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ON LIE ALGEBRAS OF GENERATORS OF INFINITESIMAL SYMMETRIES OF ALMOST-COSYMPLECTIC-CONTACT STRUCTURES

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Abstract. We study Lie algebras of generators of infinitesimal symmetries of almost-cosymplectic-contact structures of odd dimensional manifolds. The almost-cosymplectic-contact structure admits on the sheaf of pairs of 1-forms and functions the structure of a Lie algebra. We describe Lie subalgebras in this Lie algebra given by pairs generating infinitesimal symmetries of basic tensor fields given by the almost-cosymplectic-contact structure.

Introduction

The (7-dimensional) phase space of the (4-dimensional) classical spacetime can be defined as the space of 1-jets of motions, [4]. A Lorentzian metric and an electromagnetic field then define on the phase space the geometrical structure given by a 1-form ω and a 2-form Ω such that $\omega \wedge \Omega^3 \not\equiv 0$ and $d\Omega = 0$. In [5] such structure was generalized for any odd-dimensional manifold $\boldsymbol{M}$ under the name almost-cosymplectic-contact structure. The almost-cosymplectic-contact structure on $\boldsymbol{M}$ admits a Lie bracket $[\![\,,\,]\!]$ of pairs (α, h) of 1-forms and functions which define a Lie algebra structure on the sheaf $\Omega^1(\boldsymbol{M}) \times C^\infty(\boldsymbol{M})$.

In [3, 6] we have studied infinitesimal symmetries of the almost-cosymplectic-contact structure of the classical phase space. In this paper we shall study infinitesimal symmetries of basic fields generating almost-cosymplectic-contact structure on any odd dimensional manifold. We shall prove that such infinitesimal symmetries are generated by pairs (α, h) satisfying certain properties and the restriction of $[\![\,,\,]\!]$ to the subsheaf of generators of infinitesimal symmetries defines Lie subalgebras in $(\Omega^1(\boldsymbol{M}) \times C^\infty(\boldsymbol{M}); [\![\,,\,]\!])$.

In the paper all manifolds and mappings are assumed to be smooth.

1. Preliminaries

We recall some basic notions used in the paper.

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Schouten-Nijenhuis bracket. Let us denote by $\mathcal{V}^p(\boldsymbol{M})$ the sheaf of skew symmetric contravariant tensor fields of type $(p,0)$. As the *Schouten-Nijenhuis bracket* (see, for instance, [11]) we assume the 1st order bilinear natural differential operator (see [8])

$$[,]:\mathcal{V}^p(\boldsymbol{M})\times\mathcal{V}^q(\boldsymbol{M})\to\mathcal{V}^{p+q-1}(\boldsymbol{M})$$

given by

$$i_{[P,Q]}\beta=(-1)^{q(p+1)}i_P d i_Q\beta+(-1)^p i_Q d i_P\beta-i_{P\wedge Q}d\beta \tag{1.1}$$

for any $P\in\mathcal{V}^p(\boldsymbol{M})$, $Q\in\mathcal{V}^q(\boldsymbol{M})$ and $(p+q-1)$-form β. Especially, for a vector field X, we have $[X,P]=L_X P$. The Schouten-Nijenhuis bracket is a generalization of the Lie bracket of vector fields.

We have the following identities

$$[P,Q]=(-1)^{pq}\,[Q,P]\,, \tag{1.2}$$

$$[P,Q\wedge R]=[P,Q]\wedge R+(-1)^{pq+q}\,Q\wedge[P,R]\,, \tag{1.3}$$

where $R\in\mathcal{V}^r(\boldsymbol{M})$. Further we have the (graded) Jacobi identity

$$\begin{aligned}(-1)^{p(r-1)}\big[P,[Q,R]\big]+(-1)^{q(p-1)}\big[Q,[R,P]\big]&\\ +(-1)^{r(q-1)}\big[R,[P,Q]\big]=0\,.&\end{aligned} \tag{1.4}$$

Structures of odd dimensional manifolds. Let $\boldsymbol{M}$ be a $(2n+1)$-dimensional manifold.

A *pre cosymplectic* (*regular*) *structure* (*pair*) on $\boldsymbol{M}$ is given by a 1-form ω and a 2-form Ω such that $\omega\wedge\Omega^n\not\equiv 0$. A *contravariant* (*regular*) *structure* (*pair*) (E,Λ) is given by a vector field E and a skew symmetric 2-vector field Λ such that $E\wedge\Lambda^n\not\equiv 0$. We denote by $\Omega^\flat\colon T\boldsymbol{M}\to T^*\boldsymbol{M}$ and $\Lambda^\sharp\colon T^*\boldsymbol{M}\to T\boldsymbol{M}$ the corresponding "musical" morphisms.

By [9] if (ω,Ω) is a pre cosymplectic pair then there exists a unique regular pair (E,Λ) such that

$$(\Omega^\flat{}_{|\,\mathrm{Im}(\Lambda^\sharp)})^{-1}=\Lambda^\sharp{}_{|\,\mathrm{Im}(\Omega^\flat)}\,,\quad i_E\omega=1\,,\quad i_E\Omega=0\,,\quad i_\omega\Lambda=0\,. \tag{1.5}$$

On the other hand for any regular pair (E,Λ) there exists a unique (regular) pair (ω,Ω) satisfying the above identities. The pairs (ω,Ω) and (E,Λ) satisfying the above identities are said to be mutually *dual*. The vector field E is usually called the *Reeb vector field* of the pair (ω,Ω). In fact geometrical structures given by dual pairs coincide.

An *almost-cosymplectic-contact* (*regular*) *structure* (*pair*) [5] is given by a pair (ω,Ω) such that

$$d\Omega=0\,,\qquad\omega\wedge\Omega^n\not\equiv 0\,. \tag{1.6}$$

The dual *almost-coPoisson-Jacobi structure* (*pair*) is given by the pair (E,Λ) such that

$$[E,\Lambda]=-E\wedge\Lambda^\sharp(L_E\omega)\,,\qquad[\Lambda,\Lambda]=2\,E\wedge(\Lambda^\sharp\otimes\Lambda^\sharp)(d\omega)\,. \tag{1.7}$$

Here $[,]$ is the Schouten-Nijenhuis bracket (1.1).

Remark 1.1. An almost-cosymplectic-contact pair generalizes standard cosymplectic and contact pairs. Really, if $d\omega = 0$ we obtain a cosymplectic pair (see, for instance, [1]). The dual *coPoisson pair* (see [5]) is given by the pair (E, Λ) such that $[E, \Lambda] = 0$, $[\Lambda, \Lambda] = 0$. A *contact structure* (*pair*) is given by a pair (ω, Ω) such that $\Omega = d\omega$, $\omega \wedge \Omega^n \not\equiv 0$. The dual *Jacobi structure* (*pair*) is given by the pair (E, Λ) such that $[E, \Lambda] = 0$, $[\Lambda, \Lambda] = -2E \wedge \Lambda$ (see [7]).

Remark 1.2. Given an almost-cosymplectic-contact regular pair (ω, Ω) we can consider the second pair $(\omega, F = \Omega + d\omega)$ which is almost-cosymplectic-contact but generally need not be regular.

