of a CR geometry with a parabolic isotropy group in the semi-simple structure group outside of the hypersurface type CR structures provides a quite easy example of its applications, see Section 4.3 of [4].

In this paper we study another coincidence when the bracket generating distribution inherited on a generic CR submanifold $M\subset\mathbb{C}^N$ allows for only one type infinitesimal homogeneous model. The reason for this exceptional behaviour is similar to the so called free n-dimensional distributions studied intensively in geometric literature, cf. [7]. There the non-degeneracy of rank n distribution in a space, where the codimension is equal to the dimension of the space of all skew-symmetric matrices, automatically leads to isomorphic Lie algebra structures on associated graded tangent spaces at all points. In our case, the Levi form is an imaginary part of a Hermitian form valued in the space of skew-Hermitian matrices. Thus the codimension n^2 again coincides with the dimension of the whole target space. The paper [7] has been also the main inspiration for most of the algebraic technicalities here. This concerns in particular the Fefferman like construction of a circle bundle equipped with a Hermitian analog of the spinorial geometry for all generic CR dimension n and codimension n^2 geometries, in full analogy to the hypersurface case.

Our approach is based on the recent theory of parabolic geometries, as developed in [4], building itself on the Cartan-Tanaka theory, cf. [12, 13]. This turns the quite deep problems on high codimensional CR manifolds into rather straightforward applications of the general methods and results (and opens new questions at the same time). In particular, the standard technique of the exterior differential systems supported by the general results provides a very explicit and efficient approach to the basic invariants.

Acknowledgements. The research reflected in this paper has been supported by the Czech Science Foundation, grant Nr. 201/08/0397. The first named author also gratefully acknowledges the support by Max-Planck-Institut für Mathematik in Bonn. The authors benefited from numerous discussions on the topic to M.G. Eastwood, B. Doubrov, and A. Cap.

2. The homogeneous model and Cartan connections

Consider the $(2n+1)$-dimensional complex space $\mathbb{C}^{2n+1}$ with coordinates $(\epsilon_1, \dots, \epsilon_n, \zeta, \omega_1, \dots, \omega_n)$ endowed with the Hermitian form

$$h = |\zeta|^2 + \sum_{\nu=1}^{n} \epsilon_\nu \bar{\omega}_\nu + \omega_\nu \bar{\epsilon}_\nu$$

of signature $(n+1, n)$. We shall write

$$\mathbb{J} = \begin{pmatrix} 0 & 0 & I \\ 0 & 1 & 0 \\ I & 0 & 0 \end{pmatrix}$$

for the block matrix of this form in the standard coordinates on $\mathbb{C}^{2n+1}$ (here $I = I_n$ is the unit matrix of rank n).

2.1. **The homogeneous quadric Q.** Let us consider the Grassmannian of n-dimensional complex subspaces of $\mathbb{C}^{2n+1}$ and denote by Q its subset consisting of the isotropic subspaces with respect to h. By $\mathrm{SU}(n+1, n)$ we denote the special pseudo-unitary group with respect to the Hermitian form introduced above.

Lemma 1. *Q is a homogeneous CR-manifold of CR-dimension n and CR-codimension n^2 with rational transitive action of $\mathrm{SU}(n+1,n)$. The kernel of the action is $\mathbb{Z}_{2n+1}$ and so the effective homogeneous model is $Q = G/P$, where $G = \mathrm{PSU}(n+1,n) = \mathrm{SU}(n+1,n)/\mathbb{Z}_{2n+1}$ and P is the isotropic subgroup of one fixed isotropic plane V_0 in Q.*

Proof. Obviously, the standard action of $\mathrm{SU}(n+1,n)$ on $\mathbb{C}^{2n+1}$ induces an action on Q. We show that it is transitive. In order to do this we construct a pseudo-unitary basis of $\mathbb{C}^{2n+1}$ adapted to a chosen fixed plane $V \in Q$. Let $v_1, \dots, v_n$ be a basis of V. Then the $n \times (2n+1)$ matrix $(v_1, \dots, v_n)^* \mathbb{J}$ has rank n and so we can find n vectors $(w_1, \dots, w_n)$ such that

$$(v_1, \dots, v_n)^* \mathbb{J}(w_1, \dots, w_n) = I_{2n+1}.$$

The vectors $v_1, \dots, v_n, w_1, \dots, w_n$ are linearly independent. Indeed, if

$$\lambda_1 v_1 + \cdots + \lambda_n v_n + \mu_1 w_1 + \cdots + \mu_n w_n = 0$$

then multiplication of the left hand side with v_i yields $\mu_i = 0$ and the v_i were linearly independent by assumption. Next, let us write $A = (A_{ij})$ for the Hermitian Gram matrix of the basis (w_i), that is $A_{ij} = w_i^* \mathbb{J} w_j$. If we replace the basis (w_i) by $(w_i - \frac{1}{2}A(v_i))$, then

$$h(v_i, v_j) = 0, \quad h(v_i, w_j) = \delta_{i,j}, \quad h(w_i, w_j) = 0.$$

Finally choose w_{n+1} orthogonal to $v_1, \dots, v_n, w_1, \dots, w_n$. According to the signature of the inner product, it has positive length. By scaling we achieve that the length is 1.

For any isotropic planes V and V' we can now find such bases and the cooresponding coordinate change is pseudo-unitary with respect to h. On the other hand, every such basis corresponds to an isotropic subspace of dimension n. Therefore, Q can be viewed as the orbit of one isotropic space $V \in Q$ with respect to the holomorphic action of $\mathrm{SU}(n+1,n)$. Since Q is a real submanifold of the complex Grassmannian manifold $\mathrm{SU}(n+1,n)$ acts transitively on Q by holomorphic automorphisms of the ambient Grassmannian it must be a CR-manifold.

In order to see the CR structure on Q in detail, we shall pass to a local chart of the Grassmannian. Let V_0 be spanned by the standard vectors e_ν, for $\nu = 1, \dots, n$. In a neighbourhood of V_0 in the Grassmanian of n-planes we introduce coordinates $z_\nu, w_{\mu\nu}$ such that a subspace V is given as the span of

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ z_1 \\ w_{11} \\ \vdots \\ w_{1n} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ z_2 \\ w_{21} \\ \vdots \\ w_{2n} \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ z_n \\ w_{n1} \\ \vdots \\ w_{nn} \end{pmatrix} = \begin{pmatrix} I \\ z \\ W \end{pmatrix},$$

i.e. V_0 is given as $(I,0,0)^T$. An n-plane $V=(I,z,W)^T$ of this chart is isotropic if

$$(I,z^*,W^*)\mathbb{J}(I,z,W)^T=0$$

that is

$$W+W^*+z^*z=0. \tag{1}$$

Notice that (1) is the defining equation of Q in the chart.

V is the image of V_0 under the linear mapping

$$\begin{pmatrix} I & 0 & 0\\ z & 1 & 0\\ W & z^* & I\end{pmatrix}:\mathbb{C}^{2n+1}\to\mathbb{C}^{2n+1}.$$

This mapping is in $\mathrm{SU}(n+1,n)$ and it is uniquely determined it we imposed the conditions that it keeps the first n coordinates unchanged and the n+1-st coordinate unchanged modulo the first n coordinates.

Thus, using the following block structure for the matrices in the Lie group $G=\mathrm{SU}(n+1,n)$ or its Lie algebra $\mathfrak{g}=\mathfrak{su}\,(n+1,n)$,

$$\begin{array}{c} \begin{array}{ccc} n & 1 & n \end{array} \\ \left(\begin{array}{ccc} \boxed{0} & \boxed{1} & \boxed{2} \\ \boxed{-1} & \boxed{0} & \boxed{1} \\ \boxed{-2} & \boxed{-1} & \boxed{0} \end{array}\right) \begin{array}{c} n \\ 1 \\ n \end{array} \end{array}$$

we may identify the latter chart with the exponential image of the lower block triangular matrices and the action of the group $\mathrm{SU}(n+1,n)$ on Q by CR-automorphisms is just the adjoint action of $\mathrm{SU}(n+1,n)$.

In particular, the isotropy subgroup of the origin $V_0=(I,0,0)^T$ is the parabolic subgroup P of all block upper triangular matrices in $G=\mathrm{SU}(n+1,n)$. In summary, we have identified the CR-manifold Q with the compact partial flag variety $Q=G/P$.

The Lie algebra $\mathfrak{g}=\mathfrak{su}\,(n+1,n)$ of $\mathrm{SU}(n+1,n)$ enjoys the natural grading

$$\mathfrak{g}=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2$$

given by the block structure shown above. A simple computation reveals that $\mathfrak{g}_{-1}=\mathbb{C}^n$, $\mathfrak{g}_{-2}$ consists of all skew-Hermitian matrices and the Lie algebra bracket is given by $[X,Y]=X^*Y-Y^*X\in\mathfrak{g}_{-2}$ for $X,Y\in\mathfrak{g}_{-1}$.