Splitting of the tangent bundle. In what follows we assume an odd dimensional manifold $\boldsymbol{M}$ with a regular almost-cosymplectic-contact structure (ω, Ω). We assume the dual (regular) almost-coPoisson-Jacobi structure (E, Λ). Then we have $\mathrm{Ker}(\omega) = \mathrm{Im}(\Lambda^\sharp)$ and $\mathrm{Ker}(E) = \mathrm{Im}(\Omega^\flat)$ and we have the splitting

$$T\boldsymbol{M} = \mathrm{Im}(\Lambda^\sharp) \oplus \langle E \rangle , \qquad T^*\boldsymbol{M} = \mathrm{Im}(\Omega^\flat) \oplus \langle \omega \rangle ,$$

i.e. any vector field X and any 1-form β can be decomposed as

$$X = X_{(\alpha,h)} = \alpha^\sharp + h\, E , \qquad \beta = \beta_{(Y,f)} = Y^\flat + f\, \omega , \tag{1.8}$$

where h, $f \in C^\infty(\boldsymbol{M})$, α be a 1-form and Y be a vector field. In what follows we shall use notation $\alpha^\sharp = \Lambda^\sharp(\alpha)$ and $Y^\flat = \Omega^\flat(Y)$. Moreover, $h = \omega(X_{(\alpha,h)})$ and $f = \beta_{(Y,f)}(E)$. Let us note that the splitting (1.8) is not defined uniquely, really $X_{(\alpha_1,h_1)} = X_{(\alpha_2,h_2)}$ if and only if $\alpha_1^\sharp = \alpha_2^\sharp$ and $h_1 = h_2$, i.e. $\alpha_1^\sharp - \alpha_2^\sharp = 0$ that means that $\alpha_1 - \alpha_2 \in \langle \omega \rangle$. Similarly $\beta_{(Y_1,f_1)} = \beta_{(Y_2,f_2)}$ if and only if $Y_1 - Y_2 \in \langle E \rangle$ and $f_1 = f_2$.

The projections $p_2 \colon T\boldsymbol{M} \to \langle E \rangle$ and $p_1 \colon T\boldsymbol{M} \to \mathrm{Im}(\Lambda^\sharp) = \mathrm{Ker}(\omega)$ are given by $X \mapsto \omega(X)\, E$ and $X \mapsto X - \omega(X)\, E$. Equivalently, the projections $q_2 \colon T^*\boldsymbol{M} \to \langle \omega \rangle$ and $q_1 \colon T^*\boldsymbol{M} \to \mathrm{Im}(\Omega^\flat) = \mathrm{Ker}(E)$ are given by $\beta \mapsto \beta(E)\, \omega$ and $\beta \mapsto \beta - \beta(E)\, \omega$. Moreover, $\Lambda^\sharp \circ \Omega^\flat = p_1$ and $\Omega^\flat \circ \Lambda^\sharp = q_1$.

2. Lie algebras of generators of infinitesimal symmetries

We shall study infinitesimal symmetries of basic tensor fields generating the almost-cosymplectic-contact and the dual almost-coPoisson-Jacobi structures.

2.1. Lie algebra of pairs of 1-forms and functions.

The almost-cosymplectic-contact structure allows us to define a Lie algebra structure on the sheaf $\Omega^1(\boldsymbol{M}) \times C^\infty(\boldsymbol{M})$ of 1-forms and functions.

Lemma 2.1. *Let us assume two vector fields* $X_{(\alpha_i,h_i)} = \alpha_i^\sharp + h_i\, E$*,* $i = 1, 2$*, on* $\boldsymbol{M}$*. Then*

$$\begin{aligned}
(2.1)\qquad [X_{(\alpha_1,h_1)}, X_{(\alpha_2,h_2)}] = \big(& d\Lambda(\alpha_1,\alpha_2) - i_{\alpha_2^\sharp} d\alpha_1 + i_{\alpha_1^\sharp} d\alpha_2 \\
& - \alpha_1(E)\,(i_{\alpha_2^\sharp} d\omega) + \alpha_2(E)\,(i_{\alpha_1^\sharp} d\omega) \\
& + h_1\,(L_E\alpha_2 - \alpha_2(E)\,L_E\omega) - h_2\,(L_E\alpha_1 - \alpha_1(E)\,L_E\omega)\big)^\sharp \\
& + \big(\alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 - d\omega(\alpha_1^\sharp,\alpha_2^\sharp) \\
& + h_1\,(E.h_2 + \Lambda(L_E\omega,\alpha_2)) - h_2\,(E.h_1 + \Lambda(L_E\omega,\alpha_1))\big)\,E\,.
\end{aligned}$$

Proof. It follows from (see [5])

$$(2.2)\qquad [E,\alpha^\sharp] = \big(L_E\alpha - \alpha(E)\,(L_E\omega)\big)^\sharp + \Lambda(L_E\omega,\alpha)\,E\,,$$

$$\begin{aligned}
(2.3)\qquad [\alpha^\sharp,\beta^\sharp] = \big(& d\Lambda(\alpha,\beta) - i_{\beta^\sharp} d\alpha + \alpha(E)\,(i_{\beta^\sharp} d\omega) \\
& + i_{\alpha^\sharp} d\beta - \beta(E)\,(i_{\alpha^\sharp} d\omega)\big)^\sharp - d\omega(\alpha^\sharp,\beta^\sharp)\,E\,.
\end{aligned}$$

Then

$$\begin{aligned}
[X_{(\alpha_1,h_1)}, X_{(\alpha_2,h_2)}] = {} & [\alpha_1^\sharp,\alpha_2^\sharp] + h_2[\alpha_1^\sharp, E] + h_1[E,\alpha_2^\sharp] \\
& + \big(\alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 + h_1 E.h_2 - h_2 E.h_1\big)\,E
\end{aligned}$$

and from (2.2) and (2.3) we get Lemma 2.1. □

As a consequence of Lemma 2.1 we get the Lie bracket of pairs $(\alpha_i, h_i) \in \Omega^1(\boldsymbol{M}) \times C^\infty(\boldsymbol{M})$ given by

$$\begin{aligned}
(2.4)\qquad [\![\,(\alpha_1,h_1);(\alpha_2,h_2)\,]\!] = \big(& d\Lambda(\alpha_1,\alpha_2) - i_{\alpha_2^\sharp} d\alpha_1 + i_{\alpha_1^\sharp} d\alpha_2 \\
& + \alpha_1(E)\,(i_{\alpha_2^\sharp} d\omega) - \alpha_2(E)\,(i_{\alpha_1^\sharp} d\omega) \\
& + h_1\,(L_E\alpha_2 - \alpha_2(E)\,L_E\omega) - h_2\,(L_E\alpha_1 - \alpha_1(E)\,L_E\omega)\,; \\
& \alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 - d\omega(\alpha_1^\sharp,\alpha_2^\sharp) \\
& + h_1\,(E.h_2 + \Lambda(L_E\omega,\alpha_2)) - h_2\,(E.h_1 + \Lambda(L_E\omega,\alpha_1))\big)
\end{aligned}$$

which defines a Lie algebra structure on $\Omega^1(\boldsymbol{M})\times C^\infty(\boldsymbol{M})$ given by the almost-cosymplectic-contact structure (ω,Ω). Moreover, we have

$$X_{[\![\,(\alpha_1,h_1);(\alpha_2,h_2)\,]\!]} = [X_{(\alpha_1,h_1)}, X_{(\alpha_2,h_2)}]\,.$$

Let T be a tensor field of any type. An *infinitesimal symmetry* of T is a vector field X on $\boldsymbol{M}$ such that $L_X T = 0$. From

$$L_{[X,Y]} = L_X L_Y - L_Y L_X$$

it follows that infinitesimal symmetries of T form a Lie subalgebra, denoted by $\mathcal{L}(T)$, of the Lie algebra $(\mathcal{V}^1(\boldsymbol{M}); [,])$ of vector fields on $\boldsymbol{M}$. Moreover, the Lie subalgebra $(\mathcal{L}(T); [,])$ is generated by the Lie subalgebra $(\mathtt{Gen}(T);\, [\![\,,\,]\!]\,) \subset (\Omega^1(\boldsymbol{M}) \times C^\infty(\boldsymbol{M});\, [\![\,,\,]\!]\,)$ of generators of infinitesimal symmetries of T.