Then the block diagonal matrices in $\mathfrak{g}_0$ are of the form $(-C,2i\,\mathrm{Im}\,\mathrm{Tr}\,C,C^*)$ with $C\in\mathfrak{gl}\,(n,\mathbb{C})$ arbitrary, $\mathfrak{g}_1=\mathbb{C}^n$, and $\mathfrak{g}_2$ consists again of skew-Hermitian matrices.

Now, the natural complex structure on the Grassmanian of n-planes keeps the subspace $\mathfrak{g}_{-1}$ invariant, turning it into the CR-distribution on Q at the origin, while the $\mathfrak{g}_{-2}$ provides the remaining n^2 codimensions. The holomorphic transitive action of $\mathrm{SU}(n+1,n)$ extends this CR structure over the entire Q.

Finally, the elements in the reductive part $G_0 \subset P$ are the block diagonal matrices (C^{-1}, α, C^*) with $\alpha = (\det C)^2 |\det C|^{-2}$. Its action on V_0 reads (expressed in the analogy to projective coordinates $(I : Z : W)^T$)

$$\begin{pmatrix} I \\ Z \\ W \end{pmatrix} \mapsto \begin{pmatrix} C^{-1} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & C^* \end{pmatrix} \begin{pmatrix} I \\ Z \\ W \end{pmatrix} = \begin{pmatrix} C^{-1} \\ \alpha Z \\ C^* W \end{pmatrix} \simeq \begin{pmatrix} I \\ \alpha Z C \\ C^* W C \end{pmatrix}.$$

Thus the kernel of the action of G on Q consists of matrices $C = \beta I$ with $|\beta| = 1$ and $\alpha = \beta^{2n}$. The conclusion is $\beta^{2n+1} = 1$ and this proves the last claim. □

The group $\mathrm{SU}(n+1,n)$ acts holomorphically on the Grassmannian and preserves Q and so it is a subgroup of $\mathrm{Aut}\, Q$.

The expression of the general action of $\exp \mathfrak{g}_1$ and $\exp \mathfrak{g}_2$ in the coordinate patch introduced above is given by the formulae (for small Y and T)

$$\begin{pmatrix} I \\ Z \\ W \end{pmatrix} \mapsto \begin{pmatrix} I & Y & -\frac{1}{2}Y^*Y \\ 0 & 1 & -Y^* \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I \\ Z \\ W \end{pmatrix} = \begin{pmatrix} I + YZ - \frac{1}{2}Y^*YW \\ Z - Y^*W \\ W \end{pmatrix}$$

$$\simeq \begin{pmatrix} I \\ (Z - Y^*W)(I + YZ - \frac{1}{2}Y^*YW)^{-1} \\ W(I + YZ - \frac{1}{2}Y^*YW)^{-1} \end{pmatrix}$$

$$\begin{pmatrix} I \\ Z \\ W \end{pmatrix} \mapsto \begin{pmatrix} I & 0 & T \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I \\ Z \\ W \end{pmatrix} = \begin{pmatrix} I + TW \\ Z \\ W \end{pmatrix} \simeq \begin{pmatrix} I \\ Z(I + TW)^{-1} \\ W(I + TW)^{-1} \end{pmatrix}.$$

The equivalent formulae for general automorphisms have been computed also directly by the standard methods of complex analysis (see, e.g., [8]). But we shall see that actually all automorphisms of Q are of this form as a simple consequence of our general theory below.

Clearly, the case $n = 1$ recovers the real 3-dimensional hypersurfaces in $\mathbb{C}^2$. The algebraic properties of the model are quite different for this lowest dimensional case and we shall treat only the other cases $n > 1$ in the sequel.

2.2. **General facts on parabolic geometries.** Let us briefly remind some general concepts, the reader can find all details in the monograph [4].

The parabolic geometries can be viewed as curved deformations of the homogeneous spaces G/P with G semisimple and P parabolic:

Definition 1. *A* Cartan geometry of type (G, P) on a manifold M *is a principal fiber bundle* $\mathcal{G} \to M$ *with structure group* P*, equipped with an absolute parallelism* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ *which is* Ad*-invariant with respect to the principal* P*-action and reproduces the fundamental vector fields. The form* ω *is called the* Cartan connection *of type* (G, P) *on* M*.*

Most general features of the geometry in question are read off from the algebraic properties of the *flat model* $G \to G/P$, where ω is the Maurer-Cartan form. The name *parabolic geometry* refers to cases where G is semisimple and $P \subset G$ parabolic. At the level of the curved geometries, the P-invariant filtration inherited on $T\mathcal{G}$ from the absolute parallelism projects to the filtration on TM. Let us also notice that the Cartan-Killing form identifies $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ with $(\mathfrak{g}/\mathfrak{p})^*$ as P-modules and $\mathfrak{g}/\mathfrak{p}$ equals to $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ as G_0-module, where G_0 is the reductive part of the parabolic subgroup P with Lie algebra $\mathfrak{g}_0$.

Roughly speaking, the entire Cartan connection can be mostly recovered from this filtration on the manifold M using suitable normalization conditions. Thus, a parabolic geometry on a manifold M is given by a particular geometric structure visible at the manifold itself, while $\mathcal{G}$ and ω are uniquely determined by a functorial construction.

The structures in question are the so called *regular infinitesimal flag structures of type* (G, P) and they are derived from the grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of the semisimple Lie algebra $\mathfrak{g}$ giving rise to the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. In fact, the analogues of the P–invariant filtration on $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ have to be given in the individual tangent spaces.

In our case, such structure is given by a non-degenerate distribution of the right dimension and codimension, satisfying some additional conditions, and so the entire Cartan connection is constructed from these simple data by the general theory. Thus, our main technical step will be to observe that generic real submanifolds of dimension $2n+n^2$ in $\mathbb{C}^{n+n^2}$ inherit at each point such an infinitesimal structure from the ambient complex space.

The structural information on the parabolic geometries is encoded neatly in cohomological terms. The curvature form $\Omega \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of the Cartan connection ω is given by the structure equation

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

and the absolute parallelism identifies the curvature with the curvature function

$$\kappa : \mathcal{G} \to \wedge^2 \mathfrak{p}_+ \otimes \mathfrak{g}, \qquad \kappa(X, Y) = K(\omega^{-1}(X), \omega^{-1}(Y)).$$

Thus, the curvature function has values in the cochains of the Lie algebra cohomology of $\mathfrak{g}_-$ with coefficients in $\mathfrak{g}$. This cohomology is explicitly computable by the Kostant's version of the BBW theorem, cf. [9, 4]. We may compute it either by means of the standard differential ∂ or by its adjoint co-differential ∂^*. The formula in the special case of the above two-chains is

$$\partial^*(Z_0 \wedge Z_1 \otimes X) = -Z_0 \otimes [Z_1, X] + Z_1 \otimes [Z_0, X] - [Z_0, Z_1] \otimes X.$$

The normalization procedure relies on another important property of the parabolic geometries, which imposes conditions on the behaviour of the filtrations and is called *regularity*. In words, the filtrations have to respect the Lie brackets of vector fields and coincide with the Lie algebra structure $\mathfrak{g}_-$

at the graded level. In terms of the curvature, this says that no curvature components of non-positive homogeneities are allowed, cf. [4, Section 3.1].

The general theory shows that normalizing the regular Cartan connections by the co-closedness $\partial^*\kappa = 0$ of the curvature defines an equivalence of categories of certain filtered manifolds (with additional simple geometric structures under some cohomological conditions, like for all $|1|$-gradings or contact gradings) and categories of Cartan connections, cf. [4, Section 3.1]. Then the harmonic part of the curvature (which is a well defined quotient) defines all the rest and, in particular, the geometry is locally isomorphic to its flat model if and only if the harmonic curvature vanishes. Moreover, the entire curvature tensor is computable explicitly by a natural differential operator from its harmonic part, cf. [3].

2.3. **Free CR-distributions.** Let us come back to our main example, the structures modelled over the parabolic homogeneous space G/P with $G = \mathrm{PSU}(n+1,n)$, $n > 1$, and P as above. In general, the first cohomology $H^1(\mathfrak{g}_-, \mathfrak{g})$ governs the amount of data determining the regular infinitesimal structures, cf. [4, Section 4.3]. Indeed, the distribution itself encodes the entire geometry if the first cohomology $H^1(\mathfrak{g}_-, \mathfrak{g})$ appears only in negative homogeneities. A direct computation checks that this happens in our case.

Finally, we have to find out the conditions for the regularity, which is more sophisticated. At the first glance it seems that the complex structure on $\mathfrak{g}_{-1}$ and the choice of the reductive part G_0 of P should be an important part of the geometric data, but the above mentioned cohomological computation says that this cannot be true. The reason is given by the lemma below which uses the terminology of a totally real skew-symmetric form. Recall that a real skew-symmetric bilinear form ω on a complex space V with complex structure J is called totally real if $\omega(JX, JY) = \omega(X, Y)$.

Lemma 2. *The only complex structures $\tilde{J}$ on the real vector space $\mathfrak{g}_{-1}$ which make the Lie bracket $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ into a totally real skew-symmetric form valued in skew Hermitian matrices are $\tilde{J} = \pm J$, where J is the standard complex structure on $\mathfrak{g}_{-1}$. Moreover, all linear homomorphisms $A \in GL(\mathfrak{g}_{-1})$ allowing an extension $\tilde{A}$ to a Lie algebra automorphism of $\mathfrak{g}_-$ are complex linear or complex anti-linear.*

Proof. Let us observe that the Lie bracket $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is actually formed by n^2 ordinary skew-symmetric bilinear forms which are linearly independent. Thus being totally real means that every skew-symmetric bilinear form $\omega \in \Lambda^2\mathfrak{g}^*_{-1}$ in the span of the above ones has to be invariant with respect to the endomorphism J. Now, considering both ω and J as matrices in a chosen basis, this is equivalent to the property that the matrix composition $J^T\omega$ is symmetric for all skew-symmetric matrices ω in question, thus $J\omega$ is always symmetric, too.

Let $\tilde{J}$ be a real endomorphism of $\mathfrak{g}_{-1}$ with $\tilde{J}^2 = -\operatorname{id}$ and such that any J-invariant skew-symmetric bilinear form ω is also $\tilde{J}$ invariant. Thus, we request $\tilde{J}\omega$ symmetric for any skew-symmetric ω such that $J\omega$ is symmetric.

Let $\mathbb{C}^{2n}$ be the complexification of $\mathbb{R}^{2n} = \mathbb{C}^n$ with coordinates such that

$$J = \begin{bmatrix} \mathrm{i}\,\mathbb{1} & 0 \\ 0 & -\mathrm{i}\,\mathbb{1} \end{bmatrix}.$$

Then $J\omega$ is symmetric for skew-symmetric ω if and only if

$$\omega = \begin{bmatrix} 0 & B \\ -B^t & 0 \end{bmatrix}$$

for some $n \times n$ matrix B.

Now let

$$\tilde{J} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

with $\tilde{J}^2 = -\mathbb{1}$. Then $\tilde{J}\omega$ is

$$\begin{bmatrix} -QB^t & PB \\ -SB^t & RB \end{bmatrix}$$

which is symmetric for any B if and only if $Q = R = 0$ and $P = -S = \lambda\mathbb{1}$. From $\tilde{J}^2 = -\mathbb{1}$ we get $\lambda = \pm\,\mathrm{i}$ as required.

Finally, if A in $\mathrm{GL}(\mathfrak{g}_{-1})$ allows an extension into a morphism of the Lie algebra $\mathfrak{g}_{-}$, then

$$[A^{-1}JAX, A^{-1}JAZ] = \tilde{A}^{-1} \cdot [JAX, JAZ] = \tilde{A}^{-1} \cdot \tilde{A}[X, Z] = [X, Z]$$

and so $A^{-1}JA = \pm J$ by the conclusion above. □

Now, we are led to the definition of the main objects of our considerations:

Definition 2. *Consider a smooth manifold M of real dimension $2n + n^2$ equipped with a $2n$-dimensional distribution $D = T^{-1}M \subset TM$, such that $[D, D] = TM$. We call D a free CR distribution of dimension n on M if there is a fixed almost complex structure J on D such that the algebraic Lie bracket $\mathcal{L} : D \wedge D \to TM/D$ is totally real.*

Let us remind, that if such a J exists, then it is unique up to the sign in view of Lemma 2. Thus we have to understand the condition as a quite severe restriction on the distribution D only. In fact the existence of the second cohomology $H^2(\mathfrak{g}_{-}, \mathfrak{g})$ of homogeneity zero indicates, that actually we have to expect, that the generic distributions of the proper dimension and codimension will not possess such a J. We shall even see that if the dimension is bigger than two, then each such J has to be integrable (which can be understood as a differential consequence of our requirement that the curvature vanishes in homogeneity zero).

But clearly, the requirements of Definition 2 do not represent any problem for the embedded CR-structures $M \subset \mathbb{C}^{n+n^2}$, since the bracket as well as the complex structures are inherited from the ambient complex space.

Lemma 3. *Every free CR distribution of dimension n provides a regular infinitesimal flag structure of type* $(\mathrm{PSU}(n+1,n),P)$ *on the* $(2n+n^2)$-*dimensional manifold M.*

Proof. We have mentioned the properties of infinitessimal flag structures above and in the case of free CR–geometries we just want to show, that each associated graded tangent space, together with the Levi bracket $\mathcal{L}$, is isomorphic to the the Lie algebra $\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$.

Let us consider a fixed tangent space T_xM and the CR subspace $D_x\subset T_xM$ and write J for the complex structure on D_x. The requirement on the Levi bracket $\mathcal{L}(JX,JY)=\mathcal{L}(X,Y)$ says that $\mathcal{L}$ is the imaginary component of a (vector valued) Hermitian form $\tilde{\mathcal{L}}:\mathbb{C}^n\otimes\mathbb{C}^n\to\mathbb{C}^{n^2}$ (the classical Levi form). The non-degeneracy condition $[D,D]=TM$ ensures that this Hermitian form must be equivalent to the standard form mentioned in Lemma 2. Thus identifying $\mathrm{Gr}(T_xM)=D_x\oplus(T_xM/D_x)$ with $\mathfrak{g}_-=\mathfrak{g}_{-1}\oplus\mathfrak{g}_{-2}$ as graded Lie algebras is just what we need.

The regularity follows by construction. □

Theorem 1. *For each free CR distribution of dimension $n>1$ on a manifold M, there is the unique regular normal Cartan connection of type (G,P) on M (up to isomorphisms).*

The only fundamental invariants of free CR distributions of dimensions $n>2$ are concentrated in the curvature of homogeneity degree 1 *and correspond to the totally trace-free part of the* $\mathfrak{sl}(n,\mathbb{C})$-*submodule* $\mathrm{Hom}(\mathfrak{g}_{-1}\otimes\mathfrak{g}_{-2},\mathfrak{g}_{-2})$ *in the torsion.*

In the case $n=2$, the same fundamental invariant exists and, additionally, there is the Nijenhuis tensor $N(X,Y)=[X,Y]-[JX,JY]+J([JX,Y]+[X,JY])$ on the distribution, which vanishes automatically on the embedded real eight-dimensional manifolds M in $\mathbb{C}^6$.

Moreover, every smooth map between two free CR distributions respecting the distributions is either a CR morphism or a conjugate CR morphism on the connected components of M.

Proof. The existence and uniqueness of the Cartan connection follow from the general theory (see [4, Section 3.1]). For the explicit computation of the curvature see [4, Section 4.3] or follow the algorithm of Kostant, see [4, 9]. In particular, it turns out that there are two couples of complex conjugate components for the complexified cohomologies.

If $n>2$, one of them appears in homogeneity zero and the corresponding part of the curvature is excluded by the regularity condition already. The other one corresponds just to the submodule as described in the Theorem.

If $n=2$, both pairs of components appear in homogeneity one and the additional one is generated (on the complexified tangent space) by cochains mapping two holomorphic elements to the anti-holomorphic image and vice versa. Thus this part of the curvature expresses the Nijenhuis tensor of the complex structure on D, see [4, Section 4.2] for an analogous argumentation in the hypersurface CR case.

Finally, any smooth map respecting the distributions is a homomorphism of the underlying infinitesimal structures, up to a possible change of the sign of the chosen complex structures. Thus, such a map must be a morphism of the unique normal Cartan connections on the connected components of M. □

Corollary 1. *A generic real submanifold M of real dimension $2n + n^2$ in $\mathbb{C}^{n+n^2}$, $n > 1$, inherits the free CR distribution structure from the ambient complex structure. In particular, there is the canonical Cartan connection for M, and M is locally CR isomorphic to the homogeneous quadric Q if and only if the fundamental invariant torsion described in the Theorem vanishes.*

Proof. On a generic submanifold M of the given dimension, the CR structure has got CR dimension n and its Levi form is totally real. Thus, the inherited complex structure J satisfies the assumptions on the free CR distributions and the Corollary follows. □

Corollary 2. *The group of all automorphisms of the homogeneous quadric Q is $\mathrm{PSU}(n+1, n)$.*

Proof. By Corollary 1 there is the free CR distribution structure on Q and the Maurer-Cartan form on $G = \mathrm{PSU}(n+1, n)$ is clearly the normal regular Cartan connection for the geometry in question. Thus, the uniqueness result and the Liouville theorem on automorphisms of homogeneous spaces (cf. [4, Section 1.5]) show that all local automorphisms of this CR free distribution are restrictions of left shifts by elements in $G = \mathrm{PSU}(n+1, n)$. □

Corollary 3. *The complex structure J on a free CR distribution of dimension $n > 2$ is always integrable.*

Proof. The general theory of parabolic geometries ensures that the homogeneity 1 component of the torsion consists just of the harmonic components. Since the component of the torsion corresponding to the Nijenhuis tensor of J is not listed among the harmonic ones, and its homogeneity is one, this part of torsion has to vanish. □

3. The fundamental invariant of free CR distributions

In order to exploit the existence of the unique normal Cartan connection ω for free CR distributions deduced in Theorem 1, we do not need to compute the whole canonical frame in the classical way. We shall rather make only the first step in the standard prolongation procedure, i.e. we compute the necessary objects of homogeneity one. This will provide us with the fundamental invariants, as well as the splitting of the P modules into the G_0-modules if their length of grading is at most two. In particular, this provides the distinguished complementary subspaces to the CR distribution on the tangent space. The latter data will be also sufficient for the Fefferman construction later on.