Remark 2.1. Let as recall that a *Lie algebroid structure* of a a vector bundle $\pi\colon \boldsymbol{E} \to \boldsymbol{M}$ is defined by (see, for instance, [10]):

– a composition law $(s_1, s_2) \longmapsto [\![s_1, s_2]\!]$ on the space $\Gamma(\pi)$ of smooth sections of $\boldsymbol{E}$, for which $\Gamma(\pi)$ becomes a Lie algebra,

– a smooth vector bundle map $\rho\colon \boldsymbol{E} \to T\boldsymbol{M}$, where $T\boldsymbol{M}$ is the tangent bundle of $\boldsymbol{M}$, which satisfies the following two properties:

(i) the map $s \to \rho \circ s$ is a Lie algebras homomorphism from the Lie algebra $(\Gamma(\pi); [\![\, ,]\!])$ into the Lie algebra $(\mathcal{V}^1(\boldsymbol{M}); [,])$;

(ii) for every pair (s_1, s_2) of smooth sections of π, and every smooth function $f\colon \boldsymbol{M} \to \mathbb{R}$, we have the Leibniz-type formula,

$$[\![s_1, f\, s_2]\!] = f\, [\![s_1, s_2]\!] + (i_{(\rho\circ s_1)} df)\, s_2\,. \tag{2.5}$$

The vector bundle $\pi\colon \boldsymbol{E} \to \boldsymbol{M}$ equipped with its Lie algebroid structure will be called a *Lie algebroid*; the composition law $(s_1, s_2) \mapsto [\![s_1, s_2]\!]$ will be called the *bracket* and the map $\rho\colon \boldsymbol{E} \to T\boldsymbol{M}$ the *anchor*.

The pair (α, h) can be considered as a section $\boldsymbol{M} \to T^*\boldsymbol{M} \times \mathbb{R}$ and the bracket (2.4) defines the Lie bracket of sections of the vector bundle $\boldsymbol{E} = T^*\boldsymbol{M} \times \mathbb{R} \to \boldsymbol{M}$. A natural question arise if this bracket defines on $\boldsymbol{E}$ the structure of a Lie algebroid with the anchor $\rho\colon \boldsymbol{E} \to T\boldsymbol{M}$ such that $\rho \circ (\alpha, h) = X_{(\alpha,h)}$. The answer is no because, for $f \in C^\infty(\boldsymbol{M})$,

$$\begin{aligned} [\![(\alpha_1, h_1); f\, (\alpha_2, h_2)]\!] &= f\, [\![(\alpha_1, h_1); (\alpha_2, h_2)]\!] \\ &\quad + (X_{(\alpha_1,h_1)}.f)\, (\alpha_2, h_2) + \Lambda(\alpha_1, \alpha_2)\, df\,, \end{aligned}$$

i.e., the Leibniz-type formula (2.5) is not satisfied.

2.2. **Infinitesimal symmetries of ω.**

Theorem 2.2. *A vector field X on $\boldsymbol{M}$ is an infinitesimal symmetry of ω, i.e. $L_X\omega = 0$, if and only if $X = X_{(\alpha,h)}$, where α and h are related by the following condition*

$$i_{\alpha^\sharp} d\omega + h\, i_E\, d\omega + dh = 0\,. \tag{2.6}$$

Proof. Any vector field on $\boldsymbol{M}$ is of the form $X_{(\alpha,h)}$. Then we get

$$0 = L_{X_{(\alpha,h)}}\omega = i_{\alpha^\sharp} d\omega + i_{h\,E} d\omega + d i_{\alpha^\sharp}\omega + d i_{h\,E}\omega$$

and from $i_{\alpha^\sharp}\omega = 0$ and $i_E\omega = 1$ Theorem 2.2 follows. □

Lemma 2.3. *A vector field $X_{(\alpha,h)}$ is an infinitesimal symmetry of ω if and only if the following equations are satisfied:*

(1) $i_E dh + i_E i_{\alpha^\sharp} d\omega = E.h + \Lambda(L_E\omega, \alpha) = 0\,,$

(2) $d\omega(\alpha^\sharp, \beta^\sharp) + h\, d\omega(E, \beta^\sharp) + dh(\beta^\sharp) = 0$ *for any 1-form* β.

Proof. If we evaluate the 1-form on the left hand side of (2.6) on the Reeb vector field E we get $i_E dh + i_E i_{\alpha^\sharp} d\omega = E.h - i_{\alpha^\sharp} i_E d\omega = E.h - \Lambda(\alpha, L_E\omega) = 0$. On the other hand if we evaluate this form on $\beta^\sharp$, for any 1-form β, we get (2).

The inverse follows from the splitting $T\boldsymbol{M} = \mathrm{Im}(\Lambda^\sharp) \oplus \langle E\rangle$, i.e. a 1-form with zero values on E and $\beta^\sharp$, for any 1-form β, is the zero form. □

Lemma 2.4. *Let us assume two infinitesimal symmetries $X_{(\alpha_i,h_i)} = \alpha_i^\sharp + h_i\, E$, $i = 1, 2$, of ω. Then*

$$\begin{aligned} [X_{(\alpha_1,h_1)}, X_{(\alpha_2,h_2)}] = \big(& d\Lambda(\alpha_1,\alpha_2) - i_{\alpha_2^\sharp} d\alpha_1 + i_{\alpha_1^\sharp} d\alpha_2 \\ & + \alpha_1(E)\,(i_{\alpha_2^\sharp} d\omega) - \alpha_2(E)\,(i_{\alpha_1^\sharp} d\omega) \\ & + h_1\,(L_E\alpha_2 - \alpha_2(E)\,L_E\omega) - h_2\,(L_E\alpha_1 - \alpha_1(E)\,L_E\omega)\big)^\sharp \\ & + \big(\alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 - d\omega(\alpha_1^\sharp,\alpha_2^\sharp)\big)\,E \end{aligned} \tag{2.7}$$

and we obtain the bracket

$$\begin{aligned} [\![\,(\alpha_1,h_1);(\alpha_2,h_2)\,]\!] &= \big(d\Lambda(\alpha_1,\alpha_2) - i_{\alpha_2^\sharp} d\alpha_1 + i_{\alpha_1^\sharp} d\alpha_2 \\ &\quad + \alpha_1(E)\,(i_{\alpha_2^\sharp} d\omega) - \alpha_2(E)\,(i_{\alpha_1^\sharp} d\omega) \\ &\quad + h_1\,(L_E\alpha_2 - \alpha_2(E)\,L_E\omega) - h_2\,(L_E\alpha_1 - \alpha_1(E)\,L_E\omega)\,; \\ &\quad \alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 - d\omega(\alpha_1^\sharp,\alpha_2^\sharp)\big) \\ &= \big(d\Lambda(\alpha_1,\alpha_2) - i_{\alpha_2^\sharp} d\alpha_1 + i_{\alpha_1^\sharp} d\alpha_2 \\ &\quad - \alpha_1(E)\,dh_2 + \alpha_2(E)\,dh_1 + h_1\,L_E\alpha_2 - h_2\,L_E\alpha_1\,; \\ &\quad d\omega(\alpha_1^\sharp,\alpha_2^\sharp) + h_1\,d\omega(E,\alpha_2^\sharp) - h_2\,d\omega(E,\alpha_1^\sharp)\big)\,. \end{aligned} \tag{2.8}$$

Proof. It follows from Lemmas 2.1 and 2.3 and (2.4). □

According to Lemma 2.4 the Lie algebra $(\mathcal{L}(\omega);[\,,])$ is generated by the Lie subalgebra of pairs $(\alpha,h) \in (\mathtt{Gen}(\omega);\,[\![\,,\,]\!]\,) \subset (\Omega^1(\boldsymbol{M}) \times C^\infty(\boldsymbol{M});\,[\![\,,\,]\!]\,)$ satisfying the condition (2.6) (or conditions (1) and (2) of Lemma 2.3) with the bracket (2.8).

2.3. **Infinitesimal symmetries of Ω.**

Theorem 2.5. *A vector field X on $\boldsymbol{M}$ is an infinitesimal symmetry of Ω, i.e. $L_X\Omega = 0$, if and only if $X = X_{(\alpha,h)}$, where*

$$d\alpha = 0\,, \quad \alpha(E) = 0\,, \tag{2.9}$$

i.e. α is a closed 1-form in $\mathrm{Ker}(E)$.

Proof. We have the splitting (1.8) and consider a vector field $X_{(\beta,h)}$. Then, from $d\Omega = 0$ and $i_E\Omega = 0$, we get

$$0 = L_{X_{(\beta,h)}}\Omega = d i_{\beta^\sharp}\Omega = d(\beta^\sharp)^\flat = d(\beta - \beta(E)\,\omega)$$

which implies that the closed 1-form $\alpha = \beta - \beta(E)\,\omega$ is such that $\alpha^\sharp = \beta^\sharp$. Moreover, $\alpha(E) = \beta(E) - \beta(E)\,\omega(E) = 0$. □

In what follows we shall denote by $\mathrm{Ker}_{cl}(E)$ the sheaf of closed 1-forms vanishing on E. From Theorem 2.5 it follows that the Lie algebra $(\mathcal{L}(\Omega);[\,,])$ of infinitesimal symmetries of Ω is generated by pairs (α,h), where $\alpha = \mathrm{Ker}_{cl}(E)$. In this case the bracket (2.4) is reduced to the bracket

$$\begin{aligned}\llbracket (\alpha_1, h_1); (\alpha_2, h_2) \rrbracket := \big(& d\Lambda(\alpha_1, \alpha_2); \\ & \alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 - d\omega(\alpha_1^\sharp, \alpha_2^\sharp) \\ & + h_1\,(E.h_2 + \Lambda(L_E\omega, \alpha_2)) - h_2\,(E.h_1 + \Lambda(L_E\omega, \alpha_1))\big)\end{aligned} \tag{2.10}$$

which defines a Lie algebra structure on $\mathrm{Ker}_{cl}(E) \times C^\infty(\boldsymbol{M})$ which can be considered as a Lie subalgebra $(\mathtt{Gen}(\Omega); \llbracket\,,\,\rrbracket) \subset (\Omega^1(\boldsymbol{M}) \times C^\infty(\boldsymbol{M}); \llbracket\,,\,\rrbracket)$. Really, $\mathrm{Ker}_{cl}(E) \times C^\infty(\boldsymbol{M})$ is closed with respect to the bracket (2.10) which follows from

$$\begin{aligned} i_E d\Lambda(\alpha_1, \alpha_2) &= L_E(\Lambda(\alpha_1, \alpha_2)) \\ &= (L_E\Lambda)(\alpha_1, \alpha_2) + \Lambda(L_E\alpha_1, \alpha_2) + \Lambda(\alpha_1, L_E\alpha_2) \\ &= i_{[E,\Lambda]}(\alpha_1 \wedge \alpha_2) = -i_{E\wedge(L_E\omega)^\sharp}(\alpha_1 \wedge \alpha_2) = 0\,. \end{aligned}$$

Remark 2.2. Any closed 1-form can be locally considered as $\alpha = df$ for a function $f \in C^\infty(\boldsymbol{M})$. Moreover, from $\alpha \in \mathrm{Ker}_{cl}(E)$, the function f satisfies $df(E) = E.f = 0$. Hence, infinitesimal symmetries of Ω are locally generated by pairs of functions (f, h) where $E.f = 0$. Lie algebras of local generators of infinitesimal symmetries of the almost-cosymplectic-contact structure are studied in [2].

2.4. **Infinitesimal symmetries of the Reeb vector field.**

Theorem 2.6. *A vector field X on $\boldsymbol{M}$ is an infinitesimal symmetry of E, i.e. $L_X E = [X, E] = 0$, if and only if $X = X_{(\alpha,h)}$, where α and h satisfy the following conditions*

$$(L_E\alpha - \alpha(E)\,L_E\omega)^\sharp = 0\,, \tag{2.11}$$

$$E.h + \Lambda(L_E\omega, \alpha) = 0\,. \tag{2.12}$$

Proof. We have

$$0 = [X_{(\alpha,h)}, E] = [\alpha^\sharp, E] + [h\,E, E]$$

and from (2.2) we get

$$[X_{(\alpha,h)}, E] = -\big(L_E\alpha - \alpha(E)\,L_E\omega\big)^\sharp - \big(E.h + \Lambda(L_E\omega, \alpha\big)\,E$$

which proves Theorem 2.6. □

Remark 2.3. The condition (2.11) of Theorem 2.6 is equivalent to the condition

$$(L_E\alpha)(\beta^\sharp) - \alpha(E)\,(L_E\omega)(\beta^\sharp) = 0 \tag{2.13}$$

for any 1-form β.

Lemma 2.7. *The restriction of the bracket* (2.4) *to pairs* (α, h) *satisfying the conditions* (2.11) *and* (2.12) *is the bracket*

$$
\begin{aligned}
[\![(\alpha_1,h_1);(\alpha_2,h_2)]\!] &= \big(d\Lambda(\alpha_1,\alpha_2) - i_{\alpha_2^\sharp} d\alpha_1 + i_{\alpha_1^\sharp} d\alpha_2 \\
&\quad + \alpha_1(E)\,(i_{\alpha_2^\sharp} d\omega) - \alpha_2(E)\,(i_{\alpha_1^\sharp} d\omega)\,; \\
&\quad \alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 - d\omega(\alpha_1^\sharp,\alpha_2^\sharp)\big) \\
&= \big(- d\Lambda(\alpha_1,\alpha_2) - L_{\alpha_2^\sharp}\alpha_1 + L_{\alpha_1^\sharp}\alpha_2 \\
&\quad + \alpha_1(E)\,(L_{\alpha_2^\sharp}\omega) - \alpha_2(E)\,(L_{\alpha_1^\sharp}\omega)\,; \\
&\quad \alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 - d\omega(\alpha_1^\sharp,\alpha_2^\sharp)\big)
\end{aligned}
\tag{2.14}
$$

which defines a Lie algebra structure on the subsheaf of $\Omega^1(\boldsymbol{M})\times C^\infty(\boldsymbol{M})$ *of pairs of 1-forms and functions satisfying conditions* (2.11) *and* (2.12).