3.1. **Structure equations for free CR distributions.** For the sake of simplicity, we shall restrict ourselves to the case of integrable complex structure on D, which is always satisfied for embedded free CR distributions and does not represent any restriction for ranks $n > 2$.

In the Cartan-Tanaka procedure, all information related to homogeneity 1 objects is obtained after the first prolongation step (the bottom-to-top approach). The construction in [4, Section 3.1] provides the entire Cartan connection and the complete information on the structure of the curvature, without the explicit prolongation. We shall combine these two approaches by using the detailed knowledge on the curvature during the explicit computation. In the sequel, we shall always ignore the lowest dimensional case with $n = 1$ since the curvature structure is different and this case of real hypersurfaces in $\mathbb{C}^2$ is well known.

Let D be a non-degenerate rank $2n$ vector distribution on a manifold M of dimension $2n + n^2$, $n > 1$. We assume that a complex structure J on D is given such that the Levi form $\mathcal{L} : \Lambda^2 D \to TM/D$ defined by the Lie bracket of vector fields is totally real. As usual, we shall work in the complexification $TM \otimes \mathbb{C}$. The bundle $D \otimes \mathbb{C}$ splits into the holomorphic and anti-holomorphic eigen-subbundles $D^{1,0}$ and $D^{0,1}$ with respect to the complex structure J. Typical sections of $D^{0,1}$ are written as $\xi + iJ\xi$ for $\xi \in D$. The complex bilinear extension of $\mathcal{L}$ evaluates on such sections as

$$\mathcal{L}(\xi + iJ\xi, \zeta + iJ\zeta) = (\mathcal{L}(\xi, \zeta) - \mathcal{L}(J\xi, J\zeta)) + i(\mathcal{L}(J\xi, \zeta) + \mathcal{L}(\xi, J\zeta))$$

and so the assumption that $\mathcal{L}$ is totally real is equivalent to the vanishing of this expression. This is, in fact, equivalent to the fact that the (complexified) Lie bracket of two vector fields in $D^{0,1}$ lies in $D \otimes \mathbb{C}$ and similarly for $D^{1,0}$.

Now, let us fix a basis $X_1, \ldots, X_n$ of $D^{1,0}$, and the complex conjugate basis $X_{\bar{1}}, \ldots, X_{\bar{n}}$ of $D^{0,1}$. The non-degeneracy condition implies that the n^2 vector fields $X_{i\bar{j}} = -[X_i, X_{\bar{j}}], \quad 1 \le i, j \le n$, complete them to a basis of the complexified tangent space $TM \otimes \mathbb{C}$ at all points where the fields X_i, $X_{\bar{i}}$ are defined. We shall consider a choice corresponding to a complex basis $\xi_1, \ldots, \xi_n$ of D as above and the two obvious complex conjugate bases of $D^{0,1}$ and $D^{1,0}$. Moreover, since the almost complex structure on D is integrable, the distributions $D^{0,1}$ and $D^{1,0}$ are involutive and we may assume that their generators commute.

With these data at hand, we may start the computations of the fundamental torsion invariant from Theorem 1.

Let $\{X_i, X_{\bar{i}}, X_{i\bar{j}}\}$ be any local frame of $TM \otimes \mathbb{C}$ as above. To indicate the Hermitian skew symmetry $X_{j\bar{i}} = -\overline{X_{i\bar{j}}}$ we write $X_{[i\bar{j}]}$. Denote by $\{\theta^i, \theta^{\bar{i}}, \theta^{[j\bar{k}]}\}$ the dual coframe of $T^*M \otimes \mathbb{C}$. In fact, we shall be working on the real tangent space TM and so we shall need the coframe $\{\theta^i, \theta^{[j\bar{k}]}\}$ where the forms θ^i represent a real form valued in $\mathbb{C}^n$, $\theta^{\bar{i}} = \overline{\theta^i}$, and the forms $\theta^{j\bar{k}}$ represent a real form valued in skew Hermitian matrices.

Let us write $D^\perp$ for the set of all 1-forms on M annihilating D. It is clear that $D^\perp$ is generated by $\theta^{[j\bar{k}]}$.

Note that $d\theta^{[j\bar{k}]}(X_r, X_{\bar{s}}) = -\theta^{[j\bar{k}]}([X_r, X_{\bar{s}}]) = \delta_r^j \delta_{\bar{s}}^{\bar{k}}$, while $d\theta^{[j\bar{k}]}(X_r, X_s) = 0$ because of the integrability. This implies that

$$d\theta^{[j\bar{k}]} = \theta^j \wedge \theta^{\bar{k}} \mod \langle \theta^{[r\bar{s}]} \rangle.$$

So, the structure equations of the coframe $\{\theta^i, \theta^{[j\bar{k}]}\}$ have the form (here and below we use the Einstein summation convention):

$$\begin{aligned} d\theta^r =& f^r_{ij\bar{k}} \theta^i \wedge \theta^{[j\bar{k}]} + f^r_{\bar{i}j\bar{k}} \theta^{\bar{i}} \wedge \theta^{[j\bar{k}]} + f^r_{i\bar{j}k\bar{l}} \theta^{[i\bar{j}]} \wedge \theta^{[k\bar{l}]}, \\ d\theta^{[r\bar{s}]} =& \theta^r \wedge \theta^{\bar{s}} + f^{r\bar{s}}_{ij\bar{k}} \theta^i \wedge \theta^{[j\bar{k}]} + f^{r\bar{s}}_{\bar{i}j\bar{k}} \theta^{\bar{i}} \wedge \theta^{[j\bar{k}]} + f^{r\bar{s}}_{i\bar{j}k\bar{l}} \theta^{[i\bar{j}]} \wedge \theta^{[k\bar{l}]}, \end{aligned} \tag{2}$$

where $f^r_{ij\bar{k}}$, $f^r_{\bar{i}j\bar{k}}$, $f^r_{i\bar{j}k\bar{l}}$, $f^{r\bar{s}}_{ij\bar{k}}$, $f^{r\bar{s}}_{\bar{i}j\bar{k}} = \overline{f^{s\bar{r}}_{ik\bar{j}}}$, $f^{r\bar{s}}_{i\bar{j}k\bar{l}}$ are the structure functions of the coframe $\{\theta^i, \theta^{\bar{i}}, \theta^{[j\bar{k}]}\}$ on M, which are uniquely determined by the choice of the frames $X_i, X_{\bar{i}}$. The absence of terms of the form $\theta^i \wedge \theta^j$, $\theta^i \wedge \theta^{\bar{j}}$ and $\theta^{\bar{i}} \wedge \theta^{\bar{j}}$ in the first formula follows from the integrability and a suitable choice of the basis.

The integrability also implies that the $f^{r\bar{s}}_{ij\bar{k}}$ are symmetric in i, j. Indeed, we have

$$\begin{aligned} f^{r\bar{s}}_{ij\bar{k}} =& d\theta^{r\bar{s}}(X_i, [X_j, X_{\bar{k}}]) \\ =& -\theta^{r\bar{s}}([X_i, [X_j, X_{\bar{k}}]]) \\ =& -\theta^{r\bar{s}}([[X_i, X_j], X_{\bar{k}}]]) - \theta^{r\bar{s}}([X_j, [X_i, X_{\bar{k}}]]) \\ =& -\theta^{r\bar{s}}([X_j, [X_i, X_{\bar{k}}]]) = f^{r\bar{s}}_{ji\bar{k}}. \end{aligned}$$

Note that these coefficients do not form any tensor, since their transformation rule under the change of the frame involves derivatives.

3.2. **The prolongation in homogeneity one.** The natural Cartan connection associated with the distribution D will allow us to construct the coframes behaving much nicer and we shall obtain the components of the curvature tensor at the same time.

Let $\pi\colon \mathcal{G} \to M$ be any principal P-bundle on M and $\omega\colon T\mathcal{G} \otimes \mathbb{C} \to \mathfrak{g} \otimes \mathbb{C}$ be a regular and normal Cartan connection of type G/P (which always exists by Theorem 1). For any section $s\colon M \to \mathcal{G}$ we can write explicitly:

$$s^*\omega = \begin{pmatrix} \omega^i_j & \omega_i & \omega_{[i\bar{j}]} \\ \omega^j & -2i \operatorname{Im} \operatorname{tr} \omega^i_j & -\omega_{\bar{i}} \\ \omega^{[i\bar{j}]} & -\omega^{\bar{j}} & -\omega^{\bar{j}}_{\bar{i}} \end{pmatrix}$$

where $\omega^{[i\bar{j}]}$, $\omega_{[i\bar{j}]}$, ω^i_j, ω^i, $\omega^{\bar{j}}_{\bar{i}} = \overline{\omega^i_j}$, ω_j are 1-forms on M.

We say that the Cartan connection $(\mathcal{G}, \omega)$ is *adapted to the distribution* D, if $D = \langle \omega^{[i\bar{j}]} \rangle^\perp$. It is easy to see that this definition does not depend on the choice of the section s.

Let (G,ω) be an adapted Cartan connection. Then we have

$$D^{\perp} = \langle \omega^{[i\bar{j}]} \rangle = \langle \theta^{[i\bar{j}]} \rangle,$$

where, as above, the forms $\theta^{[i\bar{j}]}$ are defined by fixing a frame $\{X_1,\dots,X_n\}$ on $D^{1,0}$.

We can always choose such section $s\colon M \to \mathcal{G}$ that

$$\omega^i = \theta^i \mod D^{\perp}. \tag{3}$$

This condition defines s uniquely up to the transformations $s \to sg$, where $g\colon M \to P_+$ is an arbitrary P_+-valued gauge transformation.

Consider the component $\Omega^{[i\bar{j}]}$ of the curvature tensor:

$$\Omega^{[i\bar{j}]} = d\omega^{[i\bar{j}]} - \omega^i \wedge \omega^{\bar{j}} + \omega^i_k \wedge \omega^{[k\bar{j}]} - \omega^{[i\bar{k}]} \wedge \omega^{\bar{j}}_{\bar{k}}.$$

Let us remind that ω is regular and so only positive homogeneities may appear in the curvature. The vanishing of homogeneity zero component immediately implies that

$$d\omega^{[i\bar{j}]} = \omega^i \wedge \omega^{\bar{j}} = \theta^i \wedge \theta^{\bar{j}} \mod D^{\perp},$$

and, hence

$$\omega^{[i\bar{j}]} = \theta^{[i\bar{j}]} \quad \text{for all } 1 \le i < j \le n. \tag{4}$$

Next, let us compute the curvature coefficients of degree 1 together with the section normalizations of the form $s \mapsto \tilde{s} = sg$, where g takes values in $\exp \mathfrak{g}_1$. Any such transformation leads to the following transformation of the pull-back forms $\tilde{s}^*\omega$:

$$\begin{aligned}
\tilde{\omega}^{[i\bar{j}]} &= \omega^{[i\bar{j}]},\\
\tilde{\omega}^i &= \omega^i + p_{\bar{j}}\omega^{[i\bar{j}]},\\
\tilde{\omega}^i_j &= \omega^i_j - p_j\omega^i - \frac{1}{2}p_j p_{\bar{k}}\omega^{[i\bar{k}]},
\end{aligned}$$

where the functions $p_{\bar{j}}$ define the mapping $dg\colon M \to \mathfrak{g}_1$, $p_{\bar{j}} = \overline{p_j}$. Assume that

$$\begin{aligned}
\omega^i &= \theta^i + C^i_{j\bar{k}}\omega^{[j\bar{k}]},\\
\omega^i_j &= A^i_{kj}\omega^k + B^i_{\bar{k}j}\omega^{\bar{k}} \mod D^{\perp}
\end{aligned}$$

for some functions A^i_{kj}, $B^i_{\bar{k}j}$, $C^i_{j\bar{k}}$. They are transformed by the gauge transformation g according to the following formula:

$$\begin{aligned}
\tilde{A}^i_{kj} &= A^i_{kj} - \delta^i_k p_j;\\
\tilde{B}^i_{kj} &= B^i_{kj}\\
\tilde{C}^i_{j\bar{k}} &= C^i_{j\bar{k}} + \delta^i_j p_{\bar{k}}.
\end{aligned}$$

We can use the functions p_j to control the trace A^i_{ij}. We compute modulo $\Lambda^1(M) \wedge D^{\perp}$

$$\begin{aligned}\Omega^{i\bar{j}} \equiv&\theta^i \wedge \theta^{\bar{j}} + f^{i\bar{j}}_{rs\bar{k}}\theta^r \wedge \theta^{[s\bar{k}]} + \overline{f^{j\bar{i}}_{rk\bar{s}}}\theta^{\bar{r}} \wedge \theta^{[s\bar{k}]} - \omega^i \wedge \omega^{\bar{j}} + \omega^i_k \wedge \omega^{[k\bar{j}]} - \omega^{[i\bar{k}]} \wedge \omega^{\bar{j}}_{\bar{k}} \\ \equiv&(\omega^i - C^i_{r\bar{k}}\omega^{[r\bar{k}]}) \wedge (\omega^{\bar{j}} - \overline{C^j_{k\bar{s}}}\overline{\omega^{[k\bar{s}]}}) + f^{i\bar{j}}_{rs\bar{k}}\omega^r \wedge \omega^{[s\bar{k}]} + \overline{f^{j\bar{i}}_{rk\bar{s}}}\omega^{\bar{r}} \wedge \omega^{[s\bar{k}]} \\ &- \omega^i \wedge \omega^{\bar{j}} + A^i_{rk}\omega^r \wedge \omega^{[k\bar{j}]} + B^i_{\bar{r}k}\omega^{\bar{r}} \wedge \omega^{[k\bar{j}]} + \overline{A^j_{rk}}\omega^{\bar{r}} \wedge \omega^{[i\bar{k}]} + \overline{B^j_{\bar{r}k}}\omega^r \wedge \omega^{[i\bar{k}]}.\end{aligned}$$

It follows

$$\Omega^{[i\bar{j}]} \equiv P^{i\bar{j}}_{rs\bar{t}}\omega^r \wedge \omega^{[s\bar{t}]} + \overline{P^{j\bar{i}}_{rt\bar{s}}}\omega^{\bar{r}} \wedge \omega^{[s\bar{t}]} \mod \wedge^2 D^\perp,$$

where

$$P^{i\bar{j}}_{rs\bar{t}} = f^{i\bar{j}}_{rs\bar{t}} + A^i_{rs}\delta^{\bar{j}}_{\bar{t}} + \overline{B^j_{\bar{r}t}}\delta^i_s + \overline{C^j_{t\bar{s}}}\delta^i_r.$$

Next we compute the torsion part of the curvature

$$\Omega^i = d\omega^i - \omega^j \wedge \omega^i_j + (\omega^j_j - \overline{\omega^j_j}) \wedge \omega^i + \omega_{\bar{j}} \wedge \omega^{[i\bar{j}]}$$

modulo $D^\perp$. We have

$$d\omega^i \equiv d\theta^i + C^i_{j\bar{k}}\theta^j \wedge \theta^{\bar{k}} \equiv C^i_{j\bar{k}}\omega^j \wedge \omega^{\bar{k}}$$

$$\begin{aligned}\Omega^i &= d\omega^i - \omega^j \wedge \omega^i_j + (\omega^j_j - \overline{\omega^j_j}) \wedge \omega^i + \omega_{\bar{j}} \wedge \omega^{[i\bar{j}]} \\ &\equiv C^i_{j\bar{k}}\omega^j \wedge \omega^{\bar{k}} + A^i_{kj}\omega^k \wedge \omega^j + B^i_{\bar{k}j}\omega^{\bar{k}} \wedge \omega^j \\ &- (A^j_{kj}\omega^k + B^j_{\bar{k}j}\omega^{\bar{k}} - \overline{A^j_{kj}}\omega^{\bar{k}} - \overline{B^j_{\bar{k}j}}\omega^k) \wedge \omega^i.\end{aligned}$$

Hence

$$\Omega^i = Q^i_{rs}\omega^r \wedge \omega^s + Q^i_{r\bar{s}}\omega^r \wedge \omega^{\bar{s}} \mod D^\perp,$$

where

$$\begin{aligned}Q^i_{[rs]} &= A^i_{[rs]} - \frac{1}{2}(A^j_{rj}\delta^i_s - A^j_{sj}\delta^i_r) + \frac{1}{2}(\overline{B^j_{\bar{r}j}}\delta^i_s - \overline{B^j_{\bar{s}j}}\delta^i_r) \\ Q^i_{r\bar{s}} &= C^i_{r\bar{s}} - B^i_{\bar{s}r} - \overline{A^j_{sj}}\delta^i_r + B^j_{\bar{s}j}\delta^i_r.\end{aligned}$$

Assuming that the connection does not have the torsion in the term $\mathrm{Hom}(\wedge^2\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ (as it is implied from the normality assumption via Kostant's theorem in dimensions $n > 2$ and by our integrability assumption for $n = 2$), we get $Q^i_{[rs]} = 0$, $Q^i_{r\bar{s}} = 0$, that is

$$\begin{aligned}A^i_{[rs]} &= \frac{1}{2}(A^j_{rj}\delta^i_s - A^j_{sj}\delta^i_r) - \frac{1}{2}(\overline{B^j_{\bar{r}j}}\delta^i_s - \overline{B^j_{\bar{s}j}}\delta^i_r), \\ C^i_{r\bar{s}} &= B^i_{\bar{s}r} + \overline{A^j_{sj}}\delta^i_r - B^j_{\bar{s}j}\delta^i_r.\end{aligned}$$

It follows from the first equality that

$$A^i_{is} = (2-n)A^j_{sj} + (n-1)\overline{B^j_{\bar{s}j}}.$$

We consider the cases $n > 2$ and $n = 2$ now separately. Thus let us assume $n > 2$.