Proof. It follows from (2.4), (2.11) and (2.12). □

2.5. **Infinitesimal symmetries of** Λ.

Theorem 2.8. *A vector field* X *on* $\boldsymbol{M}$ *is an infinitesimal symmetry of* Λ, *i.e.* $L_X\Lambda = [X,\Lambda] = 0$, *if and only if* $X = X_{(\alpha,h)}$, *where* α *and* h *satisfy the following conditions*

$$[\alpha^\sharp,\Lambda] - E\wedge (dh + h\,L_E\omega)^\sharp = 0\,. \tag{2.15}$$

Proof. We have

$$L_{X_{(\alpha,h)}}\Lambda = [\alpha^\sharp,\Lambda] + [h\,E,\Lambda]\,.$$

Theorem 2.8 follows from

$$[h\,E,\Lambda] = h\,[E,\Lambda] - E\wedge dh^\sharp = -E\wedge (dh + h\,L_E\omega)^\sharp\,.$$ □

Lemma 2.9. *A vector field* $X_{(\alpha,h)}$ *is an infinitesimal symmetry of* Λ *if and only if the following conditions*

$$d\omega(\alpha^\sharp,\beta^\sharp) + h\,d\omega(E,\beta^\sharp) + dh(\beta^\sharp) = 0\,, \tag{2.16}$$

$$\alpha(E)\,d\omega(\beta^\sharp,\gamma^\sharp) - d\alpha(\beta^\sharp,\gamma^\sharp) = 0 \tag{2.17}$$

are satisfied for any 1-forms β,γ.

Proof. It is sufficient to evaluate the 2-vector field on the left hand side of (2.15) on ω,β and β,γ, where β,γ are closed 1-forms. We get

$$i_{[\alpha^\sharp,\Lambda]-E\wedge(dh+h\,L_E\omega)^\sharp}(\omega\wedge\beta) = -\Lambda(i_{\alpha^\sharp}d\omega + h\,L_E\omega + dh,\beta)$$

which vanishes if and only if (2.16) is satisfied.

On the other hand

$$
\begin{aligned}
i_{[\alpha^\sharp,\Lambda]-E\wedge(dh+h\,L_E\omega)^\sharp}(\beta\wedge\gamma) &= \Lambda(\alpha,d\Lambda(\beta,\gamma)) + \Lambda(\beta,d\Lambda(\gamma,\alpha)) + \Lambda(\gamma,d\Lambda(\alpha,\beta)) \\
&\quad - \beta(E)\,\Lambda(h\,L_E\omega + dh,\gamma) + \gamma(E)\,\Lambda(h\,L_E\omega + dh,\beta)
\end{aligned}
$$

which, by using (2.16), can be rewritten as

$$\begin{aligned} i_{[\alpha^\sharp,\Lambda]-E\wedge(dh+h\,L_E\omega)^\sharp}(\beta\wedge\gamma) &= -\tfrac{1}{2} i_{[\Lambda,\Lambda]}(\alpha\wedge\beta\wedge\gamma) + d\alpha(\beta^\sharp,\gamma^\sharp) \\ &\quad + \beta(E)\,\Lambda(i_{\alpha^\sharp}d\omega,\gamma) - \gamma(E)\,\Lambda(i_{\alpha^\sharp}d\omega,\beta) \\ &= -\, i_{E\wedge(\Lambda^\sharp\otimes\Lambda^\sharp)(d\omega)}(\alpha\wedge\beta\wedge\gamma) + d\alpha(\beta^\sharp,\gamma^\sharp) \\ &\quad + \beta(E)\,\Lambda(i_{\alpha^\sharp}d\omega,\gamma) - \gamma(E)\,\Lambda(i_{\alpha^\sharp}d\omega,\beta) \\ &= -\,\alpha(E)\,d\omega(\beta^\sharp,\gamma^\sharp) + d\alpha(\beta^\sharp,\gamma^\sharp) \end{aligned}$$

which vanishes if and only if (2.17) is satisfied.

On the other hand if (2.16) and (2.17) are satisfied, then the 2-vector field $[\alpha^\sharp,\Lambda] - E\wedge(dh + h\,L_E\omega)^\sharp$ is the zero 2-vector field. □

2.6. **Infinitesimal symmetries of the almost-cosymplectic-contact structure and the dual almost-coPoisson-Jacobi structure.** An *infinitesimal symmetry of the almost-cosymplectic-contact structure* (ω,Ω) is a vector field X on $\boldsymbol{M}$ such that $L_X\omega = 0$ and $L_X\Omega = 0$. On the other hand an *infinitesimal symmetry of the almost-coPoisson-Jacobi structure* (E,Λ) is a vector field X on $\boldsymbol{M}$ such that $L_X E = [X,E] = 0$ and $L_X\Omega = [X,\Lambda] = 0$.

Theorem 2.10. *A vector field X is an infinitesimal symmetry of the almost-cosymplectic-contact structure (ω,Ω) if and only if $X = X_{(\alpha,h)}$, where $\alpha\in \operatorname{Ker}_{cl}(E)$ and the condition* (2.6) *is satisfied.*

Proof. It follows from Theorems 2.2 and 2.5. □

Lemma 2.11. *A vector field $X_{(\alpha,h)}$ is an infinitesimal symmetry of (ω,Ω) if and only if the following conditions are satisfied*

(1) $\alpha\in \ker_{cl}(\boldsymbol{E})$, *i.e.* $d\alpha = 0$, $\alpha(E) = 0$,

(2) $i_E dh + i_E i_{\alpha^\sharp} d\omega = E.h + \Lambda(L_E\omega,\alpha) = 0$,

(3) $d\omega(\alpha^\sharp,\beta^\sharp) + h\,d\omega(E,\beta^\sharp) + dh(\beta^\sharp) = 0$ *for any 1-form* β.

Proof. It is a consequence of Theorem 2.10 and Lemma 2.3. □

The bracket (2.4) restricted for generators of infinitesimal symmetries of (ω,Ω) gives the bracket

$$\begin{aligned} [\![(\alpha_1,h_1);(\alpha_2,h_2)]\!] &= \\ &= \big(d\Lambda(\alpha_1,\alpha_2); \alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 - d\omega(\alpha_1^\sharp,\alpha_2^\sharp)\big) \\ &= \big(d\Lambda(\alpha_1,\alpha_2); d\omega(\alpha_1^\sharp,\alpha_2^\sharp) + h_2\,\Lambda(L_E\omega,\alpha_1) - h_1\,\Lambda(L_E\omega,\alpha_2)\big) \\ &= \big(d\Lambda(\alpha_1,\alpha_2); d\omega(\alpha_1^\sharp,\alpha_2^\sharp) + h_1\,E.h_2 - h_2\,E.h_1\big) \end{aligned} \tag{2.18}$$

which defines the Lie algebra structure on the subsheaf of $\operatorname{Ker}_{cl}(E)\times C^\infty(\boldsymbol{M})$ given by pairs satisfying the condition (2.6). We shall denote the Lie algebra of generators of infinitesimal symmetries of (ω,Ω) by $(\mathtt{Gen}(\omega,\Omega); [\![\,,\,]\!])$.