If we make $A^{i}_{is}=0$, using the freedom in $\mathfrak{g}_1$, we have

$$A^{j}_{sj}=\frac{n-1}{n-2}\overline{B^{j}_{\bar s j}}$$

and

$$C^{i}_{r\bar s}=B^{i}_{\bar s r}+\frac{1}{n-2}B^{j}_{sj}\delta^{i}_{r}.$$

Substituting the second equality into the expression of $P^{i\bar j}_{rs\bar t}$ we get:

$$P^{i\bar j}_{rs\bar t}=f^{i\bar j}_{rs\bar t}+A^{i}_{rs}\delta^{\bar j}_{\bar t}+\overline{B^{j}_{\bar r t}}\delta^{i}_{s}+\overline{B^{j}_{\bar s t}}\delta^{i}_{r}+\frac{1}{n-2}\overline{B^{l}_{\bar s l}}\delta^{\bar j}_{\bar t}\delta^{i}_{r}. \tag{5}$$

Notice that

$$P^{i\bar j}_{[rs]\bar t}=f^{i\bar j}_{[rs]\bar t}=0$$

automatically. Now,

$$P^{i\bar j}_{(rs)\bar t}=f^{i\bar j}_{rs\bar t}+A^{i}_{(rs)}\delta^{\bar j}_{\bar t}+2\overline{B^{j}_{(\bar r|t|}}\delta^{i}_{s)}+\frac{1}{n-2}\overline{B^{l}_{(\bar s|l}}\delta^{\bar j}_{\bar t|}\delta^{i}_{r)}. \tag{6}$$

The traces of $P^{i\bar j}_{(rs)\bar t}$ are

$$\begin{aligned}
P^{i\bar j}_{(rs)\bar j}&=f^{i\bar j}_{rs\bar j}+nA^{i}_{(rs)}+2\overline{B^{j}_{(\bar r|j|}}\delta^{i}_{s)}+\frac{n}{n-2}\overline{B^{l}_{(\bar s|l|}}\delta^{i}_{r)}\\
&=f^{i\bar j}_{rs\bar j}+nA^{i}_{(rs)}+\frac{3n-4}{n-2}\overline{B^{l}_{(\bar s|l|}}\delta^{i}_{r)}\\
P^{i\bar j}_{(is)\bar j}&=f^{i\bar j}_{is\bar j}+\frac{n}{2}A^{i}_{si}+\frac{(3n-4)(n+1)}{2(n-2)}\overline{B^{l}_{\bar s l}}=f^{i\bar j}_{is\bar j}+\frac{(2n^2-n-2)}{n-2}\overline{B^{l}_{\bar s l}}\\
P^{i\bar j}_{(is)\bar t}&=f^{i\bar j}_{is\bar t}+\frac{1}{2}A^{i}_{si}\delta^{\bar j}_{\bar t}+(n+1)\overline{B^{j}_{\bar s t}}+\frac{n+1}{n-2}\overline{B^{l}_{\bar s l}}\delta^{\bar j}_{\bar t}\\
&=f^{i\bar j}_{is\bar t}+(n+1)\overline{B^{j}_{\bar s t}}+\frac{3n+1}{2(n-2)}\overline{B^{l}_{\bar s l}}\delta^{\bar j}_{\bar t}.
\end{aligned}$$

Since we have assumed that $n\geq 3$, we can uniquely determine coefficients A^{i}_{kj} and $B^{i}_{\bar k j}$ so that the tensor $P^{i\bar j}_{rs\bar t}$ is totally trace-free. This in turn defines the coefficients $C^{i}_{r\bar s}$ as well and we are done.

In the case $n=2$ the computations are a bit simpler. Going back to our general equalities and normalizing the section so that the trace A^{i}_{is} vanishes, we immediately conclude that the trace $B^{j}_{\bar s j}$ vanishes too. Consequently,

$$\begin{aligned}
A^{i}_{[rs]}&=\frac{1}{2}(A^{j}_{rj}\delta^{i}_{s}-A^{j}_{sj}\delta^{i}_{r}),\\
C^{i}_{r\bar s}&=B^{i}_{\bar s r}+\overline{A^{j}_{sj}}\delta^{i}_{r}.
\end{aligned}$$

Thus we can again substitute into the expression for $P^{i\bar j}_{rs\bar t}$ to obtain

$$P^{i\bar j}_{rs\bar t}=f^{i\bar j}_{rs\bar t}+A^{i}_{rs}\delta^{\bar j}_{\bar t}+\overline{B^{j}_{\bar r t}}\delta^{i}_{s}+\overline{B^{j}_{\bar s t}}\delta^{i}_{r}+A^{k}_{sk}\delta^{\bar j}_{\bar t}\delta^{i}_{r}.$$

A direct check and computation of the traces again reveal that the expression is automatically symmetric in the indices r and s and the A's and B's (and thus also C's) are uniquely determined by requiring the traces to vanish.

Summarizing, we have verified the following:

Theorem 2. *The trace-free tensor $P^{i\bar{j}}_{rs\bar{t}}$ computed above is the only fundamental invariant of the non-degenerate free CR distribution D of rank $n \geq 3$ on a manifold of dimension $2n + n^2$. Hence, the Cartan connection associated with D is locally flat if and only if this tensor vanishes identically.*

In the case of non-degenerate real 4-dimensional free CR-distributions D on 8-dimensional manifolds, the Nijenhuis tensor of the complex structure on D together with the trace-free tensor $P^{i\bar{j}}_{rs\bar{t}}$ computed above form the complete system of invariants. Hence, the Cartan connection associated with D is locally flat if and only if these tensors vanish identically.

In particular, vanishing of the tensor $P^{i\bar{j}}_{rs\bar{t}}$ gives us the explicit condition when an arbitrary non-degenerate free CR-distribution D of rank $2n \geq 4$ is locally equivalent to the left-invariant distribution on the nilpotent Lie group corresponding to the algebra $\mathfrak{g}_-$ (in the case of rank 4 we have to add the integrability of the complex structure J on D). Moreover, let us notice that we obtain these complete data in the first prolongation step already.

Remark 1. *The conclusion of Theorem 2 that the complex structures are always integrable, except the lowest dimension of interest, is in fact not surprising. Just notice that the cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$ lives in homogeneities zero and one and we have excluded the zero homogeneity by the regularity assumption on the infinitesimal flag structures. A generic normal Cartan connection of our type will have its Nijenhuis tensor of the complex structure on D as a differential consequence of the homogeneity zero harmonic component (via the Bianchi identity and the BGG machinery). Thus on embedded manifolds in $\mathbb{C}^{n+n^2}$, $n \geq 2$, the Nijenhuis tensor vanishes for common reasons.*

Remark 2. *Let us also comment on the geometric meaning of the coefficients A, B and C which we computed during our prolongation step.*

In general terms, these objects correspond to the choice of partial Weyl connections (the coefficients A^i_{jk} and $B^i_{\bar{k}j}$ define the appropriate parts of the principal connection form, thus being the usual Christoffel symbols of the corresponding affine connection) and the splittings of the filtration (the improvement of the coframe by deforming θ^i with the help of the coefficients C^i_{kl}), cf. [4, Chapter 5]. *Both of these objects have to be fixed together because they influence the same curvature components in homogeneity one. This has been reflected by the explicit link between A's, B's and C's above. Of course, these partial Weyl connections are analogs of the Webster-Tanaka connections for hypersurface type CR geometries, cf.* [4, Section 5.2.12]. *The next step of the prolongation procedure along the similar lines would complete* *the connections to include also differentiation in the directions complementary to D and would compute the homogeneity two component of the so called Rho tensor.*

4. The Fefferman construction

4.1. **The abstract functorial setting.** The original Fefferman's construction of the circle bundle over a hypersurface type CR manifold, equipped with a conformal structure, can be presented in an abstract algebraic way, see [4, Section 4.5].

Let G/P and $\tilde{G}/\tilde{P}$ be two (real or complex) parabolic homogeneous spaces and let us assume

- an infinitesimally injective homomorphism $i\colon G \to \tilde{G}$ is given,
- the G-orbit of $o = e\tilde{P} \in \tilde{G}/\tilde{P}$ is open (thus, $\mathfrak{g} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ induced by $i' : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is surjective),
- $P \subset G$ contains $Q := i^{-1}(\tilde{P})$.

Consequently, there is the natural projection $\pi : G/Q \to G/P$, Q is a closed subgroup of G (which is usually not parabolic). Moreover, the homomorphism $i : G \to \tilde{G}$ induces the smooth map $G/Q \to \tilde{G}/\tilde{P}$ which is a covering of the G-orbit of o, and the latter open subset in $\tilde{G}/\tilde{P}$ carries a canonical geometry of type $(\tilde{G}, \tilde{P})$. This can be pulled back to obtain such a geometry on G/Q.

In particular, it may happen that

$$i(G)\tilde{P} = \tilde{G} \text{ and } i(P) = i(G) \cap \tilde{P}$$

i.e. $Q = P$ is the parabolic subgroup. Then both parabolic geometries turn out to live over the same base manifold $G/P = \tilde{G}/\tilde{P}$. We say that i is an *inclusion of parabolic homogeneous spaces*. This was the case of the free rank ℓ distributions with the spinorial geometry on the Fefferman space, see [7].

The two steps discussed above are instances of two general functorial constructions on Cartan geometries $(\mathcal{G}, \omega)$

- the correspondence spaces
- the structure group extensions.

The first one is given by a choice of subgroups $Q \subset P \subset G$ and it increases the underlying manifold $M = \mathcal{G}/P$ into a fibre bundle $\tilde{M} = \mathcal{G}/Q \to M$ with fiber Q/P.

The other one is based on embeddings of the structure group $G \to \tilde{G}$ and reasonable choices of subgroups $P \subset G$, $\tilde{P} \subset \tilde{G}$, and it leads to Cartan geometries on the same manifolds M, but with bigger structure groups.

Combination of these two steps yields the Fefferman-like constructions as shown above.

4.2. **$|1|$-graded parabolic geometry modelled on $\mathfrak{su}(n,n)$.** From the algebraic point of view, we have seen that the passage from free distributions to free CR distributions consisted in replacing smartly skew-symmetric matrices with the skew-Hermitian ones. Let us try to find the right way for generalizing the Fefferman construction by inspecting carefully our lowest dimensional example with $n = 1$, i.e. the hypersurface type CR-structure embedded in $\mathbb{C}^2$.

There, the original Fefferman construction provides a circle bundle $\tilde{M}$ over M, equipped with a four-dimensional conformal structure in Lorentzian signature. But the 4-dimensional Lorentzian conformal geometry, in fact, can be equally well modelled as a $\mathfrak{su}(2,2)$ Cartan geometry.

Indeed, $\mathfrak{su}(2,2)$ can be represented by block matrices $\begin{pmatrix} A & Z \\ X & -A^* \end{pmatrix}$ with $A \in \mathfrak{gl}(2,\mathbb{C})$ such that $\operatorname{tr} A$ is real and $X, Z \in \mathfrak{u}(2)$. Then the components of grade $-1, 0, 1$ correspond to the lower-left, the diagonal and upper right blocks. The induced geometric structure is a choice of $\mathfrak{u}(2)$-frames on the 4-dimensional tangent spaces with structure group consisting of all regular complex 2×2 matrices A with real determinant acting on the skew-Hermitian frames X by $(A^*)^{-1} X A^{-1}$. This action clearly preserves the conformal class of the real inner product induced on the tangent spaces by the determinant $\det X$. (For details see [4, 4.1.10].)

For a higher dimensional analogue of this geometry consider $\mathfrak{su}(n,n)$ with $|1|$-grading of $\mathfrak{su}(n,n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as follows

$$\begin{array}{cc} n & n \end{array}$$
$$\begin{pmatrix} \boxed{\;0\;} & \boxed{\;1\;} \\ \boxed{-1} & \boxed{\;0\;} \end{pmatrix} \begin{array}{c} n \\ n \end{array}$$

For a suitable choice of the pseudo-Hermitian form in the split signature $\mathfrak{g}_{\pm 1}$ are the spaces of skew-Hermitian matrices with respect to the anti-diagonal, while $\mathfrak{g}_0 = \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}$ with pairs of matrices $(A, -\bar{A}^T)$ appearing always in the block diagonal, and $\operatorname{tr} A \in \mathbb{R}$. Notice, $\mathfrak{g}_{\pm 1}$ are dual to each other with respect to the Killing form.

Now, as for any $|1|$-graded parabolic geometry, the geometric structure is given by the appropriate reduction of the frame bundle to the structure group G_0:

Definition 3. *The $|1|$-graded geometry modelled on $\mathfrak{su}(n,n)$ is given by a smooth identification of the tangent spaces T_pM to a real n^2-dimensional manifold M with the space of skew-Hermitian 2-forms $\Lambda^2_{\text{skew-H}}(\mathcal{S})$ on an auxiliary complex n-dimensional bundle $\mathcal{S}$ over M. Or, alternatively, it can be given by a smooth identification of the tangent spaces with $n \times n$*

skew-Hermitian matrices $\psi_p\colon T_pM \to \mathfrak{u}(n)$*, where two such identifications* ψ^1 *and* ψ^2 *are equivalent if* $\psi^1_p = A^*(p)\psi^2_p A(p)$ *for some* $\mathrm{GL}(n,\mathbb{C})$*-valued function* $A(p)$*.*

We shall write $\mathbb{S} \simeq \mathbb{C}^n$ for the standard fiber of the auxiliary bundle. Notice, that the purely imaginary part of the centre in $\mathrm{GL}(n,\mathbb{C})$ acts on $\Lambda^2_{\text{skew-H}}(\mathbb{C}^n)$ trivially and so our $\mathfrak{g}_0$ is the appropriate algebra to discuss here.

Still, there is the $\mathbb{Z}_2$ kernel of the action of the real multiples of identity matrices. But at the group level, we shall work with $G = \mathrm{SU}(n,n)$ and so G_0 will be the group of all regular complex matrices with real determinant. This means that we actually work with the analogue to the choice of a spin structure on M in the four-dimensional case and our structure group G_0 is a double covering of the effective group $G_0/\mathbb{Z}_2$ coming from the effective Klein geometry of the type in question.

4.3. **The Fefferman construction for the homogeneous model.** Similar to the case of free distributions in [7] and according to the construction from Section 4.1 we choose an embedding of $\mathfrak{g} = \mathfrak{su}(n+1,n)$ to the $|1|$-graded Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{su}(n+1,n+1)$

$$\begin{pmatrix} A & X & Y \\ -Z^* & 2\alpha & -X^* \\ T & Z & -A^* \end{pmatrix} \mapsto \begin{pmatrix} A & \frac{1}{\sqrt{2}}X & \frac{1}{\sqrt{2}}X & Y \\ -\frac{1}{\sqrt{2}}Z^* & \alpha & \alpha & -\frac{1}{\sqrt{2}}X^* \\ -\frac{1}{\sqrt{2}}Z^* & \alpha & \alpha & -\frac{1}{\sqrt{2}}X^* \\ T & \frac{1}{\sqrt{2}}Z & \frac{1}{\sqrt{2}}Z & -A^* \end{pmatrix}$$

where $A, Y, T \in \mathrm{Mat}_\ell(\mathbb{C})$, $X, Z \in \mathbb{C}^\ell$, $Y + Y^* = T + T^* = 0$, $\alpha = -i \operatorname{Im} \operatorname{Tr} A$.

Now, $\tilde{\mathfrak{p}}$ is the parabolic subalgebra corresponding to the only $|1|$-graded geometry as discussed above. Similar to the hypersurface CR case, the preimage $\mathfrak{q}$ of $\tilde{\mathfrak{p}}$ is nearly the entire $\mathfrak{p}$, with just one dimension in the centre $\mathbb{C}$ of $\mathfrak{g}_0$ lacking.

The general functorial construction of the Fefferman space works in the homogeneous model as follows. We first consider the quotient of $G = \mathrm{SU}(n+1,n)$ by all of the P but the centre of G_0. This will provide a complex line bundle $\mathcal{E}(1,0)$ associated to the action of central elements $0 \neq z \in \mathbb{C}$, $s \mapsto z\cdot s$. Notice, we have adopted the same convention for weights as in the hypersurface CR case, where $\mathcal{E}(a,b)$ stays for central action $s \mapsto z^a\bar{z}^b \cdot s$.

Now, write Q for the preimage of $\tilde{P}$ in the embedding. Clearly, the requested space G/Q is obtained by factorizing the action of the real part of the centre on $\mathcal{E}(1,0)$ and thus G/Q can be identified with the bundle of lines in $\mathcal{E}(1,0)$. This provides the circle bundle $\tilde{M}$, exactly as in the CR hypersurface case.

Moreover, the fixed grading of the Lie algebra $\mathfrak{g} = \mathfrak{su}(n+1,n)$ is well understood via the identification of $\mathfrak{g}$ with the skew-Hermitian matrices over the standard representation $\mathbb{C}^{n+2}$ and its splitting into $\mathfrak{g}_0$-submodules $\mathbb{C}^n \oplus \mathbb{C} \oplus \mathbb{C}^n$. This in turn provides the identification of $\mathfrak{g}/\mathfrak{q} = \mathfrak{g}_{-1} \oplus i\,\mathbb{R}$ with skew-Hermitian forms on $\mathbb{C}^n \oplus \mathbb{C}$ and, thus, the requested identification of the tangent bundle $T\tilde{M}$ with skew-Hermitian forms on the auxiliary vector bundle with standard fibre $\mathbb{C}^{n+1}$.

4.4. **The explicit construction in the curved case.** If we start with the normal Cartan connection ω associated with a free CR distribution on a manifold M and the complex line bundle $\mathcal{E}(1,0)$ over M, then the categorial construction provides the quotient manifold $\tilde{M} = \mathcal{G}/Q$, i.e. the bundle of real lines in $\mathcal{E}(1,0)$, exactly as in the flat case. In fact, we do not need the full Cartan bundle, since $\mathcal{E}(1,0)$ is a well defined quotient bundle of the complex frame bundle $\mathcal{G}_0 = \mathcal{G}/P_+$ of the defining distribution $\mathcal{D} \subset TM$, at least locally. But we have to be more careful with the requested $\mathfrak{su}(n+1,n+1)$ geometry there.