Corollary 2.12. *An infinitesimal symmetry of the cosymplectic structure (ω,Ω) is a vector field $X_{(\alpha,h)}$, where $\alpha\in\operatorname{Ker}_{cl}(E)$ and h is a constant.*

Then the bracket (2.4) *is reduced to*

$$[\![(\alpha_1, h_1); (\alpha_2, h_2)]\!] = \big(d\Lambda(\alpha_1, \alpha_2); 0\big) .$$

I.e. we obtain the subalgebra $(\mathrm{Ker}_{cl}(\boldsymbol{E}) \times \mathbb{R}, [\![\, ,]\!])$ *of generators of infinitesimal symmetries of the cosymplectic structure.*

Proof. For the cosymplectic structure we have $d\omega = 0$ and (2.6) reduces to $dh = 0$. □

Corollary 2.13. *Any infinitesimal symmetry of the contact structure* (ω, Ω) *is of local type*

$$X_{(dh,-h)} = dh^\sharp - h\, E\,, \tag{2.19}$$

where $E.h = 0$*. I.e., infinitesimal symmetries of the contact structure are Hamilton-Jacobi lifts of functions satisfying* $E.h = 0$*.*

Then the bracket (2.4) *is reduced to*

$$[\![(dh_1, -h_1); (dh_2, -h_2)]\!] = \big(d\{h_1, h_2\}, -\{h_1, h_2\}\big) .$$

I.e., the subalgebra of generators of infinitesimal symmetries of the contact structure is identified with the Lie algebra $(C^\infty_{\boldsymbol{E}}(\boldsymbol{M}), \{,\})$, *where* $C^\infty_{\boldsymbol{E}}(\boldsymbol{M})$ *is the sheaf of functions* h *such that* $E.h = 0$ *and* $\{\,,\}$ *is the Poisson bracket.*

Proof. For a contact structure we have $d\omega = \Omega$ and (2.6) reduces to $i_{\alpha^\sharp}\Omega + dh = \alpha + dh = 0$, i.e. $\alpha = -dh$. From $\alpha \in \mathrm{Ker}_{cl}(E)$ we get $E.h = 0$. □

Remark 2.4. For cosymplectic and contact structures a constant multiple of the Reeb vector field is an infinitesimal symmetry of the structure. It is not true for the almost-cosymplectic-contact structure.

Lemma 2.14. *A vector field* $X_{(\alpha,h)}$ *is an infinitesimal symmetry of* (E, Λ) *if and only if the following conditions are satisfied*

(1) $(L_E\alpha)(\beta^\sharp) - \alpha(E)\,(L_E\omega)(\beta^\sharp) = 0$,

(2) $E.h + \Lambda(L_E\omega, \alpha) = 0$,

(3) $d\omega(\alpha^\sharp, \beta^\sharp) + h\, d\omega(E, \beta^\sharp) + dh(\beta^\sharp) = 0$,

(4) $\alpha(E)\, d\omega(\beta^\sharp, \gamma^\sharp) - d\alpha(\beta^\sharp, \gamma^\sharp) = 0$

for any 1-forms β, γ*.*

Proof. From Theorem 2.6 and Lemma 2.9 $X_{(\alpha,h)}$ is an infinitesimal symmetry of (E, Λ) if and only if (2.11), (2.12), (2.16) and (2.17) are satisfied. □

We shall denote the Lie algebra of generators of infinitesimal symmetries of (E, Λ) by $(\texttt{Gen}(E, \Lambda); [\![\, ,]\!])$.

Remark 2.5. We can describe also the Lie algebras of infinitesimal symmetries of other pairs of basic fields. Especially:

1. The Lie algebra $(\texttt{Gen}(E, \Omega); [\![\, ,]\!])$ is given by pairs satisfying

(1) $\alpha \in \ker_{cl}(\boldsymbol{E})$, i.e. $d\alpha = 0$, $\alpha(E) = 0$,

(2) $E.h + \Lambda(L_E\omega, \alpha) = 0$.

2. The Lie algebra ($\mathtt{Gen}(\Lambda,\Omega)$; $[\![\,,\,]\!]$) is given by pairs satisfying

(1) $\alpha \in \ker_{cl}(\boldsymbol{E})$, i.e. $d\alpha = 0$, $\alpha(E) = 0$,

(2) $d\omega(\alpha^\sharp, \beta^\sharp) + h\, d\omega(E, \beta^\sharp) + dh(\beta^\sharp) = 0$ for any 1-form β.

3. The Lie algebra ($\mathtt{Gen}(E,\omega)$; $[\![\,,\,]\!]$) is given by pairs satisfying

(1) $(L_E\alpha)(\beta^\sharp) - \alpha(E)\,(L_E\omega)(\beta^\sharp) = 0$ for any 1-form β,

(2) $E.h + \Lambda(L_E\omega, \alpha) = 0$,

(3) $d\omega(\alpha^\sharp, \beta^\sharp) + h\, d\omega(E, \beta^\sharp) + dh(\beta^\sharp) = 0$ for any 1-form β.

4. The Lie algebra ($\mathtt{Gen}(\Lambda,\omega)$; $[\![\,,\,]\!]$) is given by pairs satisfying

(1) $E.h + \Lambda(L_E\omega, \alpha) = 0$,

(2) $d\omega(\alpha^\sharp, \beta^\sharp) + h\, d\omega(E, \beta^\sharp) + dh(\beta^\sharp) = 0$ for any 1-form β,

(3) $\alpha(E)\, d\omega(\beta^\sharp, \gamma^\sharp) - d\alpha(\beta^\sharp, \gamma^\sharp) = 0$ for any 1-forms β, γ.

Lemma 2.15. *Let X be a vector field on $\boldsymbol{M}$. Then*

$$L_X\beta^\sharp = (L_X\beta)^\sharp \tag{2.20}$$

for any 1-form β if and only if X is an infinitesimal symmetry of Λ.

Proof. Let $X = X_{(\alpha,h)}$. Then

$$\begin{aligned} L_{X_{(\alpha,h)}}\beta^\sharp = [\alpha^\sharp + h\,E, \beta^\sharp] = \big(&d\Lambda(\alpha,\beta) - i_{\beta^\sharp}d\alpha + \alpha(E) i_{\beta^\sharp}d\omega \\ &+ i_{\alpha^\sharp}d\beta - \beta(E) i_{\alpha^\sharp}d\omega + h\,L_E\beta - h\,\beta(E)\,L_E\omega\big)^\sharp \\ &- \big(d\omega(\alpha^\sharp,\beta^\sharp) + h\,i_{\beta^\sharp}L_E\omega + i_{\beta^\sharp}dh\big)\,E\,. \end{aligned}$$

On the other hand we have

$$(L_{X_{(\alpha,h)}}\beta)^\sharp = \big(d\Lambda(\alpha,\beta) + i_{\alpha^\sharp}d\beta + h\,L_E\beta + \beta(E)\,dh\big)^\sharp\,.$$

Then

$$\begin{aligned} (L_{X_{(\alpha,h)}}\beta)^\sharp - L_{X_{(\alpha,h)}}\beta^\sharp = &\big(i_{\beta^\sharp}d\alpha - \alpha(E) i_{\beta^\sharp}d\omega \\ &+ \beta(E)\,(dh + h\,L_E\omega + i_{\alpha^\sharp}d\omega)\big)^\sharp + \big(d\omega(\alpha^\sharp,\beta^\sharp) + h\,i_{\beta^\sharp}L_E\omega + i_{\beta^\sharp}dh\big)\,E\,. \end{aligned}$$

The identity (2.20) is satisfied if and only if

$$d\alpha(\beta^\sharp,\gamma^\sharp) - \alpha(E)\,d\omega(\beta^\sharp,\gamma^\sharp) = 0\,,$$
$$d\omega(\alpha^\sharp,\beta^\sharp) + h\,d\omega(E,\beta^\sharp) + dh(\beta^\sharp) = 0$$

for any 1-form γ, i.e., by Lemma 2.9, if and only if $X_{(\alpha,h)}$ is an infinitesimal symmetry of Λ. □

Theorem 2.16. *Let X be a vector field on $\boldsymbol{M}$. The following conditions are equivalent:*

(1) $L_X\omega = 0$ *and* $L_X\Omega = 0$.

(2) $L_XE = [X,E] = 0$ *and* $L_X\Lambda = [X,\Lambda] = 0$.

Hence the Lie algebras ($\mathtt{Gen}(\omega,\Omega)$; $[\![\,,\,]\!]$) *and* ($\mathtt{Gen}(E,\Lambda)$; $[\![\,,\,]\!]$) *coincides.*

Proof. (1) ⇒ (2) If the conditions (1), (2) and (3) in Lemma 2.11 are satisfied then the conditions (1), ... ,(4) in Lemma 2.14 are satisfied.

(2) ⇒ (1) From Lemmas 2.3 and 2.14 it follows that infinitesimal symmetries of (E,Λ) are infinitesimal symmetries of ω. Now let $X_{(\alpha,h)}$ be an infinitesimal symmetry of (E,Λ). To prove that $X_{(\alpha,h)}$ is the infinitesimal symmetry of Ω it is sufficient to evaluate $L_{X_{(\alpha,h)}}\Omega = d(i_{X_{(\alpha,h)}}\Omega)$ on pairs of vector fields $E, \beta^\sharp$ and $\beta^\sharp, \gamma^\sharp$, where β, γ are any 1-forms. From (1.7), (2.2), (2.3) and $\Omega(\beta^\sharp,\gamma^\sharp) = -\Lambda(\beta,\gamma)$ (see [5]) we get

$$\begin{aligned}
(L_{X_{(\alpha,h)}}\Omega)(E,\beta^\sharp) &= E.(\Omega(\alpha^\sharp,\beta^\sharp)) - \beta^\sharp.(\Omega(\alpha^\sharp,E)) - \Omega(\alpha^\sharp,[E,\beta^\sharp])\\
&= -E.(\Lambda(\alpha,\beta)) + \Lambda(\alpha, L_E\beta) - \beta(E)\,\Lambda(\alpha, L_E\omega)\\
&= -(L_E\Lambda)(\alpha,\beta) - \Lambda(L_E\alpha,\beta) - \beta(E)\,\Lambda(\alpha,L_E\omega)\\
&= i_{E\wedge(L_E\omega)^\sharp}(\alpha\wedge\beta) - \Lambda(L_E\alpha,\beta) - \beta(E)\,\Lambda(\alpha,L_E\omega)\\
&= -\alpha(E)\,(L_E\omega)(\beta^\sharp) + (L_E\alpha)(\beta^\sharp)
\end{aligned}$$

which vanishes by (1) of Lemma 2.14. Similarly

$$\begin{aligned}
(L_{X_{(\alpha,h)}}\Omega)(\beta^\sharp,\gamma^\sharp) &= \beta^\sharp.(\Omega(\alpha^\sharp,\gamma^\sharp)) - \gamma^\sharp.(\Omega(\alpha^\sharp,\beta^\sharp)) - \Omega(\alpha^\sharp,[\beta^\sharp,\gamma^\sharp])\\
&= -\beta^\sharp.(\Lambda(\alpha,\gamma)) + \gamma^\sharp.(\Lambda(\alpha,\beta)) + \Lambda(\alpha, d(\Lambda(\beta,\gamma))) - \Lambda(\alpha, i_{\gamma^\sharp}d\beta)\\
&\quad + \beta(E)\,\Lambda(\alpha, i_{\gamma^\sharp}d\omega)) + \Lambda(\alpha, i_{\beta^\sharp}d\gamma) - \gamma(E)\,\Lambda(\alpha, i_{\beta^\sharp}d\omega))\\
&= -\frac{1}{2} i_{[\Lambda,\Lambda]}(\alpha\wedge\beta\wedge\gamma)\\
&\quad + d\alpha(\beta^\sharp,\gamma^\sharp) - \gamma(E)\,d\omega(\beta^\sharp,\alpha^\sharp) + \beta(E)\,d\omega(\gamma^\sharp,\alpha^\sharp)\\
&= -i_{E\wedge(\Lambda^\sharp\otimes\Lambda^\sharp)d\omega}(\alpha\wedge\beta\wedge\gamma)\\
&\quad + d\alpha(\beta^\sharp,\gamma^\sharp) - \gamma(E)\,d\omega(\beta^\sharp,\alpha^\sharp) + \beta(E)\,d\omega(\gamma^\sharp,\alpha^\sharp)\\
&= d\alpha(\beta^\sharp,\gamma^\sharp) - \alpha(E)\,d\omega(\beta^\sharp,\gamma^\sharp)
\end{aligned}$$

which vanishes by (4) of Lemma 2.14. So $L_{X_{(\alpha,h)}}\Omega = 0$. □

2.7. **Derivations on the algebra** $(\mathtt{Gen}(\omega,\Omega); [\![\,,]\!])$**.** Let us assume the Lie algebra $(\mathtt{Gen}(\Omega); [\![\,,]\!])$ of generators of infinitesimal symmetries of Ω. The bracket $[\![\,,]\!]$ is a 1st order bilinear differential operator

$$\mathtt{Gen}(\Omega)\times\mathtt{Gen}(\Omega)\to\mathtt{Gen}(\Omega)\,.$$

Theorem 2.17. *The 1st order differential operator*

$$D_{(\alpha,h)}\colon \mathtt{Gen}(\Omega)\to\mathtt{Gen}(\Omega)$$

given by

$$D_{(\alpha_1,h_1)}(\alpha_2,h_2) = [\![\,(\alpha_1,h_1);(\alpha_2,h_2)\,]\!]$$

is a derivation on the Lie algebra $(\mathtt{Gen}(\Omega), [\![\,,]\!])$.

Proof. It follows from the Jacobi identity for $[\![\,,]\!]$. □

We can define a differential operator $L_X : \Omega^1(\boldsymbol{M})\times C^\infty(\boldsymbol{M}) \to \Omega^1(\boldsymbol{M})\times C^\infty(\boldsymbol{M})$ given by the Lie derivatives with respect to a vector field X, i.e.

$$L_X(\alpha, h) = (L_X\alpha, L_X h)\,. \tag{2.21}$$

Generally this operator does not preserve sheaves of generators of infinitesimal symmetries.

Theorem 2.18. *Let X be an infinitesimal symmetry of the almost-cosymplectic-contact structure (ω,Ω). Then the operator L_X is a derivation on the Lie algebra $(\texttt{Gen}(\omega,\Omega); [\![\,,\,]\!])$ of generators of infinitesimal symmetries of (ω,Ω).*

Proof. First, let us recall that infinitesimal symmetries of (ω,Ω) are infinitesimal symmetries of (E,Λ). Suppose the bracket (2.18) of generators of infinitesimal symmetries of (ω,Ω). We have to prove that L_X is an operator on $\texttt{Gen}(\omega,\Omega)$, i.e. that for any $(\alpha,h)\in \texttt{Gen}(\omega,\Omega)$ the pair $(L_X\alpha, L_X h)\in \texttt{Gen}(\omega,\Omega)$.

We have $\alpha\in \mathrm{Ker}_{cl}(E)$, then $L_X\alpha = d i_X\alpha$ which is a closed 1-form. Further

$$L_X(\alpha(E)) = 0 \quad\Leftrightarrow\quad (L_X\alpha)(E) + \alpha(L_X E) = (L_X\alpha)(E) = 0$$

and $L_X\alpha\in \mathrm{Ker}_{cl}(E)$.

Further we have to prove that the pair $(L_X\alpha, L_X h)$ satisfies conditions (1) and (2) of Lemma 2.3. From $L_X E = 0$ and $L_X\Lambda = 0$ we get $L_X L_E\omega = 0$ and $L_X d\omega = 0$. Moreover, $L_X dh = dL_X h$.

The pair (α,h) satisfies (1) of Lemma 2.3 and we get

$$\begin{aligned} 0 &= L_X\big(dh(E) + \Lambda(L_E\omega,\alpha)\big)\\ &= d(L_X h)(E) + \Lambda(L_E\omega, L_X\alpha) = 0 \end{aligned}$$

and the condition (1) of Lemma 2.3 for $(L_X\alpha, L_X h)$ is satisfied.

Similarly, from the condition (2) of Lemma 2.3 we have, for any 1-form β,

$$\begin{aligned} 0 &= L_X\big(d\omega(\alpha^\sharp,\beta^\sharp) + h\,d\omega(E,\beta^\sharp) + dh(\beta^\sharp)\big)\\ &= \big(d\omega(L_X\alpha^\sharp,\beta^\sharp) + (L_X h)\,d\omega(E,\beta^\sharp) + d(L_X h)(\beta^\sharp)\big)\\ &\quad + \big(d\omega(\alpha^\sharp, L_X\beta^\sharp) + h\,d\omega(E, L_X\beta^\sharp) + h\,d\omega(E, L_X\beta^\sharp)\big)\,. \end{aligned}$$

The term in the second bracket is vanishing because of the condition (2) expressed on $L_E\beta^\sharp = (L_E\beta)^\sharp$. Hence the condition (2) of Lemma 2.3 for the pair $(L_X\alpha, L_X h)$ is satisfied and this pair is in $\texttt{Gen}(\omega,\Omega)$.

Further, we have to prove

$$\begin{aligned} L_X\,[\![\,(\alpha_1,h_1);(\alpha_2,h_2)\,]\!] \;=\; &[\![\,(L_X\alpha_1, L_X h_1);(\alpha_2,h_2)\,]\!]\\ &+\;[\![\,(\alpha_1,h_1);(L_X\alpha_2, L_X h_2)\,]\!]\,. \end{aligned}$$

For the first parts of the above pairs the identity

$$L_X(d(\Lambda(\alpha_1,\alpha_2))) = d(\Lambda(L_X\alpha_1,\alpha_2)) + d(\Lambda(\alpha_1, L_X\alpha_2))$$

has to be satisfied. But

$$\begin{aligned} L_X(d(\Lambda(\alpha_1,\alpha_2))) &= d i_X d i_\Lambda(\alpha_1\wedge\alpha_2) = d i_{[X,\Lambda]}(\alpha_1\wedge\alpha_2) + d i_\Lambda d i_X(\alpha_1\wedge\alpha_2)\\ &= d(([X,\Lambda])(\alpha_1,\alpha_2)) + d(\Lambda(L_X\alpha_1,\alpha_2)) + d(\Lambda(\alpha_1, L_X\alpha_2)) \end{aligned}$$

and for $[X,\Lambda] = L_X\Lambda = 0$ the identity holds.

For the second parts of pairs the following identity has to be satisfied.

$$L_X\big(d\omega(\alpha_1^\sharp,\alpha_2^\sharp) + h_2\,\Lambda(L_E\omega,\alpha_1) - h_1\,\Lambda(L_E\omega,\alpha_2)\big)$$
$$= d\omega((L_X\alpha_1)^\sharp,\alpha_2^\sharp) + h_2\,\Lambda(L_E\omega, L_X\alpha_1) - (L_X h_1)\,\Lambda(L_E\omega,\alpha_2)$$
$$+ d\omega(\alpha_1^\sharp,(L_X\alpha_2)^\sharp) + (L_X h_2)\,\Lambda(L_E\omega,\alpha_1) - h_1\,\Lambda(L_E\omega, L_X\alpha_2)\,.$$

If X is the infinitesimal symmetry of (ω,Ω) then it is also the infinitesimal symmetry of $d\omega$ and $L_E\omega$ and we get that the above identity is equivalent to

$$d\omega(L_X\alpha_1^\sharp,\alpha_2^\sharp) + d\omega(\alpha_1^\sharp, L_X\alpha_2^\sharp) = d\omega((L_X\alpha_1)^\sharp,\alpha_2^\sharp) + d\omega(\alpha_1^\sharp,(L_X\alpha_2)^\sharp)\,.$$

By Lemma 2.15 $L_X\alpha_i^\sharp = (L_X\alpha_i)^\sharp$ which proves Theorem 2.18. □

Remark 2.6. We have

$$[\![\,(\alpha_1,h_1);(\alpha_2,h_2)\,]\!] = \frac{1}{2}\big(L_{X_{(\alpha_1,h_1)}}(\alpha_2,h_2) - L_{X_{(\alpha_2,h_2)}}(\alpha_1,h_1)\big)\,. \tag{2.22}$$

Really

$$L_{X_{(\alpha_1,h_1)}}(\alpha_2,h_2) - L_{X_{(\alpha_2,h_2)}}(\alpha_1,h_1) =$$
$$= \big(2\,d\Lambda(\alpha_1,\alpha_2);\alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 + h_1\,E.h_2 - h_2\,E.h_1\big)$$

and from (2) and (3) of Lemma 2.11 we have

$$\alpha_1^\sharp.h_2 - \alpha_2^\sharp.h_1 = 2\,d\omega(\alpha_1^\sharp,\alpha_2^\sharp) + h_1\,d\omega(E,\alpha_1^\sharp) - h_2\,d\omega(E,\alpha_1^\sharp)$$
$$= 2\,d\omega(\alpha_1^\sharp,\alpha_2^\sharp) + h_1\,dE.h_2 - h_2\,E.h_1$$

which implies (2.22).

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