Of course, we want to build the Fefferman circle bundle $\tilde{M} \to M$ including the required $|1|$-graded parabolic geometry with the minimal effort straight from the free CR distribution itself. The key is the standard tractor calculus now.

Remember that the *standard tractor bundle* is $\mathcal{V}M = \mathcal{G} \times_P \mathbb{V}$, where $\mathbb{V}$ is the standard $SU(n+1,n)$ representation. The filtration $\mathbb{V} = \mathbb{V}^0 \supset \mathbb{V}^1 \supset \mathbb{V}^2 \supset 0$ induces the filtration of the tractor bundle $\mathcal{V}M = \mathcal{V}^0 M \supset \mathcal{V}^1 M \supset \mathcal{V}^2 M \supset 0$ on $\mathcal{V}M$ with codimensions $n+1$ and n.

The standard representation carries a natural Hermitian form with respect to which the adjoint representation is identified with the space of skew-Hermitian maps on $\mathbb{V}$. As a G_0 module, the standard representation splits as $\mathbb{V} = \mathbb{C}^n \oplus \mathbb{C} \oplus \mathbb{C}^n$ and there is the P-module $\mathbb{S} = \mathbb{V}/\mathbb{V}^2$, which decomposes as $\mathbb{S} = \mathbb{C} \oplus \mathbb{C}^n$ and is dual to $\mathbb{V}^1$ as G_0-module.

Finally, the standard fiber of the tangent space to the circle bundle $\tilde{M}$ is $\mathfrak{g}_- \oplus i\,\mathbb{R}$ and it can be identified with a subspace of complex linear mappings $\mathbb{V}^1 \to \mathbb{S}$. Exploiting the duality of $\mathbb{V}^1$ and $\mathbb{S}$ as G_0-modules, we may view them as skew-Hermitian 2-forms on $\mathbb{V}^1$ as soon as we know the G_0-invariant splittings of $\mathbb{V}^1$ and $\mathbb{S}$. By the very definition, this G_0-invariant identification is compatible with the inclusion $G \to \tilde{G}$. Moreover, since the entire adjoint representation is identified as the space of skew-Hermitian maps on $\mathbb{V}$, the definition of the $\mathfrak{su}(n+1,n+1)$ geometry on $\tilde{M}$ does not depend on the choice of the G_0 reduction. Thus, the identification is carried over to the level of the appropriate associated bundles and we may identify the tangent bundle to the circle bundle $\tilde{M}$ with the skew-Hermitian second tensor product of the auxiliary bundle $\mathcal{S}$, which is the definition of the $|1|$-graded geometry introduced above. Let us finally notice that the splitting of the auxiliary bundle $\mathcal{S}$ necessary for our identification is obtained after the first prolongation already.

The necessary covering of the Cartan bundle for the free CR geometry by the $SU(n+1,n)$ geometry, ensuring the existence of the standard tractor bundle $\mathcal{V}M$ and thus also $\mathcal{S}$, is clearly equivalent to the existence of the complex line bundle $\mathcal{E}(1,0)$.

But for embedded free CR-manifolds this line bundle can be constructed similarly to the hypersurface case: There is the trivial canonical bundle $\mathcal{K}$ defined as the $(n+n^2)$-exterior power of the annihilator of the holomorphic vectors in the complexified tangent bundle. Clearly the centre in G_0 acts by the power z^{-n-2n^2} on its standard fibre, and so we can take the appropriate root and consider the dual space.

Having done this, we define the Fefferman space $\tilde{M}$ as the circle bundle obtained by real projectivization of the complex line bundle $\mathcal{E}(1,0)$. At the same time, the data from the first prolongation provide the identification of the tangent bundle of $\tilde{M}$ with the bundle of skew-Hermitian forms $\Lambda^2_{\text{skew-Herm}}\mathcal{S}$. This concludes the geometric construction of the Fefferman space, including its $|1|$-graded parabolic geometry structure.

4.5. **Concluding remarks and conjectures.** In general, the canonical normal Cartan connection $\tilde{\omega}$ on the Fefferman space $\tilde{M}$ does not need to be the one induced from the original connection ω by the functorial construction. In such a case, they both define the same underlying structure and so the difference is given by an adjoint tractor valued 1-form of positive homogeneity.

The functorial construction of the Cartan connection $\tilde{\omega}$ links the two curvatures κ and $\tilde{\kappa}$ in a simple algebraic way. In principle, $\tilde{\kappa} = i' \circ \kappa$, where we view the curvatures as the adjoint tractor valued two-forms on $\tilde{M}$ and i' is induced by the fixed Lie algebra homomorphism $\mathfrak{su}(n+1,n) \to \mathfrak{su}(n+1,n+1)$.

Moreover, we may use the same Lie algebra morphism i' to rewrite this formula at the level of cochains as a linear map

$$\phi : \Lambda^k\mathfrak{p}_+ \otimes \mathfrak{g} \to \Lambda^k\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$$

and the question becomes *'under which conditions is the image $\phi(\kappa)$ a $\tilde{\partial}^*$-closed element for a ∂^*-closed element κ?'*.

These quite tedious computations were performed in the free rank n distribution case in [7] and they can be performed with very minor extensions in our case. The conclusion reads:

Claim. The Fefferman extension of a free CR-geometry to $|1|$-graded geometry on the circle bundle $\tilde{M}$ is normal if and only if the P-invariant restriction $\kappa_{1,1}$ of the entire curvature κ to both arguments in the complex distribution $\mathcal{D} \subset TM$ vanishes identically.

The recent results in the parabolic geometry, based on the detailed understanding of the algebraic structure of the cohomology governing the curvature, allow us to go much further (see [4] for detailed treatment). In particular, the general BGG machinery implies that the entire curvature is given by a natural linear operator applied to the harmonic part of the curvature. Since our projection to the component $\kappa_{1,1}$ is invariant too, every projection of $\kappa_{1,1}$ to an irreducible component would be an invariant operator with values in an invariant subbundle of two-cochains. But Kostant's version of the Bott-Borel-Weil theorem (cf. [4]) guarantees that all the harmonic components appear in the entire space of cochains with multiplicity one. At the same time, there are only linear operators mapping bundles coming from representations in second cohomologies into higher cohomologies. Thus, such operators cannot exist as linear operators without curvature in their symbols.

Next, the Bianchi identity relates the differential and the fundamental derivative

$$\partial\kappa = \sum_{\text{cycl}} i_\kappa \kappa - D\kappa$$

and employing $\partial^*\kappa = 0$ we obtain for the lowest available nontrivial homogeneity component $\Box\kappa^0_{1,1}$ the appropriate projection of

$$\partial^* \sum_{\text{cycl}} i_\kappa \kappa.$$

Let us notice, that the homogeneity two obstruction is just a quadratic tensorial expression. If this quantity vanishes, the next homogeneity will become algebraic too, etc. The detailed understanding of the normality obstruction in terms of the harmonic curvature will require much more effort and it will be treated elsewhere.

Let us conclude with an example of nontrivial geometry satisfying the above normality condition for the Fefferman space. We adapt an example worked out for the spinorial geometry by Stuart Armstrong, see [1].

The $2n + n^2$-dimensional flat free CR manifold Q can be described in coordinates $\{z_j, w_{kl}\}$ with $1 \le j \le n$, $1 \le k \le l \le n$, where $z_j, w_{kl} \in \mathbb{C}$ and $\operatorname{Re} w_{kk} = 0$ for $1 \le k \le n$ by the $D^{(1,0)}$ vector fields

$$Z_j = \frac{\partial}{\partial z_j} - \sum_{p=j}^{n} \bar z_p \frac{\partial}{\partial w_{jp}}.$$

Then

$$\begin{aligned} W_{kk} &= [Z_k, \bar Z_k] = \frac{\partial}{\partial w_{kk}} - \frac{\partial}{\partial \bar w_{kk}} & \\ W_{kl} &= [Z_k, \bar Z_l] = \frac{\partial}{\partial w_{kl}} & \text{if } k < l \\ W_{kl} &= [Z_k, \bar Z_l] = -\frac{\partial}{\partial \bar w_{lk}} & \text{if } k > l. \end{aligned}$$

For $n \ge 4$ we modify Q by replacing Z_1 by

$$Z_1' = Z_1 + \bar w_{12} \frac{\partial}{\partial w_{34}}.$$

Notice that $[Z_1', Z_j] = 0$ for $2 \le j \le n$, hence the modified CR structure is still integrable and $[Z_1', \bar Z_j] = W_{1j}$. The only resulting change in the structure equations is that now $f^{[34]}_{1[12]} = 1$. It follows that the tensor P is already trace-free for $A = B = 0$, hence $A = B = C = 0$ in this case and the only non-vanishing coefficient in P is $P^{[34]}_{1[12]} = 1$. Since the curvature in homogeneity 1 is constant, by the Bianchi identity, the curvature of higher homogeneity vanishes automatically.

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School of Science and Technology, University of New England, 2351 Armidale, Australia

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic