Inquiry in University Mathematics Teaching and Learning

The PLATINUM Project

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Foreword

This book reports on the work carried out within the Erasmus+ PLATINUM project by eight European universities from seven countries: the University of Agder, in Kristiansand, Norway—the coordinator of the project—the University of Amsterdam in The Netherlands, Masaryk University and Brno University of Technology in Czech Republic, Leibniz University Hannover in Germany, the Complutense University of Madrid in Spain, Loughborough University in the UK, and Borys Grinchenko Kyiv University in Ukraine.

In this 21st century, projects aimed at studying and disseminating inquiry-based approaches in the teaching of STEM disciplines in primary and secondary education have proliferated in Europe, benefiting from the impulse of the publication of the Rocard's report in 2007.[1] However, university mathematics teaching has remained mainly traditional, especially in the first university years, crucial for the students' orientation and retention. As the authors point out

> Considerable evidence shows that the learning of mathematics widely is highly procedural and not well adapted to using and working with mathematics in science and engineering and the wider world; also that students learn to reproduce mathematical procedures in line with tests and examinations, rather than developing a relational, applicable, creative view of mathematics that they can use more widely." The PLATINUM project was set up to move this situation, with the aim of "developing an inquiry-based approach towards the teaching and learning of university mathematics and for the development of an international community of university mathematics lecturers who practice, explore and encourage others to use inquiry-based teaching approaches in teaching mathematics. (p. 7)

The consortium partners were well aware that they were facing a major challenge as university teaching conditions, particularly in the first university years, are not conducive to inquiry-based practices: courses gathering large numbers of students with diverse backgrounds and professional projects, loaded curricula to be covered in a short period of time, etc., not to mention the lack of pedagogical and didactic preparation and experience of such practices for the majority of university mathematics lecturers.

The way the consortium partners organised themselves to meet this challenge is particularly interesting. They have adopted a broad and flexible conceptualisation of IBME (Inquiry-Based Mathematics Education), referring rather to definitions such as that proposed by Dorier and Maaß[2] in the Encyclopedia of Mathematics Education than the more demanding characterisations proposed for Inquiry-Based Oriented (IBO) practices in the United States where such practices seem more developed in

[1]Rocard, M., Cesrmley, P., Jorde, D., Lenzen, D., Walberg-Herniksson, H., & Hemmo, V. (2007). Brussels, Belgium: Office for Official Publications of the European Communities. *Science education NOW: a new pedagogy for the future of Europe.*
`www.eesc.europa.eu/en/documents/rocard-report-science-education-now-new-pedagogy-future-europe`

[2]Dorier, J.-L. & Maaß, K. (2020). Inquiry-based mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 384–388). Springer Verlag.
`doi.org/10.1007/978-3-030-15789-0_176`

mathematics university courses. And they have created tools, especially spidercharts, providing criteria for assessing the degree of inquiry involved in student tasks and their management.

They also formed mixed teams combining a diversity of expertise, those of academic mathematicians and mathematics education specialists, and built the conceptual foundations of their project by positioning all actors, not only students, in an inquiry-based learning posture. The conceptual model which is described in detail in Chapter 2 is, in fact, made up of three nested levels. At the first level, inquiry concerns the mathematics at play in the classroom (lectures, tutorials or other devices); at the second level, it concerns teaching processes, pedagogical and didactic choices and their effects; at the third level, inquiry concerns

> the entire developmental process in which participants reflect on practices in the other two layers, and gather, analyse, and feed-back data to inform practice and develop knowledge in practice. (p. 20)

Thus Communities of Inquiry were formed which supported the work and professional development of their members, and were themselves supported by the collective work of the consortium as Chapter 7 and the various case studies show (see for example Chapters 14 and 15).

In the European IBME projects I have been involved in, the collective production of resources in the form of inquiry-based tasks and teaching units has always been an important component. This is also the case in PLATINUM and I particularly appreciated the diversity of the resources produced. As far as students are concerned, they address many mathematical domains—complex numbers, functions of one or more variables, differential equations and dynamical systems, linear algebra, geometry, statistics and numerical analysis—teaching aimed at future mathematicians and mathematics teachers, but also very often service mathematics courses, a sector where, as underlined in Chapter 8, IBME and mathematical modelling are closely linked. They also show that it is possible to engage in inquiry-based practices without revolutionising one's teaching, that many ordinary tasks, if reformulated, can engage students in more conceptual work and bring them into the spirit of inquiry aimed at.

Another interesting and original dimension of this project is the attention paid to students with special needs and the difficulties they may encounter in the different phases of an inquiry process. Chapter 4, which is very informative, is devoted to this dimension. It specifies the forms that these difficulties may take according to the students' profiles and also makes many practical suggestions. Chapter 6 devoted to the creation of teaching units for students' inquiry explains the principles of Universal Design for Learning, "a methodology adopted by PLATINUM partners to strive for an inclusive learning environment reaching the needs of as many students as possible" (p. 118), and Chapter 12 provides an insightful illustration of the use of these design principles. There is no doubt that the work carried out in the PLATINUM project should help us to make IBME more inclusive.

I enjoyed reading the pages of the pre-final manuscript I received. I appreciated its structure, the eight chapters in Parts 1 and 2 which present the project in a very detailed way, its origin, its long maturation, its implementation, its conceptual basis and the ingenious methodological tools developed, connecting these to the six main intellectual outputs structuring the project. I also very much appreciated the eight chapters in Part 3 where each partner presents in great detail one or two case studies and analyses them with great intellectual honesty. In these case studies, the authors also make clear how digital tools—both educational mathematical software already used in secondary education and professional tools used by mathematicians, and communication tools—have supported the implementation of inquiry-based approaches in their institution, and how they have also helped teams adapt to the new constraints due to the pandemic situation.

There is no doubt in my mind that PLATINUM represents an important milestone for the evolution of practices in university mathematics education. It shows that this evolution is possible if it is thought of as a progressive dynamic, adapted to the contexts, and carried out by communities combining a diversity of expertise and seeing themselves as communities of inquiry. I hope that this book will be a source of inspiration for many academics.

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CHAPTER 1

Introduction

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Mathematical demands in university courses represent a considerable hurdle for many students and for quite a few this is known to lead to dropping out or changing studies. But even among students who successfully pass mathematics exams, there are many who do not achieve the intended teaching goals. Instead of a reflected understanding of mathematical concepts and their internal and external mathematical use, rote learning often dominates. Concepts such as complex numbers or derivatives of functions are technically mastered, which is certainly important, but meanings that go beyond this, for example, reasoning of properties or relationships, are not acquired. If one does not want to attribute these results one-sidedly to deficits in the students, such as a lack of talent or commitment, then one must question the dominant teaching and also the prevailing examination practices.

IBME is a central approach to designing teaching differently. It aims directly to ensure that students are not only presented and shown some mathematics and subsequently trained in techniques, but are involved much more in the teaching-learning context from the very beginning. In the best case, for example, mathematical concepts should be (re-)discovered starting from a problem as an answer to a question. This intention is to regard teaching and learning of mathematics as closer to how mathematicians proceed in research itself and less like how mathematics lessons are regarded traditionally. This is why it is not uncommon to speak of research-oriented teaching.

The European project PLATINUM (Partnerships for Learning And Teaching in University Mathematics), a consortium of 8 universities from 7 countries (University of Agder, Norway; University of Amsterdam, Netherlands; Masaryk University, Czech Republic; Brno University of Technology, Czech Republic; Loughborough University, England; Leibniz University Hannover, Germany; Complutense University of Madrid, Spain; Borys Grinchenko Kyiv University, Ukraine) has set out to develop Inquiry-based mathematics education (IBME) approaches in the teaching of mathematics. For this purpose, the partners made their way to create a community that exchanges ideas about possibilities and limits of the implementation and realization of the different design approaches of IMBE. The partners reflect these approaches critically against respective local practical and theoretical backgrounds and, in particular, create a space for the creative development of IBME-oriented teaching units, tasks and further training. The interlocking of three levels: the learning activities of the students, the planning and design of the teaching by the teachers, and their critical academic reflection and monitoring represent, in a sense, a brand core of PLATINUM (see the Three-Level Model in Chapter 2). In this way, the project partners themselves entered into an inquiry process and formed a community of inquiry as described by Jaworski (2019).

Of course, some general problem areas of didactics played a role here as well, which we will briefly discuss in the following. A first consists of close collaboration between didacticians and mathematicians. As fruitful and necessary as such cooperation is in practice, especially for PLATINUM, it also involves different perspectives, which can be seen as complementary but also as potentially conflictual. Clearly, didacticians and mathematicians pursue the same goal, that students understand mathematics better and more deeply, but to a certain extent they speak different languages and embody different scientific cultures. In particular, their scientific discourses follow different norms of problem formulation and justification of answers to questions. Mathematicians, for example, tend to demand unequivocal evidence or proof for hypotheses and are on the search for methods to obtain such evidence through systematic procedures. Didacticians are usually aware that this must fail, that already in questions and problem formulations theories and ideas are implicitly included, which are to a certain extent unprovable, raise questions themselves, but this does not mean on the other hand, of course, that didactic research would be completely arbitrary and that justifications would not be based on rationales.

Clearly, mathematicians want clear evidence that IBME leads to better student learning outcomes. It is understandable that it is not satisfying when didacticians point out that this question addresses a major problem in a somehow undercomplex and problematic way. To outline just one aspect of this problem: Luhman and Schorr (1982) pointed out the so-called technology deficit of pedagogy. By this they meant the fundamental and insurmountable difficulty of a lack of linear causality between, say, a teacher's intention and the effect that actually occurs with learners. Learning processes can only ever be stimulated, but never directly achieved. Of course, this does not speak against efforts to make teaching more successful and to search for conditions that make desired learning possible or to question conditions that prevent it. However, concrete instructions for action with necessarily occurring learning effects are not possible and scientifically justifiable. Of course, mathematicians in general are aware of this, but usually not with regard to all the problems that arise from it in terms of scientific concepts of didactics and, for example, methods of research and evaluation.

This is one reason why didacticians, like the authors of this introduction, tend to speak of teaching goals rather than learning goals, see above. Regarding teaching goals or learning outcomes, the former are central to the teaching that takes place and therefore influence the learning that takes place (i.e., the learning outcomes). Beyond what has been said so far, unfortunately 'learning outcomes' are often expressed in a general way by those writing the curriculum and often do not match the pedagogies and the goals that teachers have for their students through the teaching-learning interactions which take place.

In particular, Holzkamp (1995) has shown that, to a certain extent, under societal pressure on educational institutions, learning theories and curricular tend to teaching-learning shortcuts, i.e., to speak in fact about teaching but attempt to formulate this in terms of learning outcomes. These then also somehow suggest that empirical research should be established with the aim of finding clear justifications for concrete instructions for teaching actions that directly ensure learning outcomes. As important and significant as this research is, a different understanding of human learning, such as in Lave and Wenger (1991), also resulted from and justifies the partial failure of this research logic. Their concept of Community of Practice also underlies PLATINUM. However, a deeper understanding of this concept, as simple as it may seem at first, is not so easy to gain. In particular, it requires an understanding of the problematic situations just outlined, not as clumsiness that can be easily overcome, but as unresolvable fields of tension in which efforts for better teaching and also IBME necessarily have to proceed.

Summarising, actions linked to teaching goals aim at learning goals, but these just cannot be ensured. Nevertheless, we believe that there are good reasons for teaching in terms of IBME. Diverse reasons and corresponding suggestions for the design of such teaching can be found in many places in this volume. In particular, this involves subject-specific proposals that are concretised in tasks and learning situations, for example. And this is precisely what makes high subject-specific demands in the field of university mathematics and cannot be mastered without mathematics experts, since general didactic considerations are not sufficient here. This is exactly the point where the complementarity of didactics and mathematics has to prove itself. A teaching design oriented towards IBME requires a different preparation of the material to be taught. It is oriented more towards questions and problems that mathematics deals with and answers, and less towards the answers themselves. The questions and problems, and how they can be approached through suitable tasks and materials, require a really good subject-specific and, above all, flexible understanding of the mathematical content. In many chapters it becomes clear that mathematics experts were active in PLATINUM. And this is especially true for the many proposals related to mathematical service-courses. Here, the authors must not only have a high command of mathematics, but also an understanding of the service subject, such as engineering, economics, or biology. Mathematics is used there in a specific way. Symbols etc. take on additional meanings and practices are, partly, justified differently, precisely from the respective subject-specific context. In other words, mathematics is not simply applied, but changes in a specific way in its use in other sciences, especially empirical sciences.

A special role is played by proofs. Clearly, the role of proof in mathematics is different from the role of proof in the didactics of mathematics, as explained above. As we inquire into mathematical processes and make our own conjectures, we aspire to mathematical proof, just as mathematicians do. However, the theory of inquiry operates 'around' the mathematics that we do. In terms of the three-layer model, mathematical proof would be a part of the central layer with students and their teachers. How to achieve such mathematical proof in teaching and learning, through inquiry, is part of the second layer as teachers make sense of inquiry approaches to explore mathematics and achieve mathematical proof. The design of teaching units takes this into account. The outer layer formalises the processes of the second layer, striving for a cohesive account of the developmental processes in the second layer and the overall complex process in the three layers.

Against this background, the present volume is a major outcome of our inquiry processes. It presents both basic theoretical concepts that accompanied our process and its organisation as well as the writing of this volume (Part 1, Chapter 2). In addition, Part 1 includes the presentation and reflection of an IBME-orientated tool developed in the project, the so-called 'spidercharts', as well as that of a general model for the consideration of students with indentified needs in IMBE-oriented teaching. Part 2 of the volume then focuses more concretely on the project as a whole: starting with the organisation and implementation of the project (Chapter 5). This is followed by concepts and examples of the development of tasks and teaching units (Chapter 6) and of the implementation of corresponding professionalisation approaches in inquiry-based teaching and learning for lecturers (Chapter 7). In Chapter 8, the development of tasks and teaching units is taken further with a focus on mathematical modelling.

The evaluation of IBME teaching is not straightforward and cannot be done through traditional quantitative instruments or essentially procedural-orientated examinations, but requires specific instruments orientated at the goals of IBME. Although there is certainly still a lot of development work and research to be done, Chapter 9 presents important ideas that can be implemented in practice.

The third and major part of the volume consists of so-called 'case studies.' After an introduction (Chapter 10), each partner describes and reflects on concrete examples of their own development within the project. According to the respective professional backgrounds and local conditions and possibilities, the contributions range from more theoretically oriented reflections to interesting practical presentations. In our view, the diversity presented here has been one of the great strengths of the project. It becomes especially clear that inquiry does not live from standardisation but from heterogeneity and different perspectives, from freedom instead of superficial normalisation.

In the final 4th part we summarise once more: Where did we start and where did we end up? What are the key experiences and insights from our project? What perspectives are emerging? We consider the journey we have taken in interpreting inquiry-based learning and teaching in PLATINUM and synthesising our activity and learning for this book.

For whom have we written this book? For ourselves, first of all. Describing our local developments, incorporating our experiences into the more theoretical parts, and reflecting on the experiencing and design of IBME approaches against a theoretical background, this interweaving of practice and theory could be taken to a new level through writing, reviewing and optimising. And then, of course, we think that what we have written here can also be interesting and fruitful for people outside. In particular the case studies offer interesting insights and suggestions for practitioners who are interested in didactic questions as well as for researchers working in the field of didactics. People interested in IBME could find answers or at least ideas regarding questions about the evaluation of measures, systematic ways of designing learning units and tasks and in doing so also taking into account special needs. Last but not least, our own professionalisation through inquiry can be seen as a basis for professionalisation more widely involving approaches to mathematics teaching and learning that benefit our students.

If we could wish for something in conclusion, it would be that this volume contributes to stimulating the development and preparation of well-reflected and evaluated IBME teaching units and tasks and to optimising existing ones. Last but not least, we also think of the teachers, mathematics colleagues, for whom teaching with more successful students also brings pleasure and contentment.

Finally, we would like to thank the European Union for the financial support of the PLATINUM project, which made all this possible.

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Part 1

Inquiry Communities in Mathematics Teaching and Learning

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CHAPTER 2

Conceptual Foundations of the PLATINUM Project

Barbara Jaworski, Inés M. Gómez-Chacón, Reinhard Hochmuth

2.1. The PLATINUM Project

This chapter addresses the conceptual background underpinning PLATINUM, a project in the EU Erasmus+ programme. PLATINUM focuses on teaching and learning in university mathematics and particularly on IBME, Inquiry-Based Mathematics Education, involving mathematics teaching and learning and their development through the use of inquiry-based processes.

PLATINUM is a European (Erasmus+) project for the development of IBME in university education. Details of the project, that is, the partners, the concrete forms of cooperation, and so on, are described in Chapter 5 and on the PLATINUM website.[1] This chapter is about the common theoretical foundations of IBME and how they relate to different parts of the project and its origins, and to other chapters in the book.

PLATINUM stands for "**P**artnership for **L**earning **A**nd **T**eaching **IN** **U**niversity **M**athematics." Our partnership, within the EU Erasmus+ project, consists of eight teams of university mathematics lecturers, educators and researchers, in universities from seven European countries (see Chapter 5 for more details). Together, we form a partnership devoted to developing the teaching and learning of university mathematics that will enable university students' better understanding of mathematical concepts related to their programmes in mathematics, science, engineering, economics and other areas of study.

PLATINUM is characterised by the fact that the development of IBME and the project processes and practices are seen not as separate from each other, but as two strands that are analytically and theoretically distinct, but closely linked. Our proposal to the Erasmus+ programme included the following statement:

> Mathematics is a discipline central and foundational to many areas of study (including natural sciences, engineering, economics and teacher education) and to national success globally in academic prestige, business and trade, active citizenship and social entrepreneurship. Mathematics education in Higher Education influences the labour market and human lives, especially for people disadvantaged in educational opportunity, limiting their access to work and leisure; several *Intellectual Outputs* [IOs] [in accord with Erasmus+] emphasise this target group.
>
> Mathematics can be experienced as difficult to learn and exclusive in terms of learning success. Considerable evidence shows that the learning of mathematics widely is highly procedural and not well adapted to using and applying mathematics in science

[1] https://platinum.uia.no

> and engineering and the wider world; also, that students learn to reproduce mathematical procedures in line with tests and examinations, rather than developing a relational, applicable, creative view of mathematics that they can use more widely. We recognise a central need to enable all students to be conceptually focused with mathematics, to work with mathematics in creative and enterprising ways, and to equip them to apply mathematics in other disciplines and the world of work.
>
> Therefore, we will develop an inquiry-based approach towards the teaching and learning of university mathematics and aim for the development of an international community of university mathematics lecturers who practice, explore and encourage others to use inquiry-based teaching approaches in teaching mathematics. These approaches will blend a range of ways of thinking, methods and technologies (including digital technology) in a well-balanced way to achieve more in-depth learning leading to meaningful application of mathematical concepts by our students. The needs of different groups of students will be in the focus of our activity, including those with special needs or other learning disadvantages.

Thus, the conceptual foundation of the PLATINUM project is *Inquiry-Based Mathematics Education* (IBME) and particularly the concept of *inquiry*. We have sought to develop an inquiry-based approach to mathematics teaching and learning at university level both theoretically and in our activity in eight universities in Europe.

The main purpose of inquiry is to engage those involved (students for example) *deeply* with concepts that they should learn or develop—in contrast with procedural learning or learning by rote—although, of course, following procedures or memorising facts or formulae can form a part of the learning process. Where mathematics is concerned, inquiry approaches in problems and tasks encourage students to get involved with the mathematics, not just using standardised rules and procedures but exploring/investigating processes and concepts, and trying to answer open-ended questions and solve problems. For their teachers/lecturers the challenge is to offer suitable problems/tasks through which their students' exploration can bring them to understanding the mathematics being presented to them in lectures. Of course, 'understanding' can mean different things for different people: Richard Skemp's (1976) position on *instrumental* versus *relational* understandings is well known; here we are rather thinking of understanding which is conceptual and relational. This challenge brings lecturers themselves into an inquiry process where their teaching of mathematics is concerned—conceiving suitable approaches to their students' engagement and bringing these into their practice with students.

In PLATINUM, we explore both didactic and pedagogic processes and practices and blend methods and resources to achieve development in teaching and learning. We utilise a developmental research approach in which partners 'walk the walk' of inquiry-based practice and share findings with others.

In this chapter we start by introducing the project briefly (above) and follow this with the reasons why new approaches to teaching and learning mathematics at university level are seen as important and necessary. We draw on relevant literature to situate our PLATINUM activity. The main part of the chapter (Sections 2.4 and 2.5) addresses our developmental approach in IBME from a PLATINUM perspective, drawing on inquiry in our seven countries, relevant literature, and explaining a three-layer theoretical model of inquiry which underpins the project. This model has acted as a framework for all our activity in PLATINUM, as we explain below. We introduce Intellectual Outputs (IOs), commensurate with an Erasmus+ programme, and discuss the activities in which we have engaged related to each IO, as a precursor to the chapters which follow.

A brief guide to the sections of this chapter follows: Section 2.2, *A Need to Redefine Teaching*, discusses some of the reasons why new approaches to teaching and learning are needed. Section 2.3, *IBME: A Brief History in the PLATINUM Countries and Beyond*, provides an outline of perceptions of IBME in the countries of PLATINUM. Section 2.4, *IBME in Mathematics Education*, presents international perspectives on which IBME is founded and as a basis for our work in PLATINUM. Section 2.5, *IBME in the PLATINUM Project*, discusses the theory of inquiry as it is used and developed within PLATINUM. In particular it introduces our Three-Layer Model of Inquiry on which PLATINUM is based and the key concepts of Inquiry Communities and Critical Alignment. Section 2.6, *Discussion and Conclusions*, concludes the chapter.

2.2. A Need to Redefine Teaching

Our focus in PLATINUM is the learning of mathematics of students in university level courses in a range of disciplines including mathematics, sciences, engineering, economics and so on. It is our overall aim that mathematics teaching should have the student in mind at all times, seeking to engender a student engagement that inspires deep levels of conceptual understanding, rather than only a superficial memorising of formulae and basic procedures. This is not to deny that a focus on formulae and procedures, or their memorisation, has its own value. Also, as we are aware, every mathematics didactic project proposal criticises in some sense the inadequate reality of existing mathematics teaching and especially the learning results. However, there is some consensus that understanding and relating mathematical concepts needs much more than memorisation and use of procedures, which is the basis of our proposed inquiry-based approach (cf., Alsina, 2002; Hawkes & Savage, 2000; Minards, 2013; Solomon & Croft, 2016; Treffert-Thomas & Jaworski, 2015). Of course, not every proposal is classified under the term inquiry. We ask, therefore, what is specific about the inquiry approach and always strive to emphasise this in our contribution.

We are aware, as the literature shows, that common practices in university learning and teaching leave many students with mathematical knowledge that does a disservice to mathematics and can be seen as inadequate for mathematical applications that depend on it (such as in the disciplines listed above (e.g., Faulkner et al., 2019)). Students themselves have reported dissatisfaction with what they are offered; for example, research into students' second-year experiences of mathematics courses in three UK universities showed many students disillusioned with their mathematics course. Solomon and Croft (2016) write:

> Student disengagement from undergraduate mathematics in the UK is widely reported ... raising basic questions as to how well-qualified students who report high levels of confidence and enjoyment at school can become so disillusioned with a subject which they have actively chosen to study at university. (p. 267)

It is in some sense common knowledge among professional colleagues that many students see the learning of mathematics as memorising formulae and procedures presented in lectures, that they expect to use in examination questions and thus contribute to their end-of-study grades and access to employment. Teachers often struggle to support students within the prevailing conditions. The following example points to a number of issues we face as university teachers:

> Recently, a colleague in linear algebra set a task that was formulated in such a way that it was recognisable that an already practised and known procedure would be useful to complete the task. But in order to implement this, it was first necessary to transform the task somewhat on a conceptual level in order to then apply the calculus. Technically, it was really only a small thing. But one had to have an idea of what it was all about.

> The result was quite bad because many students did not even get to the calculus part. This was compensated for by lowering the points required to pass. (For us, no more than 50-60% should fail. If more students do not pass the exam, there are follow-up questions, which one would like to avoid, also because the subsequent discussions are rarely productive).

There are many factors to consider, not least the culture and infrastructure of university education in which research takes academic precedence over teaching. Teaching is managed in lectures of several hundred students with exams designed to test what was presented in the lecture; and there is little time to support teaching development. Lecturers have typically teach in the ways they themselves were taught in university. For students the university teaching is very different from their experiences in school and lacks the kinds of guidance school provided. Solomon and Croft (2016) quote a response from one student who was asked how university mathematics differs from the school experience:

> It's sort of not as easy. 'Cos I used to find it easy then. I do like finding things out and getting the answers to things, but it's not as fun. So, I don't enjoy it ... sometimes when I've just got an assignment back and it's awful, I just think 'Oh no, why am I doing this?' (p. 274)

Of course, we should not necessarily assume that when students say they like to *find things* out they mean what we might mean by the same words. For example, finding things out can consist of identifying and executing the respective correct calculation steps in a strictly prescribed scheme. However, we take this statement as illustrative evidence for students' needs in terms of their mathematical appreciation.

In the study volume from the 2000 conference of ICMI (International Commission on Mathematical Instruction) focusing on teaching and learning of mathematics at university level, Claudi Alsina (2001), a professor in mathematics from Spain, quotes US historian of mathematics Morris Kline, writing about the position of university mathematics professors:

> ... appointment, promotion, tenure and salary are based entirely on status in research ... but for most of the teaching that the universities are, or should be offering, the research professor is useless. (p. 3)

We might respond here that, since 1977, there have been new conditions, new insights, and new practices. However, we might also recognise some residual elements of Kline's words. Alsina himself (ibid) writes:

> There is a need to redefine mathematical research as a university activity, combining it with soundly-based teaching excellence. ... Good teaching is according to a classic definition: "building understanding, communicating, engaging, problem solving, nurturing and organising for learning" a complete task that merits special attention and preparation (see Krantz, 1993). (p. 7)

It might be argued that, in the 20 years since the ICMI study, university teaching could have learned (and developed) internationally from what the study exposed and proposed: and to some extent we have. In the UK, for example, a government "Teaching Excellence Framework" evaluates universities on the quality of their teaching; most universities now include some generic courses for new lecturers on developing teaching. However, these generic courses are often found largely unhelpful for teachers of mathematics who claim they do not address teaching suited to mathematics itself (e.g., those related to symbolisation and proof). In Germany, for example, the three major mathematics associations (DMV, GDM and MNU) have requested that a corresponding recommendation on the subject-specific university didactic further training measures by a joint mathematics commission on the transition from school to university should be implemented.[2] In fact, some mathematics departments institute special courses for new lecturers in mathematics, to address concepts seen as directly related to mathematics teaching (see for example Winsløw et al., 2021). It remains true however that, despite such innovation, mathematics teaching at university level can benefit from further development. We are aware, of course, that not every proposal for development is classified under the term inquiry. We ask, therefore, what is specific about the inquiry approach and always strive to emphasise this in our contribution.

In PLATINUM, we have addressed the idea of development based in inquiry processes involving both lecturers and students, as we address below. This development has taken place in eight universities in seven countries in Europe, each with its own language and culture, its own higher education structure and university systems, and its own ways of approaching mathematics teaching and learning. In Section 2.3 we provide some historical information relating to IBME in these countries.

2.3. IBME: A Brief History in the PLATINUM Countries and Beyond

PLATINUM partners come from seven countries in Europe: the Czech Republic, Germany, the Netherlands, Norway, Spain, Ukraine and the United Kingdom. Details of the educational systems and specifically of university education in mathematics can be found on the PLATINUM website[3] and in the proposal to Erasmus+, also on the website.

Here we focus specifically (and in outline only) on the history and development of IBME in the countries of PLATINUM as experienced by PLATINUM colleagues. This experience relates fundamentally to *who we are* in our national situations and our personal teaching-learning activity. To some degree, all of us teach mathematics to university students in university courses. This might involve courses in mathematics for mathematics students, students of engineering or science, of economics, medicine and so on. Some of us teach prospective teachers of mathematics. Some are mathematicians, developing knowledge in mathematics through their research; some are mathematics educators, researching many aspects of teaching, learning and development in the didactics and pedagogies of mathematics. It is this latter group that has most experience of IBME through their need to study the literature of mathematics education including its history and development.

The theory(ies) behind IBME develop from some eminent educationalists and mathematicians in our history. For example, John Dewey (1859-1952), University of Chicago, and George Polya (1887–1985), Stanford University, were significant forebears to whom we can trace many of the aspects of active learning in general and IBME in particular. In our countries, we refer to significant pioneers of problem solving in mathematics, Hans Freudenthal in the Netherlands, Miguel Guzman in Spain, Erik Wittman in Germany, John Mason in the UK; Alan Schoenfeld in the US is well-known internationally and a frequent visitor to Europe. We say more about their influence in Section 2.5.

In PLATINUM, with our central focus on IBME, we are all aware of a number of high-profile European research projects into the teaching and learning of mathematics (and often of science as well) from *inquiry-based* principles, mostly at primary and secondary school levels. Colleagues at BUT in the Czech Republic point to the Fibonacci Project (Large scale dissemination of inquiry-based science and mathematics education), the PROFILES project (Professional Reflection-Oriented Focus on Inquiry-based Learning and Education through Science), and the project MaSciL

[2]http://mathematik-schule-hochschule.de/images/Massnahmenkatalog_DMV_GDM_MNU.pdf
[3]https://platinum.uia.no

(Mathematics and Science for life).[4] Research into practices of teaching and learning mathematics in schools has permeated all of our countries, with colleagues who are involved in teacher education being the researchers alongside school teachers.

For example, colleagues at UvA in the Netherlands point to the major research institute on mathematics education, the Freudenthal Institute (FI), [initiated by Hans Freudenthal (1905–1990)] which had until recently researched only in primary and secondary schools. They write:

> This is reflected in the European projects in which FI members participate(d): Fibonacci, PRIMAS, MaSciL, MERIA, and TIME. The conceptual framework for the work of the mathematics education researchers at FI always embeds Design Research and Realistic Mathematics Education (RME). It is based in engaging students in realistic (to them) problems which might be *real world problems*, perhaps involving *modelling*, or *mathematical problems* that are 'real' for the students who try to solve them.

In Mathematics Education, developmental work in the Netherlands based on RME is well known and frequently emulated internationally. For example, colleagues at UCM in Spain write, "some universities in Spain offer a conceptualisation of IBME linked to the theory of Realistic Mathematics Education (RME) (Gravemeijer & Doorman, 1999; Alsina, 2002)." They claim that "reality-based problems, as mathematical objects, promote initially a model that is context-specific. Their affordances are substantially different from those offered by the problem-solving approach."

The 'Problem-Solving Approach' refers to research and development into the use of (mathematical) problems as an introduction to mathematics learning and teaching in classrooms. Colleagues at UCM in Spain refer to "a long tradition of research and practice in our field going back to the seminal work of George Polya (e.g., Polya, 1945). Miguel de Guzmán, professor at UCM and president of the International Council of Mathematics Instruction (ICMI), encouraged teaching and learning at university level in this direction by publishing various books and developing a theoretical framework that plays an essential role in the solving of problems. The teacher training programmes under this approach were promoted with the support of the Spanish Ministry of Education and with the collaboration of international experts such as Schoenfeld (USA) who was invited to give courses and lectures."

In university education in Germany, problem solving in mathematics is seen as a specific competence, which is generally conceptualised along the lines of Polya (1945). With regard to mathematical learning processes at school as well as at university, problem-solving is considered important, especially with regard to multifaceted heuristics when working on tasks and problems, and is taught accordingly in order to make corresponding experiences possible (Bruder & Collet, 2011). However, it is assumed that the adoption of this competence in the context of the acquisition of the new and abstract material of Analysis and Linear Algebra is too big a hurdle for many students in their first semester. This is one of the reasons why this competence should initially be acquired in a special course which, in terms of mathematical content, focuses much more on school mathematics and ties in with it. With regard to teaching profession students, this has the welcome additional effect that they can acquire in-depth school knowledge on some topics. It is assumed that the problem-solving competence

[4]The Fibonacci Project—Large scale dissemination of inquiry-based science and mathematics education (`www.fibonacci-project.eu`); the MaSciL Project—Mathematics and Science for Life (`https://mascil-project.ph-freiburg.de`); the MERIA Project—Mathematics Education Relevant, Interesting and Applicable (`https://meria-project.eu`); The PRIMAS Project—Promoting Inquiry in Mathematics and Science Education Across Europe (`https://primas-project.eu`); The PROFILES project (`www.profiles-project.eu/`); the TIME Project (`https://timeproject.org`).

acquired in these courses can then be used in the context of the more abstract requirements of the classical lecture. One of the first such courses was established by Grieser (2018) at the University of Oldenburg. Other universities have subsequently established similar courses (Hochmuth et al., 2022). Such and related approaches have been attempted in recent years, especially in preparatory and bridge courses. A good overview of this is provided by the practical examples presented in (Biehler et al., 2021) from the Competence Centre for Higher Education Didactics in Mathematics.[5]

Although the European projects mentioned above, as well as RME and The Problem-Solving Approach, focused (mainly) on mathematics learning and teaching *for school students*, nevertheless, researchers from universities often led the work in these projects. These researchers were usually employed in mathematics education, perhaps in teacher education, whereas teachers of mathematics (at university level) are less likely to be involved in such research. However, it is not always so clearly distinguished. Colleagues in the Ukraine write:

> We believe that IBME refers to a student-oriented paradigm for mathematics and science teaching, in which students are invited to work in ways similar to mathematicians and scientists. The best teachers and lecturers used problem-based learning, solving research and applied problems, the case method, and the implementation of group projects in order to stimulate pupils or students to search, to conscientious and, if possible, independent construction of knowledge, thereby achieving understanding, and not formal memorisation. Although the term (IBME) was not literally used in the Ukrainian scientific community, (university) teachers, often intuitively, used certain approaches that are characteristic of IBL (posing research questions, formulating and testing/proving hypotheses, etc.).

In Norway, the national Centre for Research, Innovation and Collaboration in Mathematics Teaching[6] (MatRIC) was established to focus on mathematics teaching at university level. Researchers in MatRIC had conducted a survey of Norwegian university mathematics teachers and one colleague wrote:

> The survey focused on active learning approaches rather than IBME. My interpretation is that it does not reveal much about the incidence of IBME. As far as I am aware, IBME is more of a topic of discussion between [university] mathematics educators and lecturers, there may be some small pools of activity—for example [one colleague] developed some interesting blended learning approaches (not specifically IBME), and these were researched and reported in a PhD study and in a paper in IJRUME (International Journal for Research in Undergraduate Mathematics Education). However, these innovations came to an end when the class was incorporated into the larger cohort of first year engineering students. The mathematics team at [location], have, last semester, tried to introduce a modelling project into the first semester mathematics course. There is some intersection between this and the notion of IBME, but it was not an effort specifically designed to introduce or develop IBME, I really do not know what is happening in other institutions [in Norway], and my feeling is that there is very little substantial development of IBME approaches implemented at [university level].

In the UK, there is a history dating back to the 1960s of 'investigational activity' or 'investigations' in mathematics, often deriving from workshops and conferences of the Association of Teachers of Mathematics (ATM), and promoted by teachers in Colleges and Department of Education (Jaworski, 1994). An influential figure was Caleb Gattegno, who had written in 1960:

[5] `www.khdm.de`

[6] `www.matric.no`

> When we know why we do something in the classroom and what effect it has on our students, we shall be able to claim that we are contributing to the clarification of our activity as if it were a science.

Gattengo influenced the establishment of the Association of Teachers of Mathematics (ATM) and many publications offering starting points for explorative activity in mathematics by students, and advice for teachers. Such activity was described as follows:

> In contrast to tasks set by the teacher—doing exercises, learning definitions, following worked examples—in mathematics activity the thinking, decisions, projects undertaken were under the control of the learner. It was the learner's activity. (Love, 1988, p. 249).

While such 'activity' related mainly to school classrooms, it was promoted for university students (often school teachers) studying with the Open University in the UK, through Polya's (1945) book, "How to Solve It," developing problem-solving heuristics, and through the work of John Mason and colleagues who presented problem-solving heuristics in a book "Thinking Mathematically" (Mason et al., 2010). In the US, at this time, the problem-solving movement based on Polya's work led to research in classrooms studying students' problem-solving activity and, in particular the developing thinking of the teachers involved (e.g., Cobb et al., 1990), In the UK, in parallel, a study focused on teachers developing their use of investigational activities with students led to a recognition of teacher inquiry in the development of mathematics teaching (Jaworski, 1994) revealing issues and tensions experienced by the teachers and professional growth emerging from the activity and research. Although the term IBME was not used in the UK at this time, the ideas of inquiry in mathematics problem solving (for students at all levels) and in teachers' explorations in teaching, revealing issues and tensions for teachers, can be seen as strongly related to the IBME approaches employed in PLATINUM. These have been compared to the research approaches used by mathematicians in exploring beyond current mathematical knowledge and opening up new vistas in mathematics (e.g., Burton, 2004).

The fact that research-based learning has again become a focus of discussion, especially since the 1990s, is due to the context of "Bologna." At least in Germany, conflicting moments are seen here:

> One of the goals of these reforms was that studies should increase the general employability of university graduates. To this end, in addition to subject-specific competences, more general skills were to be taught, which were referred to as key competences. Problem-oriented forms of learning, project-oriented and also research-based learning were identified as conducive to this. (Huber & Reinmann, 2019, p. 22, translation by the authors)

Indeed, this development was complemented in Germany by an increased competitive orientation in funding policy, e.g. through the "Excellence Initiative," which encouraged universities to support research-based learning in teaching. However, the universities' public commitments to research-based learning concealed very different degrees of actual preparation, promotion and coordination of such projects (Huber & Reinmann, 2019, p. 23). Following the principles of "New Public Management," quality standards and measurements, formative and summative evaluations including statistics on student success and student evaluations of courses were introduced (cf., Wildt, 2013, p. 37) and accompanied the introduction and implementation of the projects. This in particular was accompanied critically at an early stage, e.g. with regard to inadequate content specifications, and the danger of a "didacticisation" of the university to the detriment of its scientific character was seen (cf., Mittelstraß, 1996). If one takes a look at mathematics-related initiatives for the implementation of

IBME projects, as will be done in the following, it is noticeable that these connections (which have just been hinted at) are largely ignored: The idea that students should learn concepts in depth, for example, is taken up, but without problematising the socio-institutional context and the contradictory teaching-learning conditions that go along with it. So the question arises as to whether IBME can work in this way and achieve the objectives associated with them. With regard to this issue, the largely unanswered and, under the given boundary conditions, possibly unsolvable problem of the examinations is a striking symptom, at least for Germany. With regard to teaching and learning in schools, such questions were for example analysed systematically in (Holzkamp, 1995). These analyses are taken up in Chapter 14 and discussed in more detail with regard to the concepts underlying PLATINUM.

2.4. IBME in Mathematics Education

At its simplest, inquiry involves exploring, investigating, asking questions and solving problems. In mathematics, this includes exploring mathematical relationships, investigating mathematical conjectures, asking questions about mathematical applications and solving problems in mathematics and the wider world. Artigue and Blomhøj (2013) write:

> Inquiry-based pedagogy can be defined loosely as a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work. (p. 797)

All mathematicians who do research in mathematics are familiar with the processes of inquiry, since mathematics itself, as a discipline, progresses through inquiry. For example, in 1997, the mathematician Andrew Wiles, provided a solution for the long-unsolved problem, known as 'Fermat's last theorem', posed by Pierre de Fermat a French mathematician of the seventeenth century. This achievement is described in the Foreword to Simon Singh's book (1997) addressing the proving of the problem, as "the Himalayan peak of number theory." Singh provides a gripping account of the inquiry process engaged in by Andrew Wiles (Singh, 1997). Quoting Wiles, he writes:

> I used to come up to my study, and start trying to find patterns. I tried doing calculations which explain some little piece of mathematics. I tried to fit it in with some previous broad conceptual understanding of some part of mathematics that would clarify the particular problem I was thinking about. Sometimes that would involve going and looking up in a book to see how it's done there. Sometimes it was a question of modifying things a bit, doing a little extra calculation. And sometimes I realized that nothing that had ever been done before was any use at all. Then I just had to find something completely new – it's a mystery where that comes from. (Wiles, in Singh, 1997, p. 227–228)

However, unlike the inquiry processes of researching mathematics, the processes of teaching mathematics over the centuries, and particularly in current times, have largely avoided the inquiry involved in research: they have presented mathematics as a top-down deductive process, explaining, justifying, and offering procedures to be learned, often unrelated to the inquiry processes that underpinned them. As we have mentioned above, a result of this teaching approach has often been that students memorise the presented results of the research, without understanding the underlying concepts, and consequentially struggle to apply mathematics and solve problems (see for example Alsina, 2001). Of course, "teaching approach" does not include only what happens in the classes but also the didactics and pedagogy behind what is done.

In schools, pedagogical considerations are naturally strongly oriented towards the organisation of lessons and the role of teachers: research-based work is rather difficult to squeeze into 45' or maximum 90' units. Questions and problems posed must be able to be worked through quickly, and also in such a way that possible answers or questions from the students do not overwhelm the teacher. The role that is usually attributed to teachers is that they are held responsible for the acquisition of knowledge by the students. However, research-based learning, by its very nature, must include failure, just as research does. Research without failure is not possible. A lesson in which questions are raised that teachers cannot answer, or at least cannot deal with directly, reflects badly on the teachers. Here, too, a change in thinking is necessary: A lesson in which students ask questions that are not trivial and the teacher therefore cannot answer immediately is a good lesson! Added to this is the constantly envisaged assessment of students' performance. Poor performance and assessments must be avoided. If this becomes the main goal of the pupils' activity, the content aspect of what is taught and to be learned recedes into the background, becomes secondary. A corresponding pedagogy that aims at explorative learning on the matter at hand must therefore be at least partially unassessed. However, small steps in the development of the subject matter and the best possible control over the pupils' actions do not only dominate mathematics lessons, but also the other subjects. Here, too, a change in pedagogy would be a desirable goal. Responding to an inquiry-based research project in Norway, Skovsmose and Säljö (2008) contrast inquiry-based teaching with what they called "The Exercise Paradigm":

> This [the exercise paradigm] implies that the activities engaged in the classroom to a large extent involve struggling with pre-formulated exercises that get their meaning through what the teacher has just lectured about. An exercise traditionally has one, and only one, correct answer, and finding this answer will steer the whole cycle of classroom activities and the obligations of the partners involved. (p. 40)

They suggest that inquiry-based practice takes us beyond the exercise paradigm:

> The ambition of promoting mathematical inquiry can be seen as a general expression of the idea that there are many educational possibilities to be explored beyond the exercise paradigm. (p. 4)

Some years earlier, Hiebert and colleagues in the US (1996) wrote:

> We argue that reform in curriculum and instruction should be based on allowing students to problematise the subject. Rather than mastering skills and applying them, students should be engaged in resolving problems. (p. 12)

Addressing the historical roots for inquiry and building on (Hiebert et al., 1996; Artigue & Blomhøj, 2013) attribute the concept of inquiry to John Dewey (1938), particularly his contribution in developing 'reflective inquiry' to form a basis for a pedagogical practice. They write:

> Dewey (1938) sees learning as an adaptive process in which experience is the driver for creating connections between sensations and ideas, through a controlled and reflective process, labelled reflective inquiry. The organization of students' experience and the development of general habits of mind for learning through reflective inquiry is thus an essential function of education. (p. 798)

Dewey (1933) himself has written:

> . . . reflective thinking, in distinction to other operations to which we apply the name of thought, involves (1) a state of doubt, hesitation, perplexity, mental difficulty, in which thinking originates, and (2) an act of searching, hunting, inquiring, to find material that will resolve the doubt, settle and dispose of the perplexity (p. 12) . . . Demand for the solution of a perplexity is the steadying and guiding factor in the entire process of reflection. (p. 14)

The concept of reflective inquiry and the language of Dewey here capture well the perspectives and approach we have taken to our work in PLATINUM that we have called *inquiry-based education.*

Particularly, also because of our focus on university mathematics, we see the need to complement Dewey's approach somewhat: As Artigue and Blomhøj (2013) point out, referring to Bachelard's concept of epistemological obstacles, successful IBME requires the "careful organisation of students' experience and inquiries to allow them to face the limitation of common sense," which again emphasises the teacher's responsibility from a more content-related point of view. Also the notion of students' acquisition of general discovery and problem-solving competences must always be complemented by concrete content-related activities that promote "incorporating local constructs into more regional perspectives" (Artigue & Blomhøj, 2013, p. 800).

We mention other sources highly relevant to our work in PLATINUM. Cochran Smith and Lytle (1999), working with teachers in the US, developed a construct that they call "Inquiry as Stance." This captures the ways in which teachers took on a mantle of inquiry-based practice and 'made it their own;' it relates strongly to the idea of "Inquiry as a Way of Being" (Jaworski, 2004), used in our PLATINUM model (see below). Gordon Wells (1999) wrote about "Dialogical Inquiry," in which teachers (or students) engage together in dialogue that is inquiry-based. Alrø and Skovsmose (2002) have also related dialogue and inquiry in mathematics classrooms, writing of the critical nature of inquiry in "landscapes of investigation" (see also Skovsmose & Säljö, 2008). Wells (1999) talks specifically of *communities of inquiry* through which dialogue encourages concept formation. Here again, this is highly relevant for PLATINUM.

In much of the literature cited above, authors have discussed aspects of inquiry-based practice with relation to teaching and learning in school classrooms. While we might see inquiry-based practices offering didactics and pedagogy desirable for students' deeper engagement with mathematics, there are factors in both learning and teaching within educational infrastructures that promote less desirable activity and outcomes. For example, in schools, classes of 30 students taught by one teacher might favour prescriptive teaching and learning, they also leave little responsibility for the students, protecting them from taking initiative and thinking for themselves. When students who have always been nurtured in such ways then go on to university, it is not uncommon for them to wish to be taken more by the hand and, understandably, to expect that the learning they have experienced at school will also be successful at university. That the university needs to organise a pedagogical transformation process is not well understood. For example, for the student, the concept of lecturing to large groups of students (often several hundreds in one lecture theatre) can look very different from teaching in classrooms of 30 students. Simply holding lectures with difficult content and expecting students to swim their way through them doesn't work for many students. The expansion of many small-step examinations in this situation and the offer of very tightly guided assistance, while accommodating the students and showing "success," achieve the opposite of what they actually want in pedagogical terms, in particular no transformation towards inquiry, or research-based learning.

It is fair to say that, while a wealth of literature addresses development in mathematics learning and teaching at school level, often promoting processes based in inquiry, there is yet correspondingly little at university level. The transformation to inquiry-based teaching and learning builds on similar motives and theoretical reasoning at both school and university levels, but, at the practical level, has to take account of differences in both culture and infrastructure. While many aspects of IBME have the same meanings and relevance, their application in university teaching can look different from their application in school classrooms.

Although yet small in comparison with school-based education, a literature relating to teaching and learning mathematics in Higher Education (HE) is growing. For example, in 1998, in Singapore, the 7th study conference in the ICMI study series was held, focusing on *the teaching and learning of mathematics at university level* (see the study volume, Holton, 2001). In the second edition of the Encyclopedia of Mathematics Education (Lerman, 2020), we see entries on *University Mathematics Education, Teaching practices at University Level and Preparation and Professional Development of University Mathematics Teachers* (see pp. 670–674). A chapter on *Research on University Mathematics Education* (Winsløw et al., 2018) is included in a book *Developing Research in Mathematics Education: Twenty Years of Communication, Cooperation and Collaboration in Europe* (Dreyfus et al., 2018). The INDRUM conference (International Network for Development of Research in University Mathematics—an offshoot of CERME, Congress of the European Society for Research in Mathematics Education) has, in 2020, had its third conference. A book from INDRUM (Durand-Guerrier et al., 2021) has been published. Although IBME-related activity is touched-on in some of these sources, there is as yet very little published that specifically addresses IBME in HE. We hope that our sources from PLATINUM will stimulate the beginnings of an IBME corpus of research and development in Higher Education.

As with the differences in growth of literature relating to school and university mathematics learning and teaching, it is also the case that professional development programmes for university teachers are less common and less well-developed than those compulsory for school teachers. Although many universities in different countries have their own non-subject-specific professional development programmes there is much less emphasis on subject teaching development (see for example Winsløw et al., 2021). In the PLATINUM project we have taken seriously the need to think about teacher education for IBME teaching and learning, rooted in the motives and principles introduced above. In Section 2.5, we address these motives and principles, and the sorts of practices that we have developed over the time of our project.

2.5. IBME in the PLATINUM Project

The conceptual foundation of the PLATINUM project is an *inquiry-based* approach to mathematics teaching and learning, recognising *inquiry* very much in the way expressed by Dewey as '*reflective inquiry*' (above). In this project we explore (inquire into) both didactic and pedagogic processes and practices, and blend methods and resources to achieve development in mathematics teaching and learning. We utilise a developmental research approach in which partners 'walk the walk' of inquiry-based practice, share findings with others and look critically at what they are doing as they do it.

The main purpose of *inquiry* is to engage our students *deeply* with concepts that they should learn or develop, in contrast with procedural learning or learning by rote. Where mathematics is concerned, inquiry approaches in problems and tasks encourage students to get involved with the mathematics, asking and trying to answer questions, and exploring/investigating processes and concepts in contrast with an exercise paradigm as mentioned above. For their teachers/lecturers the challenge is to offer problems/tasks such that, through inquiry-based engagement, their students can come to understand the mathematics being presented to them in lectures. This challenge brings lecturers themselves into an inquiry process where their teaching of mathematics is concerned—conceiving suitable approaches to their students' engagement and bringing these into their practice with students.

Within our PLATINUM partnership, *collaboration* has been at the heart of our activity: it has been our intention to form *Communities of Inquiry* (CoI) at university level. In fact, the entire partnership can be regarded as one large community of inquiry. Together we inquire into the processes of learning and teaching with which we engage; we design inquiry-based tasks for our students and associated teaching units related to the mathematical topics we teach. Moreover, each partner group itself constitutes an inquiry community. Its members work together within their particular system and culture (both mathematical and educational) and the ways of working and underlying assumptions that this entails. Part of our inquiry process involves inquiring into and becoming more aware of the ways of thinking and reasoning which underpin our educational activity in mathematics whether in large lectures or other kinds of grouping. We ask *why* things are the way they are, and whether there are alternatives that might be more effective for our students' learning of mathematics. We explore possibilities and use inquiry where we can, looking critically at practices and their outcomes, both established and innovative.

It seems fair to say that an inquiry community, in the PLATINUM context, means a group of people (lecturers and researchers) who sincerely explore possibilities for the use of inquiry-based activities with our students to achieve students' more conceptual understandings of mathematics. In so doing, we hope to create teaching units in which students explore mathematical questions and work together to discuss their inquiry processes and argue potential solutions. Through such practices, we hope that students, together, will develop their approaches towards exploring and solving problems, feel the satisfaction of understanding what they are doing and why they are doing it, and will enjoy their mathematics. We therefore extend our concept of inquiry community to the students themselves.

We recognise that the transition from a more traditional to an inquiry-based approach to teaching and learning mathematics in not easy. It is not easy for the students who have been enculturated into a top-down didactic approach in which their involvement is to listen, make notes, learn and reproduce theorems and their proofs, and tackle stereotypical problems. In fact, many students resent being asked to explore mathematical ideas rather than being told exactly what to learn and how to solve. It is thus also not easy for the teachers to encourage their students to work in these new ways, especially considering that the teachers themselves are not so familiar with what an inquiry approach can mean. The transition can destabilise the entire *didactic contract* between teachers and their students (see for example Alsina, 2001; Brousseau, 2002). A didactic contract creates (often implicitly) the expectations of teacher and students of each other: in more traditional forms of teaching, the teacher has responsibility to tell students what they have to learn and understand, and to guide them in ways of achieving what is required. Students have responsibility to work in the ways guided by the teacher, to make effort to learn what is presented and to become skilled in working on tasks and solving exercises. In inquiry-based activity, it is the student's responsibility to work on a task, asking questions, seeking patterns, making conjectures, proving or disproving. The teacher's responsibility is to design tasks through which students' activity may reveal key concepts and relationships to progress mathematical understanding. Inquiry activity is far less well-defined than traditional activity and, when new to students and teacher alike, can present difficulty and confusion.

In PLATINUM, therefore, we both recognise the difficulties of transition, towards inquiry, and seek to identify their nature and outcomes. Recognising that we have set out to undertake practices which are both demanding and challenging, we reflect on our practices to identify issues and tensions in what we are seeking to achieve and the particular outcomes of what we do (following Dewey, quoted above). By identifying the challenges and sharing with our colleagues in our inquiry communities, we come to know more about the inquiry-based processes we seek to engage. We learn what it means to work in inquiry ways in practice as well as to adapt in theory or belief. Thus, we walk the walk as well as talking the talk! In Part 3 of this book each chapter offers a case study from one of the partners in PLATINUM. These cases document the thinking and experiences of the partners. Of course, each one relates to the situation and culture in which it takes place, so together they offer a panorama of inquiry-based university mathematics development in Europe.

We have written above about two 'layers' of inquiry: that is inquiry in mathematics between teachers and students, and inquiry in teaching as teachers explore new approaches to teaching and learn from their own reflective inquiry. Thus, teachers within the project seek, not only, to engage with their students in inquiry in practice, but also, to provide an account of their activity, the outcomes, issues and tensions arising from it, that they can share with other practitioners beyond the project (for example, in the cases of Part 3 of the book). The process here is developmental. As teachers participate in inquiry activity, learning from their engagement with students, their practice develops.

This indicates a third, developmental layer in which reflection and analysis provide insights into the processes and practices of inquiry and their outcomes. What we see here is what we call developmental research. Here we raise questions related to our practice and its development and address these questions in a systematic way: our aim is to provide insights into aspects of mathematics learning and teaching and the issues and tensions that arise for us as practitioners and researchers. This creates a new layer in our model. Thus, our inquiry model—see Figure 2.1, developed from (Jaworski, 2006, 2019)—constitutes three layers as follows.

Inquiry in:

(1) engaging with mathematics in inquiry-based teaching-learning situations with students; students will engage with inquiry in mathematics with the aim of developing their own in-depth understanding of mathematical concepts.
(2) exploring teaching processes, the didactics and pedagogies involved in student inquiry, and their use in teaching-learning situations to achieve desired student outcomes; teachers/lecturers will reflect on their own practices and seek to understand better the teaching approaches that enable their students' understanding.
(3) the entire developmental process in which participants reflect on practices in the other two layers, and gather, analyse, and feed-back data to inform practice and develop knowledge in practice. Communities of inquiry will both enable and progress this development and share with others beyond the project. Sharing of insights gained and issues and tensions addressed is an important vehicle for promoting development more widely.

These layers are deeply interrelated. Teachers/lecturers, inquiring into their teaching, focus centrally on their students' learning through inquiry. Teacher-researchers, reflecting and analysing data from the other two layers, feedback what is learned to the practices they are in the process of developing. The whole constitutes an interrelated developmental process represented by the figure below.

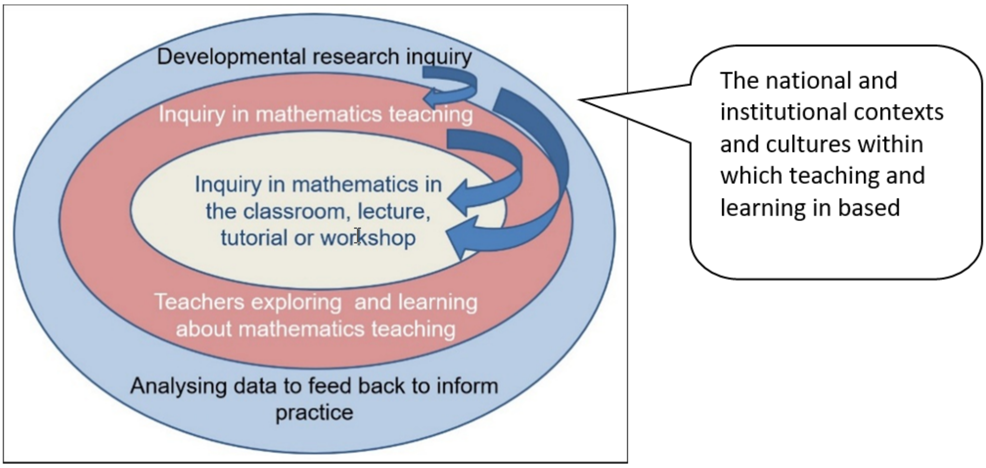

FIGURE 2.1. The Three-Layer Model of Inquiry.

Important to the inquiry process in its three layers, is the concept of 'community'. An *inquiry community* in PLATINUM consists of lecturers and/or students working together in inquiry ways to achieve learning and development. Central to our model is the belief that engagement with others, into concepts we seek to learn/develop, enriches engagement and provides opportunity for individuals to broaden their own thinking and to clarify their own conceptions. This provides opportunity for colleagues to look critically at the practices in which they engage, and to introduce and explore changes to practice. Such a critical approach within a supportive environment enables participants to address problems and issues with teaching and learning which might otherwise be beyond individual resolution. In the project, we expect our analysis of data from our own teaching practices to allow us to report on the outcomes of our activity with evidence to support what we claim regarding our developments in teaching-learning through inquiry.

Associated with the important concept, *Community of Inquiry* (CoI), is a concept we call *Critical Alignment* (e.g., Jaworski, 2006): While following community norms and expectations, we look critically at outcomes and achievements; we question, or inquire into, alternative practices as part of our engagement. These two concepts are strongly related to Etienne Wenger's (1998) *Community of Practice* (CoP). Wenger postulates that a Community of Practice builds on three constructs: *mutual engagement*, *joint enterprise*, and *shared repertoire*. In our developmental practice, we engage with each other in the agreed practices of our community and with the same ways of being and doing. However, we also start to introduce new practices/activities and to explore their development. Wenger conceptualises 'belonging' to a CoP as involving *engagement*, *imagination*, and *alignment*: *Belonging* entails engaging with others, using *imagination* in forging our own trajectories in the practice, and *alignment* with the norms and expectations of the practice. Thus, our practice changes as we engage with and explore the new elements, side-by-side with established practices. Thus, a Community of Inquiry (CoI) can be seen to encompass *mutual engagement*, *joint enterprise*, and *shared repertoire* as in a CoP, and *belonging* to a CoI to involve engagement and imagination as in a CoP. The main point of difference between CoP and CoI, lies with *alignment*. Simply aligning with the norms and expectations of a practice can result in perpetuation of elements of the practice that are not what we would like to see. It is very difficult, often, not to align with existing practices (the ways in which all our colleagues are engaged), but, while we align we can be aware of the need for change, and seek out ways of achieving the changes we would like to see: looking critically at what we are doing as we do it, with our eyes on possibilities for change. Thus, in an inquiry community, it is proposed that *belonging* presupposes an element of *critical alignment*: While following community norms and expectations, we look critically at outcomes and achievements, and question, or inquire into, alternative practices as part of our engagement. This concept of critical alignment can be seen as central to developmental activity in PLATINUM; indeed, one partner group uses the concept to present activity in their case study (see Chapter 14).

In PLATINUM, in our Communities of Inquiry, we work within university systems in which teaching and learning practices in mathematics have been embedded over decades, centuries or even millennia. Changing these is hugely demanding and very difficult for many reasons, some articulated above. To work within these practices, it is impossible not to align with the ways in which teaching and learning practices are shared and accepted. However, we do not have to do this uncritically. This is where inquiry comes in. By changing the idea of Community of Practice to Community of Inquiry, we start to open up possibilities. We can make small changes – for example, ask students more open questions, offer them an inquiry-based task and discuss it with them, start some occasional small-group work in which inquiry activity becomes the norm. It is through such efforts for small changes towards inquiry approaches that we engage with critical alignment—we align critically—and we start to appreciate elements of inquiry-based practice and to engage with its demands.

The three-layer model of inquiry has been used as a framework to structure our work in PLATINUM and to describe the results according to the intellectual outputs (IOs). The PLATINUM project has worked with six IOs as agreed within the Erasmus+ programme. These are as follows:

IO1 focuses on the conceptual underpinnings of the project, providing a framework for dealing with *inquiry*, *inquiry communities*, *inquiry-based learning and teaching*, and *critical alignment*. This chapter (2) explains conceptual underpinnings in the PLATINUM Project and sets the scene for the contents and structure of the following chapter.

IO2 focuses on *Communities of Inquiry*. The whole PLATINUM project as well as each partner group is committed to establishing a *community of inquiry* between its participants. Chapter 3 explores some of the underpinnings of the concept of community of inquiry. Through the idea of a *spiderchart*, it explores concepts and constructs that can contribute with differing degrees to a CoI. The central features of a CoI are the ways we work together to promote inquiry-based practices for the benefit of our students' learning and understanding in mathematics. Discussion in our CoI around the constructs of the spidercharts can help the community to develop. We do not need all to think in exactly the same ways—inquiry is about exploring possibilities and learning from our exploration whether in mathematics or in mathematics teaching; it is not about everyone working or thinking identically. Inquiry-based practice can look different for different people, but the principles of inquiry can be shared for mutual advantage and individual development.

IO3 focuses on *Inquiry-Based Tasks and Teaching Units* that are designed for use by teachers with their students. Chapter 6 addresses this design and the examples that have emerged. In each of our partner groups we have engaged with practices of teaching and learning: giving lectures, tutorials, seminars, small group work; working with our students on different courses, setting examinations and marking them, awarding grades. This is our practice. Inquiry can enter into any of these elements of practice in many different ways. One of the tasks we have undertaken is that of designing inquiry-based tasks and teaching units for our students' engagement. These look different depending on the designer, the students, the mathematical topic for design, the nature of the teaching/learning event (lecture, tutorial etc.) and the desired learning outcomes. We are in the process of developing a compendium of tasks and teaching units with contributions from all partners. The activity in each partner group is specific to that group, relating to local context and culture and to ways of being and doing. We have no intention to make the groups look all the same. The case study chapters in Part 3 of this book present activity, perspectives and the learning and development in the different partner communities. One important focus here is the identified needs of students. Although every student is different with their own particular characteristics and needs, some students have well defined needs, either physical or psychological (e.g., sight or hearing; dyslexia or autism). Two of our partner universities have special centres for the support of students with identified learning needs. Colleagues in these centres have guided the rest of us in making provision for these students. Chapter 4 focuses specifically on provision for identified needs and Chapter 6 includes identified needs in relation to the tasks and teaching units.

IO4 focuses on *Professional Development* for new and experienced teachers/lecturers who are interested in developing inquiry-based learning and teaching in mathematics. Chapter 7 addresses such professional development by offering methods and materials for professional use. We are well aware, in PLATINUM, that inquiry-based practice is itself an important source of professional development as can be seen in the chapters in Part 3. As we have worked together during nearly three years, it is fair to say that we have all developed in differing ways, depending on our starting points and the activities in which we have engaged. We are all more knowledgeable in what inquiry means for us, in *what we can do* to engage with inquiry, and the *differing ways* in which we can engage. If new colleagues join us, we can draw them into our communities and they can learn through working alongside others with critical alignment.

However, it is sometime necessary, or requested, that we offer some professional development workshops or seminars in inquiry-based practice for the benefit of others who wish to hear about our experiences and associated practices. For this purpose, we experimented with three workshops, each in one of our partner settings. The workshops had some activities in common and some specific to the setting. What we learned from these forms the basis of Chapter 7 in which we address *professional development in* PLATINUM and provide ideas for workshops etc to embrace and inform interested colleagues.

IO5 focuses on *Mathematical Modelling* addressing the design and use of inquiry-based tasks that relate to the world around us. Chapter 8 addresses this design and the examples that have emerged.

IO6 focuses on *Evaluation.* This includes both evaluation of our use of inquiry-based materials with students in our Partner locations as well as the wider evaluation of the project as a whole. Chapter 9 addresses issues in evaluation.

We highlight some of these interrelations among IO's according to the chapters.

Relations between IO1 and representative local cases (IO2). All partners contributed to the development of a generic framework for inquiry-based teaching of mathematics at the university level and in different contexts of higher education. It is worth noting that Chapter 15 reflects a long history of implementing the model, while other chapters express differing levels of experience in inquiry-based practice.

Relations between IO2, IO3, and IO5. The development of local university mathematics lecturers' communities of inquiry in which the university lecturers have explored inquiry-based teaching and learning and reflected on the effects in the design and implementation of teaching units. For example, Chapter 18 reports inquiry-based teaching approaches including expertise from industry and presentation of worked examples of realistic mathematical models. Also, see Chapters 13 and 17 written by mathematicians and experts in diversity (identified needs). These show productive results in contributing to IBME elements typically in large courses (including mathematics and statistics for economists).

Relations between IO4 and IO3 in the development of a professional development programme. Here, not only theoretical frameworks of professional development, but also the results and experience of the cycles concerning the design of the tasks or units in IO3 were taken into account. The development of collections of teaching units at the local level promoted mathematical conceptual learning through an inquiry approach. The nuances brought by each implementation in the specific regular courses in different Bachelor degrees (biomedical science, mathematics, computer engineering, etc.) at several universities have enriched PLATINUM's global approach to professional development of mathematics lecturers (see Chapters 11, 14, and 16). The contributions of these chapters allow us to give answers from PLATINUM to different questions such as: What does inquiry-based mathematics education means and why do we prefer inquiry-oriented education for professional development courses? What went well (or even above expectation), what was less successful, and what kind of obstacles were encountered? What ideas came up to improve the course in terms of contents, learning outcomes, and pedagogy (in particular opportunities for student inquiry)?

Relationship of the IO6 with other IO's, Developing the third layer of the CoI. PLATINUM project has made an effort to transmit the evaluation experience of the different inquiry communities (See Chapter 7). Some concepts related to IBME such as (1) evaluation of conceptual learning and teaching of mathematics; (2) monitoring students' engagement in IBME; (3) evaluation of the teaching practice by the CoI; and (4) evaluation of experiences about professional development of university mathematics lecturers. Each partner has contributed to different layers of the model. We have been able to share evaluation commonalities, but also characterise differences. The common aspect of the cases presented is the commitment in the CoI and the subsequent commitment to develop research on Inquiry based teaching and learning in light of what was learned in the process. This, as we have seen above, is the third layer of the CoI and the one that needs to be developed in time.

2.6. Discussion and Conclusions

Our writing. above, addresses not only the theoretical basis of our chapter, our inquiry approach towards mathematics learning and teaching at university level, but also the reasons why such an approach could be needed and should be considered. Although, in PLATINUM, our Communities of Inquiry bring together colleagues from different traditions within Europe, the needs for improvement of mathematics teaching and learning at Higher Education level are largely commonly understood and agreed. Despite national differences, we all find students who may not experience mathematics in the ways we would like to see, as articulated above. Although this might be widely recognised in most environments, the developmental solutions are not widely understood or agreed. However, through conceptualisation of the PLATINUM project, our eight partners came to agreement on many of the issues involved and a commitment to addressing them. IBME provided a possible basis for our exploration, which included our theoretical model and its characterisation as a framework for our developmental activity.

One, very obvious difference, in our PLATINUM partnership, was that some of us are mathematicians (doing research in mathematics) and some are mathematics educators (doing research in mathematics education and, in some cases, in educating new teachers). A small number are both. All of us are teachers. In this book we emphasise our teaching and its development, the learning that we have experienced through exploring new forms of practice and considering carefully the outcomes and issues. This is what we mean by 'walking the walk'. The case studies in Part 3 offer insights to the developmental processes in which we have engaged. These insights have brought us closer together as partners, exploring what inquiry can mean in our teaching and for our students. Together we have designed tasks and used them with students, in our teaching. Researching the nature of this process of design-use-feedback has enabled us to gain insights leading to new knowledge in practice. The other chapters in this book address the various dimensions of our work, reflected in the intellectual outputs (IOs).

The *essence* of the PLATINUM project lies in its interpretation of inquiry-based learning and teaching in mathematics. This was not pre-given. Working from the three-layer model, our framework for activity developed throughout the project. At workshops in each of our countries, we shared our thinking and our associated activity. Design of inquiry-based mathematical tasks became central to that activity and provided a rich common ground for discussion and experimentation. The concept of Inquiry Community—working together collaboratively, both in and across our partner groups—brought with it fellowship and understanding, a willingness to learn with and from each other in a variety of ways. Other chapters, in Parts 1 and 2 of this book, provide details of this work and its outcomes. They are written by teams of authors from across our partner groups. They bring together perspectives from across the project, unifying partner perspectives in identifying common aims and practices, and providing examples. Part 3 consists of eight case studies, each one presenting key elements of the activity and development of one partner group. Here we see diversity, both in terms of the starting points for each group, and also the developmental directions their activity and learning have taken.

As you read this book, we hope you will enter into our activity, its modes of inquiry, and the issues we have addressed. For us, the new insights we have gained provide a rich basis for further activity in applying and understanding elements of inquiry-based progress in learning and teaching to provide, we hope, better learning experiences of mathematics for our students.

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CHAPTER 3

Spidercharts: A Tool for Describing and Reflecting IBME Activities

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3.1. Introduction

The main goal of PLATINUM was to strengthen IBME in university mathematics teaching within a collaborating network of European partner universities, which saw itself as a *Community of Inquiry* (CoI) (see Section 2.5). As the visions of IBME present among project partners as well as local conditions at the partner universities were very diverse, efforts were made to account for local specificities in the development and implementation of ideas and concepts of IBME—in accordance with the notion of CoI. In this process of community building, its diversity became a fruitful resource, which enriched interactions at jointly conducted workshops (cf., Chapter 7). A key factor for successful cooperation within the community was good project communication.

Successful project communication relies on participants' familiarity with basic terms and concepts of relevance to project goals, and on their ability to identify and communicate parallels and differences between the many views that are present among partners. This enables project partners to locate their respective objectives and local conditions within the shared project context and to relate them to each other. With this in mind, we developed three spidercharts as a tool to facilitate project-wide thinking and communication about activities of local groups and to promote reflection on and further elaboration of the common vision to integrate IBME in our teaching. The spidercharts were developed both theory-based and inductively. We will say more about this in the next section.

This chapter is structured as follows: In the next section, we first describe theoretical considerations that inspired the development of the spidercharts. In Section 3.3, we explain the meaning of the labels in the spidercharts. After this, we present in Section 3.4 the results we obtained from each partner when we asked them to fill in the three charts with respect to the specific PLATINUM case they present in their respective chapter. Lastly, we shortly describe some patterns and distinctive features that became visible when comparing our partners' completed charts.

3.2. Developing the Spidercharts

We came across the idea of spidercharts in a contribution by Lübcke, Reinmann, and Heudorfer (2017). The double-wheel model presented there was used in a large research project to classify research-oriented teaching at university level into types. The double-wheel model is a further development of the wheel model by Brew (2013) that was created to facilitate "the identification of choices to be made in developing research-based pedagogies" (p. 612). Both, the wheel and the double-wheel model subsume curriculum design under pedagogic choices and propose dimensions for reflection on pedagogic choices in inquiry-oriented teaching settings. The double-wheel model by Lübcke et al. (2017) differentiates between a *micro-level* that focuses on didactical choices and a *meso-level* that has a stronger focus on curricular considerations. Neither model focuses explicitly on mathematics education, but both are specific to university contexts. We borrowed the spiderchart idea with the intention of designing a similar model as a reflection tool for the PLATINUM community. For this, we had to adapt the model to the specific needs and theoretical basis of the PLATINUM project. Specifically, our model needed to reflect important general dimensions of IBME. Here, we had to strike the following balance in our design: on the one hand, we wanted to take into account the wide variety of IBME approaches and activities present in PLATINUM; on the other hand, we wished to maintain a certain degree of specificity, in the sense that the charts' collection of aspects proposed for reflection should have specific relevance to mathematics as a university subject. In the following we will explain in more detail the intentions and ideas that guided the development and design of the spidercharts.

The spidercharts are intended as a supportive tool for (groups of) people who engage in the development of IBME activities and their implementation in teaching-learning environments (e.g., lecturers, teacher-researchers; henceforth *teachers*).[1] They are meant to guide teachers through a structured session of reflection about their teaching—or, more precisely, about one of their IBME activities. This guided reflection can spark new ideas that help flesh out the notion of IBME underlying the teaching project and can create opportunities to uncover formerly overlooked potentials in the local teaching-learning setting. Inspired by the theoretical context of our PLATINUM project, we decided to create three charts that address three perspectives on a teaching-learning scenario that are of major importance in the development of IBME activities: the perspective of students and their scope of possible activities within the learning environment is addressed in the chart *inquiry learning*; the perspective of teachers when working in the classroom is captured in the chart *inquiry teaching*; and the perspective of teachers in their work environment outside the classroom is the topic of the third chart, *group of inquiry*. In other words, each spiderchart proposes as focus of reflection a set of aspects relevant to the following questions concerning the processes and decisions involved in organising IBME activities:

- Chart *inquiry learning*: What possibilities of engaging with the IBME activity are available to students?
- Chart *inquiry teaching*: What methods of conducting an IBME activity are realised by the teacher? What restrictions are present in the specific learning scenario?[2]
- Chart *group of inquiry*: What structures and dynamics are at play in the group's work to support and reflect the IBME teaching of their members?

The questions focus mainly on methodical-didactical design elements in the development and implementation of IBME activities: At the centre of attention of the charts *inquiry learning* and *inquiry teaching* is not the mathematical subject matter that is the object of the inquiry activity unfolding in the classroom, but the spectrum and quality of opportunities for engagement with the (mathematical) object of inquiry that the teaching-learning environment opens up to students (and teachers). Thus,

[1]This can but does not necessarily include researchers who research the implementation of IBME activities.

[2]Such restrictions might be set by the teacher or be imposed on students and teacher alike by the institutional context.

these two charts draw attention to the types of activities that are allowed or expected to be conducted (in the classroom) in order to approach and investigate the (mathematical) subject matter at hand. Filling out these two charts leads to a visual display of didactical choices made with the intention to engage students in (mathematical) inquiry. Activities and didactical choices may involve tasks, a specific setting, curriculum design, considerations in relation to the student group, the culture, etc. To what extent it is possible for teachers to integrate IBME into their regular teaching (e.g., constructing a whole module, creating one session within a course) depends of course on local conditions. The third chart *group of inquiry* similarly addresses methodical aspects, but this time of the teachers' work environment. It focuses on the collaborative work of the local group of inquiry and serves to reflect and visually display how the local communities within PLATINUM organise their teamwork. In particular, it calls to attention teachers' possibilities within their professional environment to engage in inquiry into their IBME teaching.

With these key questions and ideas in mind, we will now elaborate in more detail on the aspects and quality nuances addressed by the spidercharts: Each chart covers eight *aspects* of IBME that may be relevant to the development of IBME activities within the local group and their implementation in the classroom. Aspects can have different qualities in different teaching-learning settings. This is expressed by the *quality nuances* assigned to each aspect. The aspects and their respective quality nuances displayed on the spidercharts mirror basic principles of inquiry-based education (Artigue & Blomhøj, 2013; Dorier & Maaß, 2020; Lübcke et al., 2017) and also address collaborative developmental work within communities.

For instance, Dorier and Maaß (2020) characterise inquiry-based mathematics education (IBME) as follows:

> *Inquiry-based mathematics education* (IBME) refers to a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions (like carrying out experiments, systematically controlling variables, drawing diagrams, calculating, looking for patterns and relationships, and making conjectures and generalizations), interpret and evaluate their solutions, and communicate and discuss their solutions effectively. (p. 384)

The quotation formulates an idealised vision of teaching and learning. Although it is mainly learning that is explicitly described, the word "invite" indicates a vision of teaching whose specifics are not elaborated in the quotation, but which is nevertheless indirectly indicated by its ambition.

In addition, Artigue and Blomhøj (2013) provide an overview of existing forms of IBME with a strong focus on approaches typically pursued in school contexts. They present a broad variety of different understandings of IBME and their relation to different constructs, institutional contexts and educational cultures. In summary, they point out different prioritisations in the context of IBME of the following concerns:

[1] the 'authenticity' of inquiry questions, the connection of students' activities with their real life, links between everyday-life questions and activities;
[2] the epistemological relevance of inquiry questions from a mathematical perspective and the cumulative dimension of mathematics;
[3] the progression of knowledge as expressed in the curriculum;
[4] extra-mathematical questions and the modelling dimension of the inquiry process;
[5] the experimental dimension of mathematics;

[6] the development of problem-solving abilities and inquiry habits of mind;
[7] the autonomy and responsibility given to students, from the formulation of questions to the production and validation of answers;
[8] the guiding role of teachers' and teacher-students' dialogic interactions;
[9] the collaborative dimension of the inquiry process;
[10] the critical and democratic dimensions of IBME.

To illustrate how these concerns contributed to the spidercharts' development, we will now present an exemplary collection of connections between them and concrete aspects and quality nuances displayed on the spidercharts. We will do this by explicitly stating possible questions and thoughts for reflection that arise from the respective concern and naming the aspect(s) or quality nuance(s) connected to this question:

In what way is the connection between mathematical knowledge and real life relevant? What relevance does the modelling dimension[3] of the tasks we provide have (see [1] and [4])? Do teachers in their teaching focus on *applied* or *theoretical mathematics*? [chart *inquiry teaching*; aspect: *type of mathematics*]

If mathematics is taught with a stronger emphasis either on application or on formal/scientific criteria, is this a didactical choice made by the teacher or is this situation predetermined by the curriculum? [This is an underlying concern of the spiderchart *inquiry teaching* that should be kept in mind when using it.]

What is the epistemological relevance of the questions or the content that we address in class from a mathematical point of view? And what importance do the questions or does the content have for the cumulative dimension of mathematics (see [2] and [3])? Is the manner in which asking questions, communication of findings and justification take place in class open to student choice, or determined by the teacher or other factors (i.e., closed)? Is the form of justification open for choice or is it required to follow formal/scientific criteria? [chart *inquiry learning*; aspects: *asking questions*, *communication of findings*, *justifying*]

The experimental dimension of mathematics points to similarities between inquiry learning in mathematics and inquiry-based education in other disciplines (see [5]). We highlight the importance of making observations, asking questions, planning investigations and using tools within an inquiry process in the spiderchart *inquiry learning.* [chart *inquiry learning*; aspects: *making observations*, *asking questions*, *planning investigations*, *using tools*]

What degree of autonomy and responsibility is given to and demanded from students (see [7] and [8])? This question motivated the definition of certain quality nuances we attached to the charts' aspects (esp. in the chart *inquiry learning*). If the didactical choices grant students a high degree of autonomy and responsibility, the quality nuance assigned to the aspect in question (e.g., aspects: *asking questions*, *communication of findings*) is open. If a low degree of autonomy and responsibility is granted to students then the quality nuance of an aspect is closed. Regarding the quality nuances open and closed, it should be kept in mind that the autonomy and responsibility given to students bears implications for the role of the teacher in the respective teaching-learning environment. [This is a general concern connected with the spiderchart *inquiry learning.*]

[3]For the development of modelling tasks in PLATINUM, see Chapter 8.

So far we have described how we have selected aspects and quality nuances based on theory. Additionally, the development of the spidercharts was done inductively in that the concrete selection of aspects was also based on discussions at our international PLATINUM meetings (discussion notes, feedback by PLATINUM partners): We repeatedly asked partners about their local cases at different stages of their development. With the help of this information we checked whether the categories of the spidercharts and their characteristics could be interpreted in a meaningful way and whether, from our and our partners' point of view, essential aspects of local cases could be addressed. In the workshop in Brno we could observe that the spidercharts could be successfully used as a tool for reflection. When they were filled in, they triggered a wide range of discussions about the current design of the cases and about the possibilities for further adaptations. However, the Brno workshop also produced incentives to revise some of the categories. After this revision, we approached the partners a second time with the updated spidercharts and filled them in together with them.

All in all, our efforts yielded the following eight aspects for each of the spidercharts:

- *Inquiry learning:* exploration, planning investigations, communication of findings, justifying, asking questions, cooperation, using tools, and making observations.
- *Inquiry teaching:* assessment format, media, type of mathematics, content, teaching methods, tasks, scaffolding, and feedback.
- *Group of inquiry:* deciding on objectives, access to group, discussions, evaluation, reflection on professional growth, reflection on teaching, joint planning of teaching, and organisation of group work.

Moreover, each of these aspects has specific quality nuances assigned to it. Some quality nuances appear in connection with a number of different aspects, some appear in connection with only one single aspect. The different quality nuances that appear are:

- open & closed/& standardised/& structured/& formal/scientific
- essential & non-essential
- formative & summative
- weakly formatted & strongly formatted
- reduced facilitation & increased facilitation
- student-centred & teacher-centred
- student-chosen & teacher-chosen
- digital & analogue
- applied & theoretical

Each quality nuance is intended to represent a continuum of possibilities. Therefore, we additionally introduced the two middle options of *quantitative* ("quan.") and *qualitative* ("qual."): *quantitative* expresses that with regard to all occurrences of an aspect (in an IBME activity), each end of its quality nuance applies to (roughly) equally many of these occurrences; *qualitative* expresses that with regard to the occurrences of an aspect, both ends of its quality nuance are equally important, in the sense that most occurrences of this aspect are located between both quality nuance ends in terms of gradation.

In the following section, we explain the use of the spidercharts and provide detailed descriptions of each chart's aspects and their respective quality nuances. Afterwards we present results of a survey conducted with the spidercharts among the partners of the PLATINUM project.

3.3. The Three Spidercharts and How to Work With Them

Because the spidercharts are intended as a tool for reflecting IBME activities, the first thing to do in order to use them is to decide, which IBME activity should be focused and what exactly constitutes this activity. Additionally, the group of inquiry relevant for the third spiderchart needs to be specified. These choices impact the significance of each of the three spidercharts for the description and reflection of the chosen IBME activity. The spectrum of eligible IBME activities is broad and can range from a single exercise conducted in one specific session of a course to an entire course which was designed to contain IBME. Instead of a course it is also possible to pick a course plan which has not yet been implemented, if it is sufficiently detailed. The relevant criterion for selecting the group of inquiry is the group's participation in the development, implementation or research of the activity in question. The group could for example be a group of researchers who developed an IBME activity or a team of teachers who work together to promote IBME in their courses.

After this first step has been completed, the charts can be consulted. If we take a quick look at the charts on the upcoming pages, we can see that the central label of each spiderchart shows the perspective the chart addresses: *inquiry learning*, *inquiry teaching*, or *group of inquiry*. The outermost ring of each chart shows the *aspects* of the respective perspective. The three rings between the centre of the chart and the outermost ring are divided into fields that will be ticked in the process of filling in the chart. The four fields between each aspect and the chart's centre name the *quality nuances* of the respective aspect (cf., for example, in the chart *inquiry learning*, the quality nuances "open" and "closed" of the aspect *exploration*, and their middle options "quan." and "qual.").

In order to characterise a specific IBME activity and its context using the spidercharts, we ask the user to tick one quality nuance field for every aspect with the selected IBME setting in mind: Of the four fields for every aspect, ticking the inner- or outermost one means, that the descriptor in the respective field describes that aspect of the chosen setting best. The middle should be ticked, if the descriptors on the two ends of the quality continuum both apply equally strongly in the context in question. In this case, a choice must be made between ticking the field "quan." (for quantitative) or the field "qual." (for qualitative). The rule of thumb is that "quan." should be ticked if the decision to tick the middle was taken because the teaching-learning setting that is being reflected with the help of the charts hosts a quantitative mix of features or elements of which some are best described by the innermost field and others best described by the outermost field. The field "qual." should be ticked, if the decision to tick the middle was taken because both inner- and outermost fields are qualitatively equally important or applicable to the feature(s) of the user's setting referred to by the respective aspect (i.e., if said features are really located in the middle of the continuum defined by the two quality nuances at its two ends).

The aspects in each spiderchart may overlap in meaning and may also be correlated with each other. This can be more or less the case depending on the specific scenario chosen to be described or reflected with the help of the spidercharts. However, this does not pose a problem to the central goal of the activity of filling in the three charts, which is to bring to the attention of the user different aspects which are relevant to IBME activities, to inspire thought about their nature in the context of the user's own IBME project and to start a conversation about a specific IBME activity among the members of a group of inquiry. Furthermore, the aspects of the spidercharts are generally not intended to be interpreted strictly within the context of mathematics as a subject.

For example, *exploration* can mean any type of exploration, it is not restricted to designate a purely mathematical activity. Some aspects might have different meanings or are likely to be interpreted differently in different subjects or faculties. Which interpretation is relevant for the IBME activity to be reflected is up to the members of the involved group of inquiry.

With a view to the application of the three spidercharts in practice, we now formulate their foci as action-oriented as possible and explain the meanings of all aspects. Afterwards we look at the quality nuances and illustrate them by examples.

Spiderchart inquiry learning. This spiderchart focuses on aspects which are relevant for IBME from the perspective of the learner. Filling in the chart can be facilitated by keeping the following questions in mind:

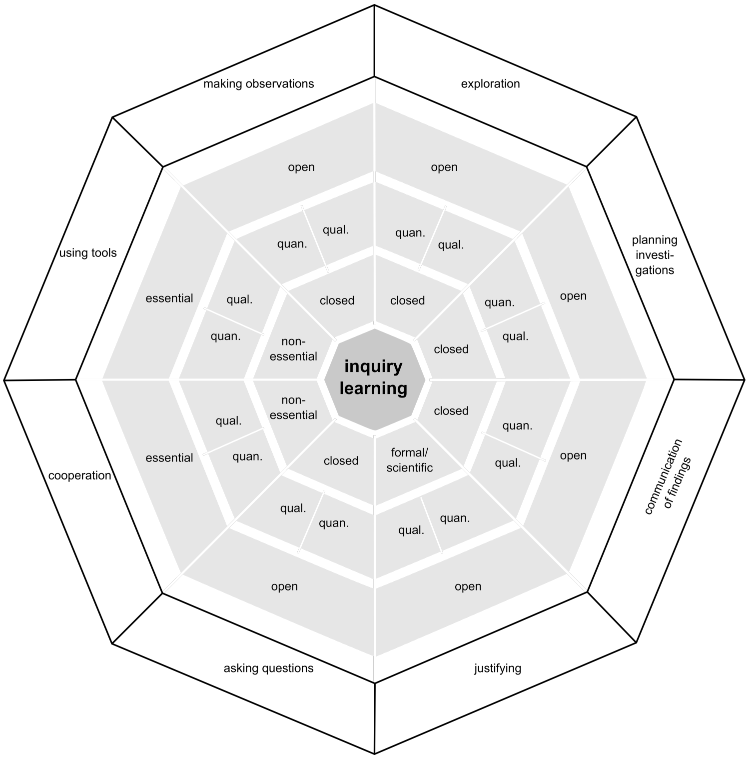
- What do students need to do in order to participate in my IBME activity?
- In what ways are students engaged in inquiry learning?

Spiderchart inquiry teaching. This spiderchart brings to attention aspects of the instructional setting which may or may not be under the control of the teaching staff responsible for the chosen IBME activity. The following questions might help with filling in the chart:

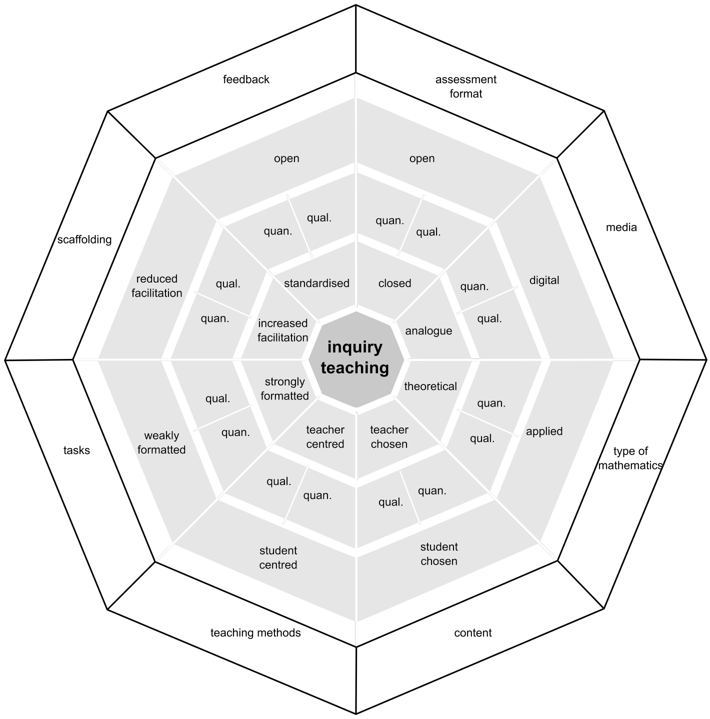
- What did the instructional setting of my IBME activity look like?
- What does the envisioned instructional setting of my IBME activity look like?

Spiderchart group of inquiry. This spiderchart is intended to help describe the collaboration in the group of inquiry who develops, teaches and/or does research on the chosen IBME activity. The following questions might help bring to mind information relevant to filling in this chart:

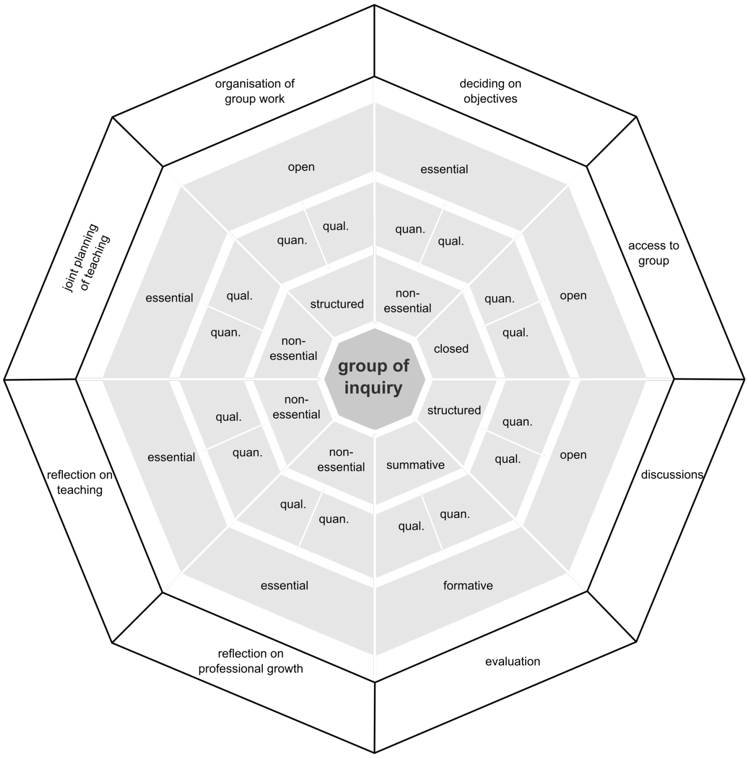
- How does my group of inquiry work on projects and tasks?
- How is collaboration in our group organised?



Aspect	Definition
exploration	Learners engage in exploratory activities in view of an unfamiliar problem/mathematical phenomenom/etc.
planning investigations	Learners plan a structured investigation into a (larger/more complex) mathematical topic or problem.
communication of findings	The manner in which learners communicate results of investigations or thought processes to peers, the teacher or other people. They might talk informally, write a proof, hold a presentation, etc.
justifying	Learners give justifications for statements or choices they made. Justifications can be mathematical, scientific, common sense, naive, etc.
asking questions	Learners ask questions at different stages of an inquiry process: to initialise an inquiry, to refine a question, to question findings, etc.
cooperation	Learners form groups and interact in order to jointly work on an IBME activity.
using tools	Learners make use of digital tools, algorithms, specific heuristics or strategies, etc. as a means to achieve some goal (e.g., representing or depicting something, solve a task, explore a phenomenon, etc.).
making observations	Learners make and possibly articulate observations about some phenomenon of interest or some problem situation.



Aspect	Definition
assessment format	Any mechanism of evaluation of student performance, knowledge or skills. This may but does not have to produce a grade.
media	The means of communication (e.g., text on paper, video, speech, whiteboard, etc.).
type of mathematics	We distinguish between pure or theoretical mathematics (e.g., proofs, logic) and applied mathematics (modelling, mathematics used in other subjects).
content	The knowledge, topics and questions of the IBME activity.
teaching methods	The intended ways of knowledge creation and exchange that are supported/managed/implemented by the teacher.
tasks	Exercises and problems given to learners in the context of an IBME activity in order to structure the educational process. Learners are typically expected to provide some sort of task solution as a result of working on the task.
scaffolding	Additional hints, instructions or information that accompany a given task in order to facilitate the process of solving it (e.g., provision of subtasks). Scaffolds are often part of the task instruction.
feedback	Any form of feedback or evaluation by students, teachers or others on actions/statements/homework/etc. by students, teachers or other people. The feedback must be relevant to or part of the IBME activity.



Aspect	Definition
deciding on objectives	The process of deciding about objectives the group wants to work towards.
access to group	The readiness of the group to invite/welcome temporary visitors or new members.
discussions	The conversations that take place in group meetings and that may or may not follow a previously fixed agenda.
evaluation	The process of implementing some mechanism or approach to determine the group's success in achieving their common goals and the effectiveness of the group's collaboration.
reflection on professional growth	As a place of professional growth for the teachers participating, a group of inquiry can choose to explicitly think about or even promote the professional growth of their members.
reflection on teaching	The process of thinking about and evaluating past teaching experiences. It can be done individually in private or through exchanging and discussing experiences in a group.
joint planning of teaching	The preparation of course sessions, the design of courses and curricula, etc. – individually or in a group.
organisation of group work	The manner in which of the group of inquiry cooperates. The way in which the group's collaboration evolves.

Next, we describe in detail the quality nuances and illustrate some of them by examples.

Open & closed/& standardised/& structured/& formal/scientific. The quality nuance "open" generally refers to an absence of restrictions or a lack of guidance. The counterparts ("closed", etc.) tend to mean the opposite, namely that specific rules or circumstances limit the number of choices for an involved person and that advice or rules are given which specify what to do and/or how to do it. These quality nuances apply to the following aspects:

- *Spiderchart inquiry learning*: exploration (Table 3.1),[4] making observations, planning investigations, justifying (Table 3.2), asking questions, communication of findings (Table 3.3).
- *Spiderchart inquiry teaching*: feedback (Table 3.4), assessment format.
- *Spiderchart group of inquiry*: organisation of group work, access to group, discussions.

Quality nuance	Example-scenario corresponding to the selected quality nuance
open	Students may explore a question/topic/situation by consulting sources of their choice and using any strategies available.
quan./qual.	quan.—There are about as many situations of exploration of the type "open" as there are of the type "closed". qual.—Most situations of exploration are neither of type "open" nor of type "closed"; the freedom to explore is generally situated somewhere in the middle between the two.
closed	Students may explore a question/topic/situation in a very limited environment, for example testing certain types of input in a given program and observing what happens to the output, to form a hypothesis.

TABLE 3.1. Spiderchart *inquiry learning*, aspect *exploration*.

Quality nuance	Example-scenario corresponding to the selected quality nuance
open	How students justify their results or solve a problem is open for choice. Non-formal arguments are accepted as justifications.
quan./qual.	See Example 1 in Table 3.1.
closed	Students' justifications have to adhere to standards of formal mathematics or some scientific discipline.

TABLE 3.2. Spiderchart *inquiry learning*, aspect *justifying*.

[4]Regarding notation: Exemplary illustrations can be found in the given tables. Where no reference is made to a table the respective nuance does not have an illustration.

Quality nuance	Example-scenario corresponding to the selected quality nuance
open	Students can choose freely how to articulate or communicate their findings. (e.g., choosing a manner of presentation, creating a picture/diagram/poster, writing a proof/text/poem/etc.)
quan./qual.	See Example 1 in Table 3.1.
closed	Students have to follow some pre-structured format to articulate their findings. (e.g., multiple-choice questionnaire, writing a formal proof, using specific (technical) language or images, etc.)

TABLE 3.3. Spiderchart *inquiry learning*, aspect *communication of findings*.

Quality nuance	Example-scenario corresponding to the selected quality nuance
open	Feedback is given more or less spontaneously and in a rather unstructured manner. It is not oriented towards predefined criteria.
quan./qual.	See Example 1 in Table 3.1
closed	Feedback is given in response to specific contributions, e.g., as a fixed element of every student presentation or by students after every course/semester, and it is given in a structured manner, e.g., by going through a list of items to be commented on or by filling a questionnaire.

TABLE 3.4. Spiderchart *inquiry teaching*, aspect *feedback*.

Digital & analogue. If the interaction between teacher and students relies heavily on digital media, the quality nuance “digital” applies. The quality nuance “analogue” is used, if the interaction between teacher and students rely on analogue media. This quality nuance applies to the aspect *media* in the spiderchart *inquiry teaching* (Table 3.5).

Quality nuance	Example-scenario corresponding to the selected quality nuance
digital	STACK; internet; programming; etc.
quan./qual.	See Example 1 in Table 3.1.
analogue	Printed paper; blackboard; etc.

TABLE 3.5. Spiderchart *inquiry teaching*, aspect *media*.

Essential & non-essential. This pair of quality nuances refers to the necessity of actions in the context of the IBME activity. Is it essential for a student to do something in order to be able to adequately or successfully participate in the IBME activity? As a member of a group of inquiry, how essential are certain aspects of cooperation for the overall success of the group's project? The quality nuance "essential" is the only one for which the rule of thumb for the choice between "quan." and "qual." in the middle box does not apply. The rule that holds for this quality nuance is explained in Tables 3.6 and 3.7. These quality nuances apply to the following aspects:

- *Spiderchart inquiry learning*: cooperation, using tools.
- *Spiderchart group of inquiry*: deciding on objectives (Table 3.6), reflection on professional growth, reflection on teaching, joint planning of teaching (Table 3.7).

Quality nuance	Example-scenario corresponding to the selected quality nuance
essential	The group of inquiry discusses common objectives and changes objectives if necessary. Joint decision making with the involvement of all group members is considered essential for the group's work.
quan./qual.	quan.—The group discusses common objectives and/or changes objectives frequently. However, this is not considered essential for the group's work. qual.—Joint discussion of common objectives is considered to be essential for the group's work. However, it happens only very rarely.
non-essential	Common objectives are not discussed (anymore) in group meetings (e.g., objectives could be defined by one group member and followed by the others). Joint discussion of common objectives is not considered essential for the group's work.

TABLE 3.6. Spiderchart *group of inquiry*, aspect *deciding on objectives.*

Quality nuance	Example-scenario corresponding to the selected quality nuance
essential	The group of inquiry designs teaching units/materials together. The involvement of all group members in the planning of teaching is considered essential for the group's work.
quan./qual.	quan.—The group regularly plans teaching together. However, this is not considered essential for the group's work. qual.—Planning teaching as a group is considered essential for the group's work, but happens rather infrequently.
non-essential	Each group member has full responsibility for a course of their own. They may talk with other members about teaching, but each member designs their own course curricula, tasks, etc.

TABLE 3.7. Spiderchart *group of inquiry*, aspect *joint planning of teaching.*

Formative & summative. The quality nuance "formative" applies when the evaluation of an IBME activity focuses on the work process in the group of inquiry. The quality nuance "summative" applies when evaluation focuses on the outcomes of the group's work. This quality nuance applies to the aspect *evaluation* in the spiderchart *group of inquiry* (Table 3.8).

Quality nuance	Example-scenario corresponding to the selected quality nuance
formative	The group discusses regularly whether and how effectively/ easily goals have been achieved. If needed, the activity or the group's work organisation are changed.
quan./qual.	See Example 1 in Table 3.1.
summative	On a regular basis questionnaires or criteria checklists are used to determine if goals have been achieved.

TABLE 3.8. Spiderchart *group of inquiry*, aspect *evaluation*.

Weakly formatted & strongly formatted. If task instructions together with contextual information (e.g., didactic contract) do not or only slightly lead the solution process into a certain direction, the quality nuance of the aspect *task* is "weakly formatted". If task instructions in the given context hint at some intended solution or solution format, the quality nuance "strongly formatted" should be chosen. This quality nuance applies to the aspect *tasks* in the spiderchart *inquiry teaching* (Table 3.9).

Quality nuance	Example-scenario corresponding to the selected quality nuance
weakly formatted	A modelling problem that does not have fixed quality criteria for a solution and that allows for non-mathematical arguments and knowledge to be applied. This leaves room for a wide variety of different solutions or different types of solution.
quan./qual.	See Example 1 in Table 3.1.
strongly formatted	The task of formally proving a mathematical theorem limits the range of acceptable task solutions (arguments must adhere to specific quality standards, the language must adhere to specific norms of communication, there are correct and incorrect solutions, etc.).

TABLE 3.9. Spiderchart *inquiry teaching*, aspect *tasks*.

Reduced facilitation & increased facilitation. The quality nuance "reduced facilitation" applies, if no or very few scaffolds are used in a task instruction. The quality nuance "increased facilitation" applies when a broad array of scaffolds is provided in a task instruction. This quality nuance applies to the aspect *scaffolding* in the spiderchart *inquiry teaching* (Table 3.10).

Quality nuance	Example-scenario corresponding to the selected quality nuance
reduced facilitation	A typical Fermi problem does not guide the learner towards a specific solution path and does not provide much help.
quan./qual.	See Example 1 in Table 3.1.
increased facilitation	A mathematical task that consists of a sequence of consecutive subtasks, splitting the tasks in smaller, more manageable parts.

TABLE 3.10. Spiderchart *inquiry teaching*, aspect *scaffolding*.

Student-centred & teacher-centred. If a teaching format/approach relies on the active participation and engagement of students, it is "student-centred". If a teaching format/approach relies mostly on the teacher's actions and activities, it is "teacher-centred". This quality nuance applies to the aspect: *teaching methods* in the spiderchart *inquiry teaching* (Table 3.11).

Quality nuance	Example-scenario corresponding to the selected quality nuance
student-centered	A group project.
quan./qual.	See Example 1 in Table 3.1.
increased facilitation	A typical lecture.

TABLE 3.11. Spiderchart *inquiry teaching*, aspect *teaching methods*.

Student-chosen & teacher-chosen. If learners choose what problems to investigate and have control over the content studied, then the content (of the IBME activity or course) is "student-chosen". If a rigid curriculum exists that is followed very closely in the course of the IBME activity, the content is "teacher-chosen" (even if the curriculum was not designed by the teacher him-/herself). This quality nuance applies to the aspect *content* in the spiderchart *inquiry teaching* (Table 3.12).

Quality nuance	Example-scenario corresponding to the selected quality nuance
student-chosen	Group project with relatively open choice of question.
quan./qual.	See Example 1 in Table 3.1.
teacher-chosen	The contents of a specific book have to be studied in a fixed amount of time.

TABLE 3.12. Spiderchart *inquiry teaching*, aspect *content*.

Applied & theoretical. The quality nuance "applied" is used, if the type of mathematics students and teachers inquire into is essentially shaped by its context of use. The quality nuance "theoretical" is used, if the inquiry focuses on innermathematical topics. This quality nuance applies to the aspect *type of mathematics* in the spiderchart *inquiry teaching* (Table 3.13).

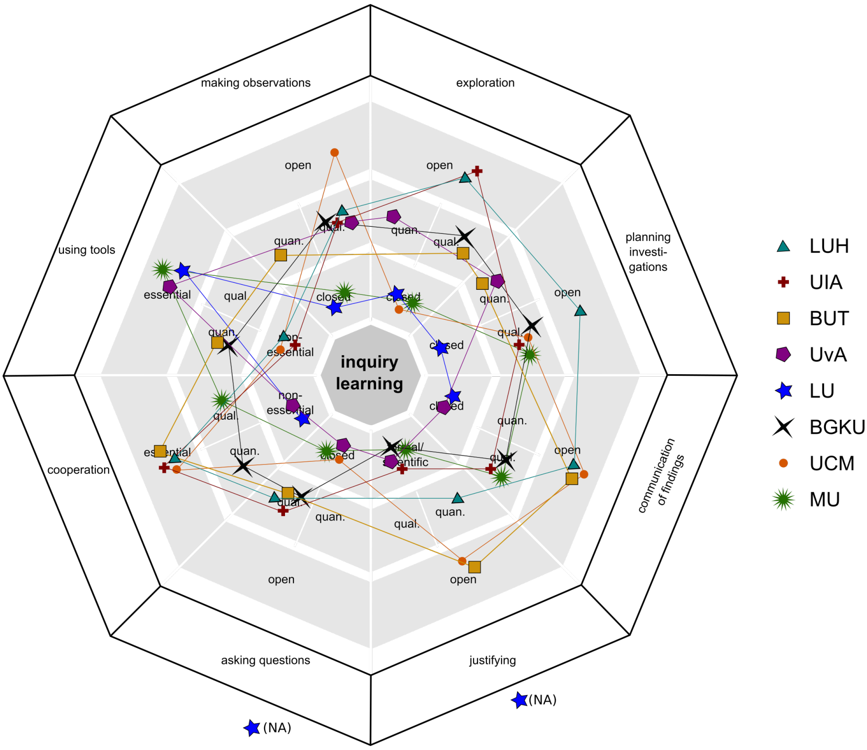
Quality nuance	Example-scenario corresponding to the selected quality nuance
applied	Mathematical modelling; mathematics teaching scenarios.
quan./qual.	See example 1 in Table 3.1
theoretical	Any topic of pure mathematics; foundations of mathematics.

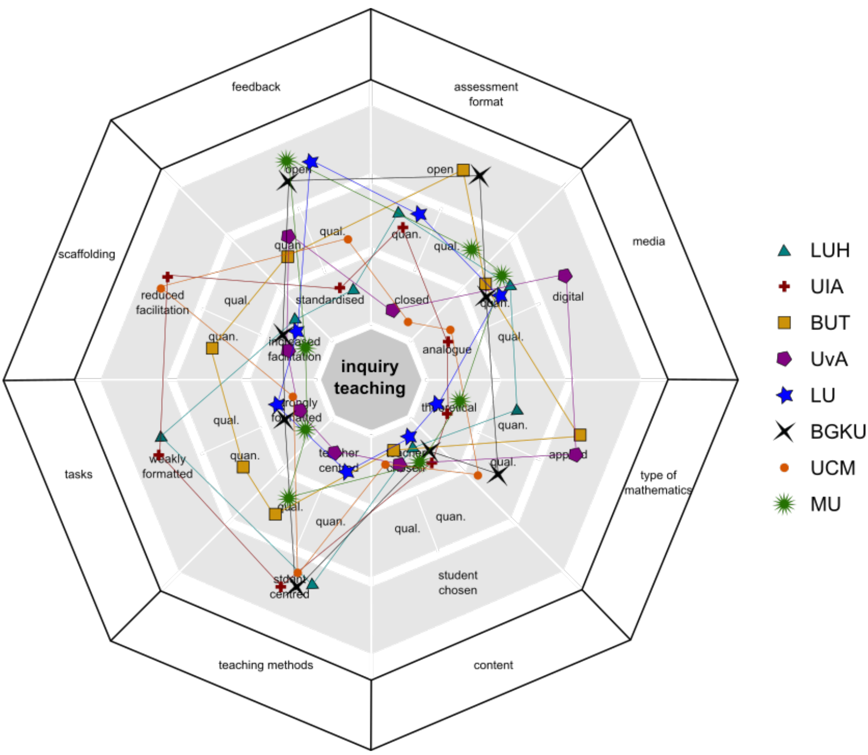
TABLE 3.13. Spiderchart *inquiry teaching*, aspect *type of mathematics.*

3.4. The Spidercharts of the PLATINUM Cases

In order to fill in the spidercharts with each of our PLATINUM partners, we organised online meetings of about an hour each. These meetings served to assist the partners in filling in the spidercharts and to clarify ad hoc questions. Before the meeting, we sent an email to our partners asking them to look at the charts and to look through a preliminary manual providing more detailed information about the aspects and nuances than we have just presented in this text. In addition, we asked the partners to recall their local case study or an example from it that was suitable for filling in the charts. Thus, the completed spidercharts represent a structured survey of the partners' understanding and interpretation of the different aspects and quality nuances in view of their respective case studies.[5] The spidercharts do not serve us here to evaluate the individual activities or to classify whether inquiry has taken place or not. Instead they illustrate the wide variety of possible inquiry activities. We integrated the choices of all individual partners into one chart (see Figure 3.1, spread over the next two pages). In this way we can graphically illustrate similarities and differences between the different local cases of our partners of the PLATINUM project. In the following we describe some initial observations. More detailed analyses are of course possible but will be reported elsewhere.

[5]The spidercharts are primarily designed as a reflection tool for the groups and in the interaction between groups. The activity of filling in, discussing and reflecting on the individual aspects within the group is essential. The completed spidercharts themselves are rather insignificant with regard to this function. The completed spidercharts also do not represent "objective" scientific data and should not be used to assess or evaluate the groups or the associated case studies or IBME activities. A comparison is also only possible to a limited degree, as the aspects and quality-nuances are open and require the interpretation of the respective partners in their context.


inquiry learning
making observations
exploration
planning investi-gations
communication of findings
justifying
asking questions
cooperation
using tools
open
quan.
qual.
closed
essential
non-essential
critical/ scientific
(NA)
LUH
UIA
BUT
UvA
LU
BGKU
UCM
MU


inquiry teaching
feedback
assessment format
media
type of mathematics
content
teaching methods
tasks
scaffolding
open
qual.
quan.
digital
analogue
theoretical
applied
student chosen
teacher chosen
student centred
teacher centred
weakly formatted
strongly formatted
reduced facilitation
increased facilitation
standardised
closed
LUH
UIA
BUT
UvA
LU
BGKU
UCM
MU

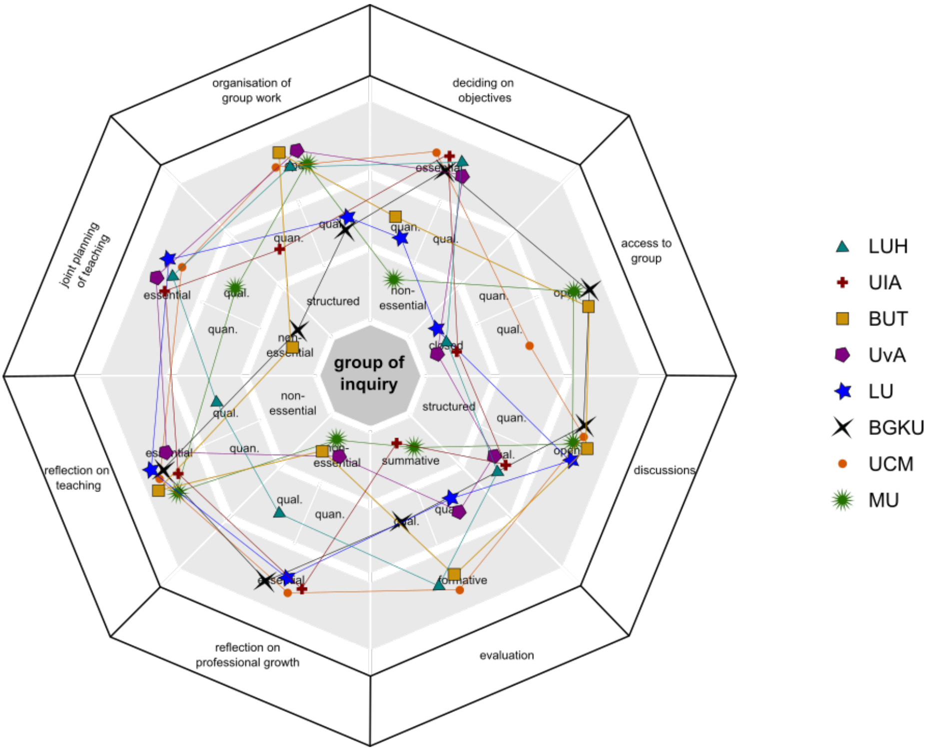


FIGURE 3.1. Spidercharts with options chosen by the PLATINUM partners (coloured version in the ebook).

If we take a look at the spiderchart *inquiry learning*, it is striking that no partner chose *asking questions*[6] as "open." For most partners students' cooperation was considered to be essential for the IBME activities. Only the partners from UvA and LU considered "cooperation" as "non-essential." Instead both partners report about the importance of their computer-based setting. It also seems to be due to their general institutional conditions that usually lecture rooms are equipped with individual computer workstations, which makes cooperative approaches at least more difficult. The issue of individual computer workstations for IBME activities was also discussed by BUT. Designs of lecture rooms that are equipped for computer-based group work are rare (Rønning, 2019). The aspects *using tools*, *communication of findings*, and *exploration* show a great variance, which reveals a scope for didactic decisions.

Considering *inquiry teaching* it is noticeable that, in comparison to the other spidercharts, for many aspects quality nuances in the inner ring are chosen. The aspects *content*, *teaching methods*, *tasks*, and *scaffolding* form a cluster that could be described as a type of "guided teaching": teacher chosen content, teacher centred teaching methods, strongly formatted tasks that guide the inquiry process together with increased facilitated scaffolding. The combination of those quality nuances guide the IBME activities in a desirable and manageable direction. For further explanation, we will discuss the options chosen by the UiA as an example of this type and contrast the

[6]The partners from Loughborough considered the aspects *asking questions* and *justifying* to be inapplicable to their case. In their setting students did not communicate openly their questions and justifications during the IBME activity.

options chosen by the UvA with them: UiA describes two cases for inquiry teaching. They chose a rather unconventional teaching scenario by deliberately choosing "unusual situations." One attempt is valued as a successful implementation and one attempt was abandoned. LUH has chosen a similar task design and also reports difficulties and resistance in implementation due to the given institutional conditions. UvA, in their case study, also points to several but different challenges that led them to opt for a more "guided" teaching-approach: Firstly, they refer to challenges that are rooted in the digital environment. The digital environment seems to be very challenging for their students, which made more scaffolding and a teacher-centred approach necessary.[7] Secondly, they draw on different types of mathematical content (statistics and mathematics) and its instrumental genesis.[8]

In contrast to the spiderchart *inquiry teaching*, the spiderchart *group of inquiry* has many options chosen in the outer ring. Two types of groups emerge: Firstly, "planning groups" that are not open in access and tend to reflect on professional growth within the group. And "open groups" who do not see planning as an essential group activity and who generally do not consider reflecting on professional growth as an essential group activity. The options chosen by BGKU show that it is also possible to choose a design for inquiry group activities that is open in access and provides spaces for reflecting professional growth collectively. MU does not reflect professional growth collectively. The group activities described in the case study show a strong connection to individual reflection of professional growth. UCM can be considered to be a planning group. They describe several inquiry teaching scenarios for different types of courses in their case study. In their work, reflection on professional growth takes place through joint theory-based planning.

3.5. Discussion

The spidercharts intend to support reflection processes revolving around the concepts of inquiry learning and inquiry teaching and the work in groups of inquiry. The quality nuances of their aspects represent different locations along a continuum. However, they are not prescriptive of what constitutes an appropriate choice: Per se, none of these options are better or worse choices to foster IBME or to bring forward the work in our local groups of inquiry. Instead of categorising, the charts are intended to inspire reflection processes: The forced choice of one out of four options along the continuum motivates reflection on aims of, choices made in, broader conditions of (etc.) local IBME activities. The charts do not intend to serve as an evaluation tool that judges the "success" of a local group or IBME activity. In the best case, they help to reveal hitherto unconsidered opportunities for intervention and to make hitherto implicit decisions explicit with regard to the respective local cases.

The interviews we conducted on the occasion of filling in the spidercharts showed us that the three charts actually cover a very wide range of possible IBME activities. There was no case that could not be located in terms of important aspects from the point of view of the interview partners. Moreover, the results really show the great diversity of IBME activities developed and implemented in the PLATINUM project. Thus, central goals of our development of the spidercharts could be achieved.

[7]From the UvA case study: "This teacher direction is called instrumental orchestration in the instrumental approach" (p. 227).

[8]From the UvA case study: "We also had not realised that the instrumental genesis of students during the statistic part of the course had a different orientation than the one needed for the tool use in the module" (p. 227).

In summary, the spidercharts could help our project partners and us to articulate ideas, to adequately characterise each of the various contributions in this project, to grasp and compare creative leeways and thus to express specificities of the local cases within PLATINUM. Thus, we believe that the spidercharts are helpful to structure the presentations of our work in the PLATINUM project and to facilitate communication between local groups of inquiry. Moreover, a first rough comparison already yields some results that appear typical, for example with regard to the inner or outer concentration of ticks on the various spidercharts. It has to be understood, though, that behind similar lines there could be different situations or reasons for didactic-methodical decisions. This suggests that very narrow interpretations of the content of trajectories should be taken with caution. For the time beeing, we will leave open the question of whether further conclusions can be drawn from more detailed analyses of the completed spidercharts. Overall, we believe that the spidercharts are helpful to structure the presentations of our work in the PLATINUM project and to facilitate communication between local groups of inquiry.

At the same time, the spidercharts can easily be applied as reflection tool outside of PLATINUM, by any (group of) teacher(s) that wishes to engage in developing their teaching further in the direction of enhanced inquiry-orientation. Also, the charts are not strictly research-oriented, in the sense that they are intended to be integrated in or followed by a developmental research project. Their usage can pursue aims not related to research in a narrow sense.

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CHAPTER 4

Students With Identified Needs and IBME

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4.1. Introduction

In this chapter we will focus on students with identified needs and try to analyse the differences they may display when participating in inquiry-based mathematics education. Students in universities are diverse by their characteristics and it is difficult to comply with all their needs individually and on demand. Therefore, we will follow the social model of disability and its main idea emphasising the responsibility of educational institutions as the key factor in creating an inclusive learning environment (see Section 4.2). According to our own perspective as explained in Section 4.3 it is much easier to be prepared in advance to try to satisfy all the possible students' needs, preferences and requirements. In Section 4.6, we will introduce readers to the principles of *Universal Design*, a methodology to follow if one wants to create an inclusive learning environment that reaches the needs of as many learners as possible. "However, it is necessary to know the specific needs of each student, and therefore the categorisation of the needs should rather help to find appropriate support, methods, forms and tools in the learning process" (Čerešňová, 2018, p. 16). Such typology of students with identified needs will be introduced in Section 4.4 and followed by a detailed study of students' differences when they undertake an inquiry within university mathematics courses (Section 4.5).

4.2. Diversity of Students' Characteristics and Needs

It is difficult to comprise the meaning of special (educational) needs in a few sentences. We find a definition of special needs in the Cambridge dictionary[1] as "the particular things needed by or provided to help people who have an illness or condition that makes it difficult for them to do the things that other people do." Another, very similar explanation is offered by the United Nations in the Convention on the Rights of Persons with Disabilities (Article 1): "Persons with disabilities include those who have long-term physical, mental, intellectual or sensory impairments which in interaction with various barriers may hinder their full and effective participation in society on an equal basis with others" Unites Nations (2006). Many definitions focused on educational aims and delivery are based on the same medical model: special educational needs are caused by an individual's health state and affect his/her ability to learn. "In many countries ... students are required to be formally diagnosed and identified, and adjustments are made so that they can fit in to existing approaches to learning and teaching" (Pollak, 2009, p. 270).

On the other hand, based on our experience, there are a lot of students who choose not to disclose their learning differences, behavioural problems, mental health issues,

[1] https://dictionary.cambridge.org/dictionary/english/special-needs

physical disabilities etc. They try to cope with their studies without any special help. Furthermore, a need for help can arise from different circumstances than long-term health conditions and difficulties mentioned above and described in more detail in Section 4.5. For example, students may need to study a course remotely as they were accepted for a study placement or traineeship in a company. Or they are not able to study for some time as they need to deal with the death of a family member. We can add further characteristics such as different learning and cognitive styles or language difficulties of non-native speakers and continue with other examples. Should we consider these needs as special too?

It is clear every student is diverse by his/her characteristics and needs and it is very difficult to address this heterogeneity individually, one by one. Flexibility in all the teaching and learning activities is the key idea to provide this group of diverse people with an opportunity to achieve desired goals.

Taking this diversity into account, we prefer to use the social model of disability whereby difficulties are seen as a product of social circumstance, removing the onus from the individual and giving the responsibility for inclusive learning environments to educational institutions. "This is in contrast to the medical model of disability that concentrates on the impairment as the cause of the disability" (Drew, 2016, p. 30).

According to the social model there are two key terms with a different meaning:

- impairment (biological limitations in a person's functioning) and
- disability (limitations in access and inclusion caused by barriers educational institutions produce).

The social model requires that educational institutions take on responsibility and break down barriers in order to ensure an inclusive learning environment. And based on our experience, individual's disability is in many cases caused by society's response to his/her impairment. With respect to our preference, from now on we will use the term 'identified needs' instead of 'special needs' as we consider all the students' differences in respect of their health condition and/or their learning preferences as nothing 'special' anymore.

4.3. PLATINUM Partners' Perspectives

Partners of the Platinum project come from seven different European countries and many of them experience teaching students with identified needs. During the first year of the project they were asked to describe the communication between teachers, students with identified needs and offices for their support in the partners' universities. In most cases, teachers receive information on students with identified needs regarding their functional specificities and reasonable adjustments that should be offered in order to meet these specificities implied by their learning differences, behavioural problems, mental health issues, physical disabilities etc. Such information is given by the office for students with identified needs and

(1) has to be approved and confirmed by students, including its wording shared with teachers;
(2) may be reduced (at some universities teachers receive only a list of students and specification of extra time for them during written tests);
(3) is offered at the beginning of the semester or before the period of assessment, and occasionally during the semester if it is a new student or his health condition changed rapidly. Some information can be given during semester if the student is newly identified.

Confidentiality plays a part here and it must be agreed what information is shared and with whom.

Teachers develop their courses well in advance, usually not at the beginning of the semester it is delivered to students. When planning the course's curriculum, designing study materials and teaching units, or deciding on methods of assessment, teachers often do not have any information about students and their differences or needs. In many countries, adjustments for students with identified needs are made 'on demand' and with the intention to fit into existing approaches to learning and teaching because the course's design has already been finalised. "This process is better than leaving them to 'sink or swim', but the assumption is that the procedures of higher education are immutable" (Pollak, 2009, p. 270). Taking the diversity of students into account when designing the very processes of learning, assessment, and organisation, is a much more inclusive way and may result in benefits for all learners.

4.4. Typology of Students With Identified Needs

Although we constitute this chapter on the principles of the social model of disability, the perspective of the medical model is taken as an input aspect to describe groups of students whose ability to learn is affected by their health condition or specific learning difference. Because this book is focused on education which could be understood as a continuous interaction between teachers and students, the following categorisation of students with identified needs is proposed based on functional principles with the emphasis on working and communication procedures. The categories are: students with

(1) specific learning differences, for example Dyslexia, Attention Deficit (Hyperactivity) Disorder—AD(H)D, Dyscalculia, Dyspraxia;
(2) Autism spectrum disorder including Asperger Syndrome;
(3) mental health issues, mostly represented by anxiety, depression and low resistance to stress;
(4) physical/mobility disabilities, for example those using a wheelchair and/or crutches (lower limbs impairment), or with disturbed fine motor skills (upper limbs impairment);
(5) visual impairment;
(6) hearing impairment;
(7) disturbed communication skills; and
(8) other chronic conditions, for example diabetes, epilepsy, multiple sclerosis, etc.

When preparing the categorisation we draw on three different typologies originating from the US (Lee, n.d.), United Kingdom (UK Department for Education & Department of Health, 2014), and Czech Republic (MŠMT ČR, 2018). We believe these eight groups of students differ in needs when studying at university. However, it is important to say, many individuals can fit into more than one category with their primary health condition since this can affect their abilities in different ways. As an example, we could consider a student with multiple sclerosis included in the last category. "The initial symptom of Multiple Sclerosis is often blurred or double vision, red-green colour distortion, or even blindness in one eye. Most MS patients experience muscle weakness in their extremities and difficulty with coordination and balance" (NINDS, 2018). These indicators affecting individual's vision and mobility may convince us to include students with multiple sclerosis in the 4th or 5th category. This example can also help us to understand the necessity to identify the full range of an individual's needs, not only those resulting from the primary health condition or specific learning difference. As we described above, the proposed categorisation is based on functional principles and therefore does not include some of the categories you may feel are missing, for example Cerebral Palsy. We didn't create more groups as we assumed students with Cerebral Palsy are mostly affected in their physical abilities and hence belong to the 4th category.

4.5. Inquiry-Based Instruction and Students With Identified Needs

In this section we look at pedagogical processes during inquiry-based instruction. Firstly, we identify which processes a teacher should pay attention to with regard to student differences. Secondly we discuss what this means for specific learning differences of students when they are engaged in inquiry.

4.5.1. Pedagogical Processes During Inquiry-Based Instruction. In the following, we try to look at the pedagogical processes that are present during inquiry-based instruction and how students with identified needs can take an active part with regard to their learning and working, communication, attention, behavioural and emotional, sensory or physical and other differences. Although we have just introduced the categorisation of students with identified needs based on the medical model with regard to functional principles, we try to investigate differences in their engagement to inquiry-based activities according to the social model of disability. Different ways of engagement should be respected and it is the responsibility of teachers to include actively all the learners into educational activities during instruction.

"Inquiry is about asking questions and seeking answers, recognising problems and seeking solutions, exploring and investigating to find out more about what we do that can help us do it better" (Goodchild et al., 2013, p. 396). During inquiry-based activities, students interact in small groups or together with teachers, examine textbooks and other sources of information to see what is already known, use tools to gather, analyse, and interpret data, make observations, propose explanations and predictions and communicate results of their work (Artigue & Blomhøj, 2013). When starting an inquiry-based activity a teacher may announce goals and questions to answer, s/he can also advice on methods and tools to start with in order to get to the 'end' of inquiry, s/he can help with interpretation of results.

Even though it is a student who conducts his/her learning, there are several pedagogical processes a teacher should pay attention to with regard to student differences. Let's identify the processes chronologically by breaking down an inquiry activity into some typical sub-activities:

(1) Start of the inquiry-based activity including instructions given by teachers;
(2) Students' collaborative work in small groups with plenty of discussions, communicating observations, conjectures, uncertainties etc.;
(3) Using tools and software applications to gather, analyse, and interpret data;
(4) Presentation of results;
(5) Final discussion leading to an answer to the questions given or formulated in the beginning of the activity;
(6) Summary of the activity.

In the remainder of this section, we try to capture student differences when they undertake any of the previous processes according to the categories of students with identified needs described above. In Section 4.6 we will try to give advice on how to deal with such student differences.

4.5.2. Students With Specific Learning Differences. Specific learning differences are neurodevelopmental in nature and affect individual's verbal and/or visual abilities. Research shows people with specific learning differences process information in the brain differently and use contextual understanding to the maximum possible extent without attention to details. Such students may display differences in performing everyday tasks such as learning and remembering, information perception, time management, attention span (Pollak, 2009; Barnová et al., 2020).

The most common specific learning differences are listed below.

- *Dyslexia* (sometimes called Reading Disorder) displays in many areas such as reading skills (slower pace of reading, errors in reading, reading comprehension), spelling and writing.[2]
- *Dyspraxia* (sometimes called Developmental Coordination Disorder) affects organisation of movements and body coordination as brain messages are not being properly or fully transmitted.
- *Dyscalculia* can be understood as weaker mathematical abilities displaying in problems with number manipulations and understanding number concepts and relationships, performing mathematical calculations and mathematical conceptualisation.
- *Attention Deficit (Hyperactivity) Disorder*, in short AD(H)D, is indicated by problems with concentration, lower resistance to distraction, deficits in behavioural inhibition and poor regulation of one's activity within the demands of a situation.
- *Auditory Processing Disorder* (APD) manifests itself in problems with processing information one can hear; it is difficult for his/her to recognise and interpret sounds.

In many cases, these specific learning differences come together and may result in mental health issues (anxiety or depression) and isolation.

If we think about pedagogical processes during inquiry, we should pay attention to the opening phase when a teacher introduces participants to a problem and gives instructions. When following the specification of the inquiry-based activity in written form, students with specific learning differences may read text at a slower pace, experience difficulties with the surrounding text or understanding the content (dyslexia) and can be distracted both by internal and external stimuli (ADHD). On the other hand, listening to spoken instructions may also be difficult for them as they are easily distracted by background noise or sudden, loud noises, and having to simultaneously listen and write down notes, recalling details being heard. In order to be focused on the problem students need to understand the purpose of the inquiry.

During the inquiry, students with specific learning differences may have problems with organising their activities. It is difficult for them to read the words in which the problem is embedded, so particularly wordy problems and tasks in which mathematical notation is mixed with text. They also struggle with writing solutions in logical order and aligning mathematical statements. Memory, in particular, is an issue such as remembering formulae, theorems and mathematical terms that are not firmly cemented yet. Students focus on contextual understanding of the problem and therefore are considered to be good problem-solvers with creative and original ideas although writing down their solutions can be problematic with little mathematical flow. If highly motivated by the problem, they are very determined and don't give

[2]In some countries, difficulty in writing is known as Dysgraphia and is caused by lower fine motor skills and lower visual motor coordination.

up. Students, especially those with auditory processing disorder and ADHD, may have problems in following a conversation when listening to more than one speaker or hearing lots of background noise. If participants of inquiry are supposed to use tools or software applications they are not familiar with, it could be difficult for them to manipulate/use these effectively (dyslexia, dyspraxia).

When presenting inquiry results, students with weaker reading/writing skills prefer to describe their findings verbally or diagrammatically. Because of problems with recalling the details of what they heard during the final discussion and difficulties in simultaneously listening and writing, students may forget to make notes about important facts they learnt from others. They may get behind in note taking or need a slower pace. All the students, not only those with specific learning differences, need to know they can make mistakes and come up with incorrect or incomplete answers and comments so they are not afraid to offer their opinion. It is the responsibility of a teacher to conduct the conversation and motivate all the students to share their ideas.

4.5.3. Students With Autism Spectrum Disorder Including Asperger Syndrome. Autism spectrum disorders (including Autism and Asperger syndrome) are neurodevelopmental in nature and affect the 'triad of autistic impairments': communication, social interaction and flexible thinking (i.e., ability to adapt to new situations). These three abilities are limited most frequently, however autism spectrum disorders can display some (not necessarily all) of the following indicators (Burgstahler & Russo-Gleicher, 2015; Pollak, 2009):

- problems with concentration and hyper-focus;
- issues with time management, planning activities, self-organisation;
- differences in adapting to unexpected situations (preference of classroom activities which are predictable/routine, lower ability to react intuitively);
- ineffective participation in conversations (asking too many questions, making comments not related to the topic of discussion, poor nonverbal communication including lack of eye contact, facial expression, and body postures and gestures to regulate social interaction);
- hyper-sensitivity (they can be easily distracted by visual or auditory stimuli or by touch);
- misunderstanding particular forms of language (e.g., sarcasm, jokes, irony, metaphors, humour, abstract concepts).

Students with autism spectrum disorder need to receive structured and accurate instructions, as clearly as possible, in order to focus their activities to the task and not spend too much time trying to understand the meaning of what was written/spoken. It is difficult for them to 'read between the lines,' to understand things not being said explicitly, to make intuitive decisions during the work. To avoid unexpected situations as much as possible, they prefer to have an opportunity to prepare for the session and get to know in advance what will be discussed, which tools/applications they are supposed to use, and so on.

Working in small teams and interacting with other members of the group could be a challenging activity for students with autism spectrum disorder. They need more time to reply as they have to be fully concentrated to understand what is discussed. They may interrupt the discussion by an off-topic question or may be afraid to pose questions at all because of negative reactions from other members of the group. It is difficult for them to lead other members of the team or to anticipate their role and responsibilities. Careful management of the group is needed. On the other hand, if they work independently they are likely to lose track of time and focus on researching details that are not very important for finishing their inquiry. Despite these differences in communication, students with autism spectrum disorder may come with valuable contributions especially when they find the topic of inquiry interesting or even fascinating. In such a case they deeply immerse themselves in research and achieve noticeable outcomes. They can prove to be excellent mathematicians.

When presenting their results or participating in the final discussions, students with autism spectrum disorder need more time to formulate their thoughts as they carefully choose each word to express accurately what they have in mind. They have difficulties in identifying the appropriate moment to enter or link to an existing discussion and sometimes need to be clearly invited to offer their opinion and comments. On the other hand, they can sometimes talk too much as they become 'experts' on the subject in question or they are overloaded by expressing their thoughts and do not have capacity to control their performance. In order to let others speak they need to be tactfully advised to finish so they don't lose motivation to express themselves on another occasion.

4.5.4. Students With Mental Health Issues. "Mental health problems range from the worries we all experience as part of everyday life to serious long-term conditions" (Mental Health Foundation, n.d.). The most common emotional issues students in higher education can experience are stress, anxiety and depression. "More serious and less common mental health difficulties include conditions such as schizophrenia and bipolar disorder, but students with these conditions are likely to arrive at university with their conditions under control" (Pollak, 2009, p. 201). People under medical treatment may take prescribed medication and/or attend one-to-one regular sessions with mental health support staff such as therapists, clinical psychologists, psychiatrists, social workers, etc. Despite that, many students try to keep their feelings hidden and deal with their emotional problems alone without any special support as they may be afraid of other people's reactions. The most common characteristics students with mental health issues display are

- lack of energy, weariness and inhibition (may be caused by medicine, insomnia, anxiety, etc. and may affect regular daily routines);
- withdrawing into oneself, inconstant motivation to work, low self-confidence;
- weakening of cognitive functions (concentration, working memory, information processing and sorting, pace of thinking, etc.); and
- low resistance to stress, restlessness and intrusive thoughts that distract from the immediate task.

"Although certain symptoms are common in specific mental health problems, no two people behave in exactly the same way when they are unwell" (Mental Health Foundation, n.d.).

Students with mental health issues need to feel they are safe and in control of the situation in order to avoid stressful moments, make rational decisions and stay focused. It is very helpful to provide them with clear and widely available information about the availability of resources, to indicate a time schedule of inquiry and to discuss the possible flow of activities leading to achieving desired goals. In such a case they can start the inquiry more easily as they can anticipate what is going on and may organise their activities in a better way. The process of forming groups can be challenging for students with mental health issues as they may be afraid of another member's reaction to their participation in the group.

"Assessed group work can also present difficulties for students who feel that they cope best when they can work on their own at their own pace" (Pollak, 2009, p. 205).

They may feel bounded by the fact that other members of their team rely on results of their work. On the other hand, being a member of a group may compensate for an individual's weaker abilities to organise the process of inquiry. Sharing the workload with others may relieve the pressure and enable students with mental health issues to focus on activities they feel confident in performing. Weaker cognitive functions can display in differences with generalisation and transfer of knowledge to problems experienced for the first time. Students with mental health issues often perform better in specific 'routine' tasks as they are not so overloaded by thoughts of failure and other doubts (Abels, 2014). For the same reason, they need to verify their inquiry is in the right direction. Dr. Vicky Klima (n.d.) from Appalachian State University pointed out another problem many students face: "Oftentimes when people read a problem and they don't know how to do it, they just skip it." In such a moment, students with mental health issues may tend to slow down/stop their work as they are not focused and are overwhelmed by thoughts of failure.

"Most students with mental health issues will have no difficulty with the research and reading that goes into preparing for presentations and seminar discussion, but the performance anxiety that is present for most people in public speaking can present huge difficulties for them" (Pollak, 2009, p. 205). They are worried by other students' reactions and teachers' critical comments in case they come up with incorrect or incomplete answers or remarks. Despite these concerns they need to receive feedback as soon as possible in order to control the situation and avoid uncertainty. Generally, if they need to wait a long time for any kind of support (assessment of their work, answers to their questions, etc.), they tend to occupy themselves with concerns and may start to feel anxious about that.

Mathematics anxiety can be viewed as a serious issue that sits alongside more generic anxiety. It can be defined as: "Feelings of tension, apprehension, or even dread that interferes with the ordinary manipulation of number and the solving of mathematical problems" (Ashcraft & Faust, 1994). Mathematics anxiety can be characterised in three ways:

- physiological (nausea or increased heart rate),
- psychological (confusion or mind chaos), and
- biological (preoccupation with worry or reduced working memory).

Students with mathematics anxiety will not be able to study effectively and may avoid their studies entirely.

4.5.5. Students With Physical/Mobility Disabilities. A physical disability is defined as a limitation on a person's physical functioning, mobility, dexterity or stamina. We can distinguish two major types of physical disabilities:

- Musculoskeletal Disability is caused by muscular or body deformities, diseases or degeneration (loss or deformity of limbs, muscular dystrophy, brittle bone disease);
- Neuro Musculo Disability is defined as an inability to perform controlled movements of affected body parts due to diseases, degeneration or disorder of the nervous system (Cerebral Palsy, Spinal cord injury, Spina bifida, etc.).

If lower limbs are affected, a person's mobility is limited as well as the body coordination and balance. Such people may use wheelchairs to enhance their mobility or walk with the aid of callipers, crutches or walking stick. In the case of upper limbs impairment, individual's fine motor skill can be disturbed and such a student may display differences in writing (taking notes by hand or on keyboard) and manipulating physical objects and equipment (for example printed material and stationery). Cerebral

Palsy may also affect people's speech and vision as well as their ability to concentrate and other cognitive functions. "Some students may experience chronic fatigue and for others there will be extreme fluctuations of energy from day to day" (ADCET, n.d.).

Some students may experience reading/writing differences when working with a print version of instructions—an inability to write using a pen, involuntary head movements which affect the ability to read standard-sized print, reduced ability to manipulate books and other printed material, etc. (ADCET, n.d.). They may prefer, especially those with upper limbs impairment, the digital editable version of instructions in order to write their notes and computations on a computer-based device equipped with supportive tools such as adapted mouse and keyboard, or system for eye tracking and typing. As they may display differences in writing pace due to reduced fine motor skills, the opportunity to make/access an audio or video recording of a lesson is beneficial for them (they may complete their notes later in case they did not manage that during the instruction).

Students may display differences in using software applications to gather, analyse and interpret data and manipulating physical objects or instruments in laboratories. Together with experts in assistive technology[3] they may search for more effective ways to use the applications (for example to access parts of the program more quickly or to set the program interface to be more user-friendly for those using adapted input tools). Limitations in the use of hands (their shaking, damage of finger motor capacities) or difficulties of access for students using wheelchairs and other walking aids may radically affect their possibilities to manipulate instruments in laboratories or to work with standard ICT available in classrooms (Čerešňová, 2018). In general, any non-standard arrangement of furniture and other equipment in the classroom may cause an accessibility issue for students with physical/mobility disabilities.

When working in groups, presenting results of an inquiry or discussing, students mostly do not display any differences.[4] "They may have frequent or unexpected absences from class owing to hospitalisation or changes in their rehabilitation or treatment procedure. When there is limited time to move between venues, students may miss the beginning of a class" (ADCET, n.d.).

4.5.6. Students With Visual Impairment. People with visual impairment have decreased ability to see to such a degree that it cannot be corrected by usual means (e.g., glasses or contact lenses). "The impairment may be the result of a range of conditions and its impact will depend on the type, extent and timing of vision loss" (ADCET, n.d.). There exist a lot of different visual impairment categorisation models but we take the most simple one regarding the possibility of using sight or not.

- Partially sighted students can use the sight to read but need to modify the visual display of information or adapt conditions for reading. In the case of electronic documents, the modification is based on zooming in/out and other optical changes such as colour layout and setting the display of important system or application elements[5] to be more visible. They may also display differences in working with a printed version of a study material/written

[3]Assistive technology is a tool, software or equipment that helps people with identified needs to learn, communicate or generally to perform activities of daily living (e.g., a wheelchair, an application that reads text aloud, a magnifying glass, etc.).

[4]Individual's communication and speech skills can be affected due to Cerebral Palsy or traumatic brain injury, but we will discuss such a difference in the next part of this chapter.

[5]Partially sighted users of software magnifiers can change the size, colour and border of mouse cursor and can ask for highlighting of the line which is actually spoken by synthetic human voice. This feature is also available to users of some software magnifiers such as ZoomText or Magic.

test. Generally, they need to receive enlarged print and set up to their reading preferences concerning the font and line attributes (font size and face, line spacing, etc.). If they have to read printed documents that are not adapted according to their needs they may use magnification devices such as magnifying glasses or video magnifiers to enlarge the area they try to recognise.

- Blind students cannot use their sight to read and use other senses to process information. The most common way is computer-based; therefore, they need digital editable documents to work with. Pieces of text and textual information about visual elements are reproduced verbally by a synthetic human voice or tactually on the refreshable braille display. This text-to-speech and text-to-braille service is part of a program called *screen reader*, the most important assistive technology for blind users of ICT which enables them to read/write electronic documents, e-mails, or textual data offered by operating systems or other programs, play audio or video files, work with common applications, etc. Some technologies work well with mathematical notation, which allows such students to study mathematics. Blind students may sometimes prefer to work with printed braille version of documents. Such a print can be produced by braille embossers and supplemented by tactile graphics replacing diagrammatic information such as schemes, graphs or maps.

People with visual impairment display differences in reading as they work in a linear way and are therefore able to follow a very limited amount of information at a time. However, mathematics and other STEM disciplines are very visual in their nature. When solving a mathematical problem we often receive input data/information. We may observe some of these items "concurrently and put them into suitable positions in space or a plane, which helps us understand the relationship between them better and enables us to work with them more effectively" (Másilko & Pecl, 2013, p. 99). This fact causes them to work at a slower pace with documents and applications if their reader/user needs to follow more than one source of information in parallel (Barnová et al., 2020).

When following the specification of the inquiry-based activity in written form, students with visual impairment need to receive instructions electronically, in editable format and well in advance in order to work with them using assistive technologies on their computers, laptops, tablets, etc.[6] They have limited opportunities to follow visual sources of information on black (white) boards or video projector screens or to understand the nonverbal activities of a teacher or students during the session.

Working in a group may be a challenge for the student with visual impairment and other members his/her team when they need to share results of their inquiry, for example to show others their computations, graphs, schemes. Having no access to what is being discussed, students with visual impairment may consider their active participation to be difficult as they do not have an idea of the problem's context. They display differences in using software applications and manipulating physical objects or instruments in laboratories. They need more time to familiarise themselves with the interface of programs and may need help from experts in assistive technology in order to search for effective ways to use them. They may need more time and individual help to learn how to work with instruments in laboratories. As for students with physical/mobility impairment, any non-standard arrangement of furniture and other

[6]We will summarise the requirements for accessible electronical documents later, in the chapter on Universal Design.

equipment in the classroom may cause an issue for the orientation and mobility of students with visual impairment.

4.5.7. Students With Hearing Impairment. Partial or total inability to hear is a common characteristic of people with hearing impairment (in other words with hearing loss, deaf or hard-of-hearing). They display differences in communication and may rely on spoken language with possible support by speech to text transcribers or the use of sign language and the need for an interpreter when 'talking'/listening to someone else. "Students with a hearing loss may require accommodations and assistive devices to have the best access to education. Accommodations may be as simple as preferential seating or as complex as wireless assistive listening devices in the classroom" (ADCET, n.d.). There is one more important aspect impacting on an individual's literacy and mathematics: the age of a person when s/he lost their hearing. We distinguish two groups of people with hearing impairment with regard to this aspect:

- pre-lingually deafened are those who lost their hearing before the development of spoken language;
- post-lingually deafened lost their hearing after they acquired language skills.

Most of the people with pre-lingual deafness experienced communication barriers during their pre-school age which influenced their language development. "Language deficits and differences have cascading effects in language-related areas of development, such as theory of mind and literacy development" (Lederberg et al., 2013).

When receiving the specification of the inquiry-based activity, many deaf or hard-of-hearing students are able to follow only one source of information at a time. They need a closer visual contact with the faces of speakers and minimised background noise in order to use assistive listening devices properly and/or have optimal conditions for lip-reading. Watching the teacher speaking and pointing at some part of formulae, diagram, or graph simultaneously is another big challenge for students with hearing loss as they have limited opportunity to follow more than one visual source and therefore can easily misunderstand the problem's context. For the same reason, they

- easily miss the beginning of a teacher's speech when they are focused on writing down their notes or making computations; and
- find it difficult to follow the teacher simultaneously speaking and demonstrating the use of some computer application or the manipulation with some instrument in a laboratory.

They may also display differences in working with different formats of information sources such as video/audio recordings or written materials. They may require the instructor to supplement such a video file with sign language interpretation or subtitles/captions. In case of audio files they may need text transcription. The language deficits may cause differences in getting oriented in extensive texts, interpreting information in wordy problems and tasks or producing their own written work. Students with hearing impairment may prefer visual learning strategies, such as organising information into diagrams or emphasising important facts by different colours, which fits with the nature of mathematics and other STEM disciplines.

Interaction based on discussions with members of one team may be a challenge for students who cannot hear the flow and nuances of rapid verbal exchange (ADCET, n.d.). It becomes even more difficult to react within a reasonable time if students with hearing loss follow sign language interpretation or speech to text transcription which is usually a few seconds delayed in comparison with speech flow during discussions.

4.5.8. Students With Disturbed Communication Skills. "Communication disorders can affect how a person receives, sends, processes, and understands concepts. They can have weaker speech and language skills, or impair the ability to hear and understand messages. There are many types of communication disorders" (Giorgi, 2019). There are:

- speech disorders—differences in articulation of speech sounds, fluency and/or voice (cluttering, stuttering, atypical production of speech sounds characterised by substitutions, omissions, additions or distortions, mutism);
- language disorders—differences in comprehension and/or use of spoken, written and/or other symbol systems (affect listening, speaking, reading, writing and doing math calculations);
- hearing disorders (more details in Section 4.5.7);
- central auditory processing disorders—deficits in the information processing of audible signals not attributed to impaired peripheral hearing sensitivity or intellectual impairment (more details in Section 4.5.2) (ASHA, 1993).

Students with disturbed communication skills need to receive an alternative to spoken instructions given by a teacher during the start of an inquiry-based activity. They prefer clear and structured information to the activity setting and its time plan in order to understand properly what they are supposed to do. They may need more time to research information sources.

Differences in communication may cause problems with active participation in discussions during small groups' collaborative work, presentation of results or final summary of the activity. They need more time to express their opinions or comments, to reply to questions and so on. They may be worried by other students' reactions to their verbal performance and such a concern may decrease the quality and fluency of the speech during the presentation of results in front of others. It is important to be patient and give frequent positive feedback in order to reduce their stress with 'public' speaking.

4.5.9. Students With Other Chronic Conditions. The last category of students with other chronic conditions seems to be the broadest one but when summarising their characteristics we can assume their needs as more 'organisational' in terms of their study as a whole without any specific projection to mathematics and other STEM disciplines. It is impossible to give a complete and structured list of all the chronic conditions. Let us name the most common among students with identified needs at universities: diabetes, autoimmune diseases of digestive tract (e.g., Ulcerative Colitis, Crohn's Disease, and Celiac Disease), migraine, tick-borne diseases (e.g., Borreliosis, Encephalitis), chronic fatigue syndrome, epilepsy, multiple sclerosis, and traumatic brain injuries.

Students with chronic conditions may take prescribed medication to regulate indicators of a disease. It can cause lack of energy, weariness, problems with concentration, memory and disturbed daily routines. Regular appointments with medical doctors or lengthy periods of hospitalisation may result in frequent nonattendance at lectures and seminars. Stressful situations like final tests and oral exams can worsen the symptoms of a disease and can cause mental health issues affecting the student's preparation for the assessment and his/her concentration/energy to successfully pass it.

Students with chronic conditions may need more frequent rest breaks during lectures and seminars to revive their energy or take medication. They prefer more flexible scheduling of homework and the delivery of projects. They need to optimise their time schedule of lectures and seminars and set the dates for final exams in order to have enough time for preparation and rest (Barnová et al., 2020; ADCET, n.d.).

4.6. Universal Design of Inquiry-Based Mathematical Education

When we described pedagogical processes during inquiry-based instruction, we ordered them chronologically (Section 4.5.1). Later we tried to explain the differences between students with identified needs when they undertake any of these processes (Sections 4.5.2 to 4.5.9).

In this section, we introduce the readers to the principles of *Universal Design*, a methodology we can follow in order to create an inclusive learning environment reaching the needs of as many learners as possible. We will not offer a detailed explanation of the methodology. Instead, we try to project general principles of Universal Design onto the inquiry-based education of mathematics at universities. We will select/interpret the most important ideas of this methodology and offer them in the form of recommendations relevant to

- inquiry-based education of university mathematics, and
- students with identified needs and their active engagement.

4.6.1. Universal Design Principles. The fundamental design concept is called *Universal Design* and was introduced in 1985 by the architect Ronald L. Mace and his research group at the North Caroline State University as a "design of products and environments to be usable by all people, to the greatest extent possible, without the need for adaptation or specialised design" (Burgstahler, 2020b). The methodology is described by 'Seven principles of Universal Design': Equitable Use, Flexibility in Use, Simple and Intuitive Use, Perceptible Information, Tolerance for Error, Low Physical Effort, Size and Space for Approach and Use. Guidelines and examples of implementation according to the principles are available at (Burgstahler, 2015, 2020a; Čerešňová, 2018).

Universal Design for Learning (UDL) is a framework more closely associated with education. It was developed at the Center for Applied Special Technology located in Wakefield near Boston, led by Anne Meyer and David Rose. It aims to improve and optimise teaching and learning for all people based on scientific insights into how humans learn. UDL guidelines as an implementation of UDL is a structured set of suggestions and recommendations providing a reader with multiple means of

(1) engagement to support the motivation of learners,
(2) representation to address the needs and preferred learning styles of as many students as possible,
(3) action and expression to optimise the learning process and to offer a variety of options how to demonstrate knowledge and skills.

A lot of suggestions are proposed to reduce barriers to the educational process and create a learning environment more inclusive for students with identified needs. See more details at (CAST, 2018). We also refer to Section 6.7 of this book for a more detailed explanation of this framework and examples of the above mentioned UDL principles' implementation based on our own experience.

Accessibility of digital content is another big issue. The World Wide Web Consortium (W3C) is an international community leading the development of standards to ensure the accessibility of web sites and other digital documents including multimedia. The W3C Web Accessibility Initiative (WAI) produced detailed guidelines and other support materials to understand and implement accessibility. A lot of useful and practical information on this topic can be found at *Web Content Accessibility Guidelines*

(WCAG) or *Authoring Tool Accessibility Guidelines* (ATAG) based on the following essential principles:

(1) perceivable information and user interface (e.g., text alternatives for non-text content; captions and other alternatives for multimedia; content can be presented in different ways);
(2) operable user interface and navigation (e.g., users have enough time to read and use the content; users can easily navigate, find content, and determine where they are);
(3) understandable information and user interface (e.g., text is readable and understandable; content appears and operates in predictable ways; Users are helped to avoid and correct mistakes);
(4) robust content and reliable interpretation (e.g., content is compatible with current and future user tools) (W3C, Web Accessibility Initiative, 2021a).

Each of the following sections 4.6.2 to 4.6.8 focuses on one of the pedagogical processes during inquiry-based instruction (described in Section 4.5.1) and includes recommendations followed by additional resources. We offer these resources for two reasons: (1) as a source of the recommendations' interpretation/citation and (2) as a link to more detailed explanation of their implementation. Section 4.6.9 gives examples and general hints that might be useful when Universal Design is not enough and additional individual accommodations are needed for students with identified needs.

We believe some of these recommendations are well-known and respected by teachers of university mathematics. Based on our experience and supported by the general resources on Universal Design (Pollak, 2009; Burgstahler, 2015; Čerešňová, 2018), we can confirm the usefulness of Universal Design not just for students with identified needs, but for all learners.

4.6.2. Teacher's Talks and Presentations. Even though it is a student who conducts his/her learning during inquiry-based instruction, there may be situations a teacher talks in front of learners and shares visual sources of information on black (white) boards or video projector screens. It can happen at the beginning of the inquiry when the teacher gives instructions or during the final summary of the activity. We give several recommendations to facilitate the teacher's inputs:

(1) provide written instructions and other information resources ahead of time in a digital format, if possible in an editable version;
(2) speak clearly, minimise background noise, avoid unexpected moments of surprise or embarrassment (in motion, gestures, speech), keep eye contact with the audience when speaking, use a microphone, repeat comments and questions from others, wait a second to talk about details you refer to visually;
(3) cover all displayed text, describe visuals, comment on actions that can be recognised only by sight;
(4) enable video recording of your lecture, mainly the parts when all the participants communicate with one another;
(5) make sure everything is clear to students, i.e., convince them to create a summary of the goals and discuss the possible flow of activities leading to achieving them, indicate the time plan of the inquiry, give tips on useful sources or applications they can use during inquiry;
(6) enable different means of interaction during the talk (e.g., students may write questions and deliver them to you on paper or via shared document).

Additional resources: (W3C, Web Accessibility Initiative, 2021b)

4.6.3. Information Resources. The only way to meet the Universal Design principles when preparing information resources is to fix the content of any study and teaching material correctly in a digital way. For many reasons, described in Section 4.5, it is important to provide students with information resources well in advance. Such documents should not be based on just one perception 'channel' and should enable users' perception by different modalities (e.g., through vision, hearing or touch). Moreover, each of these 'channels' should be autonomous (i.e., clear for a person following just that 'channel'). Other recommendations are:

(1) create content that can be presented in different ways: use standard formatting tools, offer documents or multimedia in a flexible format to enable users to customise the display of the visual content (text, images, tables, etc.) such as its size, colour layout or contrast with the background, the volume and speed or timing of video/audio recordings and animations, the properties of a printed version;
(2) enhance the visual readability of a document, i.e., use larger character/word/line/paragraph spacing [follow the parameters indicated in the WCAG or (British Dyslexia Association, 2018)], set left alignment of the text (no justification), avoid multiple columns layout, underlining and italics, text in uppercase/capital letters, do not use colours as the only visual means of conveying information, define clearly rows and columns of a table and avoid using merged cells;
(3) help users navigate and find content, i.e., use headings and labels to describe the topic or purpose of the document's components (text, tables, images) and to organise the content, include a table of contents, consider using bulleted or numbered lists rather than continuous prose;
(4) make the content more comprehensible, i.e., provide clear instructions, give students a summary of goals, details of resources and knowledge they will gain, highlight how complex expressions are composed, summarise or visualise structural relations, provide readers with glossaries of abbreviations, new terms and symbols, unusual words (e.g., idioms, jargon);
(5) provide text alternatives for non-text (visual) content (images, video, animations, and so on);[7]
(6) provide captions and other alternatives for multimedia (e.g., captions for video documents or written transcripts for audio files);
(7) use standard tools to input maths symbols and enable access to the source of your mathematical document, usually created by a mark-up language such as LaTeX, MathML, etc.[8]

Additional resources: (W3C, Web Accessibility Initiative, 2021a; CAST, 2018; British Dyslexia Association, 2018; Čerešňová, 2018).

4.6.4. Collaborative Work and Projects in Small Teams. Learning how to communicate and collaborate effectively within small teams is a very important part of the educational process in order to prepare students for such situations that are very common in their future employment. It is easier for some rather than others as

[7] "The need for a description depends very much on the purpose of the visual information, i.e., pictures used for decoration may not need to be described, but pictures that convey meaning may need to be described" (Čerešňová, 2018).

[8] Students with identified needs may read/write mathematical expressions in special applications and need access to the document's source in order to convert it to the desired input format for such assistive technology.

students have different personal characteristics, communication skills and executive abilities (see details in Section 4.5). It is therefore a responsibility of a teacher to set out clear expectations of how the team inquiry should be organised. Moreover, s/he should carefully monitor all the processes of such peer cooperation and be prepared to help students if their collaboration gets stuck. Regarding students with identified needs a teacher should not pass her/his teaching responsibility to other people during the students' group work as such teams may easily behave inappropriately towards individuals with identified needs.

Teamworking may be a short in-class activity or a long-term project-based collaboration. In both cases, students with the help of their teacher should learn to accept and respect differences and abilities of each member of the team. We give several recommendations to facilitate group work so that the participants feel more comfortable during this type of learning activity:

(1) enable flexible rather than fixed grouping, ensure that no one is isolated or disadvantaged (allowing groups to self-select their members is not always the best policy);
(2) do not force participation in a group if it is possible to pass the inquiry activities without interaction and teamworking is not one of the main learning goals;
(3) define clearly goals of the inquiry, offer possible information resources and let the students know the time reserved for the activity;
(4) create expectations for group work and help students to establish their collaboration effectively (e.g., rules explaining your ideas on the team's organisation and all the participants' active participation, giving examples of the team member's roles and responsibilities, and so on);
(5) provide the team with constructive feedback which is frequent, in time, and specific;
(6) support long-term group work, i.e., provide checklists and project planning templates for understanding the problem, division of long-term goals into short-term objectives, setting up prioritisation, scheduling the activities including due dates and indication of who is responsible for the work.

Additional resources: (CAST, 2018; Pollak, 2009).

4.6.5. Software Applications to Gather, Analyse and Interpret Data. Students of mathematics and statistics are supposed to use statistical software, computer algebra systems and other specialised applications in order to manipulate data, perform their visualisations, analyse them and interpret their properties. A teacher should optimise access to such tools. As we described in Section 4.5, students with visual impairment and physical/mobility disabilities need help from experts—so they can use assistive technologies effectively when working with applications they have not used before. Such students may access applications only with keyboard and use keyboard key strokes for any mouse action. Blind users need to access applications via a screen reader while partially sighted students will zoom in on visual content using software magnifiers. This means providing text as a pure text (not as images), enhancing the quality of graphics so it is not distorted when magnified and giving appropriate text labels on all buttons, menus and menu items, icons, sliders, and all other interface objects (Čerešňová, 2018). All of this and much more should be examined by experts in assistive technology who may determine unexpected barriers and/or give advice on how to use specialised applications effectively.

On the other hand, there are different computer-based devices with different operating systems and any student, not only that with identified needs, may ask if s/he can use the tool with her/his computer, laptop, tablet, or mobile phone, and how the application interface looks like in order to be prepared and not solve any technological issue during inquiry.

We give several universal recommendations to facilitate using above mentioned software tools:

(1) let the students know well in advance about specialised software applications they will actively work with;
(2) choose and offer suitable support materials to help students with installation and ask them to familiarise themselves with the interface of the tools;
(3) consult accessibility of applications with experts in assistive technology in order to find the best option for students with identified needs.

Additional resources: (W3C, Web Accessibility Initiative, 2021a).

4.6.6. Physical Environments and Products. Appropriate classroom layout and placement of its elements should support the diversity of students and teachers, different activities and a variety of learning and teaching styles. During inquiry-based activities, when students interact in small groups or together with teachers, they might need to use computer-based devices in order to work with data, examine information resources, prepare outputs of their inquiry, manipulate physical objects and instruments in labs. Flexibility and adaptability of educational environment and its physical components is therefore the key requirement for inquiry-based instruction.

We give several universal recommendations a teacher should take into account when asking for a classroom or lab that best suits the different types of inquiry-based activities s/he plans with the aim of offering an inclusive and comfortable instructional space. Some of this advice is related to appropriate room utilisation and addresses the very important requirement of safety for students with identified needs which also applies to all students:

(1) ask for adaptable and movable furniture that can be arranged in order to enable effective communication of small groups undertaking inquiry;
(2) when arranging physical facilities try to minimise distractions and remember to return all equipment to its original position when finishing your lecture/seminar;
(3) eliminate elements and objects protruding from the walls that cannot be identified by a blind learner when using a white cane;
(4) check if the classroom layout including furniture allows sufficient space for manoeuvring of a person in a wheelchair[9] and convenient access to the equipment such as computers, lab instruments, and so on;
(5) check if you can modify lighting and acoustic conditions and give each student a clear line of sight to the instructor and visual aids used during instruction, and avoid strip lighting or flickering lights;
(6) ensure it is a suitable place for wheelchair users and do not separate them from other students, allow room for personal assistants, sign language interpreters, and speech-to-text transcribers.

Additional resources: (Čerešňová, 2018; Burgstahler, 2020a).

[9]The size of the manoeuvring space with wheelchair is 150 cm in diameter.

4.6.7. Discussion. Inquiry-based teaching and learning is based on asking questions, reflecting on given problems and their solutions. A teacher or students may initiate a discussion for different reasons: to make clear the goals of an inquiry, offer their own perspective and experiences with the inquiry's subject, advise on appropriate information resources and the tools needed for an investigation, reflect on students' or teams' findings, predictions or results, or to summarise activities. Hence discussions play a fundamental role during inquiry-based instruction and its participants should have an equal access to interact. Students with identified needs display differences in communication for different reasons described in Section 4.5 and it is therefore important to follow several recommendations in order to avoid unnecessary barriers and/or individual's concerns for interacting:

(1) establish a welcoming environment, encourage the sharing of multiple perspectives, value each individual's contribution and respond patiently so students will not be afraid to offer their opinion in front of others.
(2) indicate basic behavioural expectations and rules for common discussion so that all students can participate equally regardless of their communication differences and preferences;
(3) offer multiple options for communication, enable students to interact not only verbally, but also in written form (delivering questions and comments on a sheet of paper or online in a shared document or discussion forum) or another way (e.g., hands-on activities);
(4) pause slightly before letting other person speak so students have some time to process and summarise the information received and sign language interpreters or speech-to-text transcribers can finish the transfer of real-time spoken content to people with hearing impairment; and
(5) paraphrase a comment/question from a previous speaker or explain actions that can be recognised only by sight—it can help students with visual or hearing impairment or persons who did not understand all the details that were spoken/visualised.

Additional resources: (ADCET, n.d.; Burgstahler, 2020a; Pollak, 2009).

4.6.8. Students' Presentation of Results and Knowledge. Students often present results of their inquiry in front of their peers, sometimes informally (e.g., through discussions), sometimes more formally. If it is an assessed piece of work, they may have concerns about a lack of ability or opportunity to speak. Not only students with identified needs should know and understand well in advance all the teacher's expectations, presentation dates, what are the opportunities in case of the individual's or the team's failure. Such predictability together with flexibility of deadlines and variable opportunities to demonstrate the knowledge and skills can help to establish a more inclusive educational environment for students with mental health issues, specific learning differences and other identified needs:

(1) enable variability of informal and formal assessed presentations in order to give students opportunities to learn how to present their knowledge;
(2) give students alternatives to public on-site presentations if possible (e.g., posters or other types of written summary shared with others, pre-recorded video presentations, delivery of the presentation to the teacher only or by more than one student in case of a teamwork inquiry results presentation);
(3) ensure that time constraints are minimised and announce the date of the presentation well in advance (long-term projects) or give students/teams enough time to prepare a demonstration of the inquiry results (in-class activities);

(4) provide students with constructive and well-structured feedback which is delivered in time in order to improve their work and its presentation; give comments about the work and not the individual, proposals to change should be balanced by positive reflections. The feedback should be multi-modal such as written, audio or video recorded;
(5) offer corrective opportunities and options to resubmit the work or re-deliver the presentation in case students or teams did not meet your requirements.

Additional resources: (ADCET, n.d.; Burgstahler, 2020a; Pollak, 2009).

4.6.9. Individual Adjustments. "The goal of universal design is to create products and environments that are usable by everyone, regardless of ability or other characteristics, to the greatest extent possible, without the need for adjustments" (Burgstahler, 2015, p. 38). In other words, following the principles described in Section 4.6.1 should result in a more inclusive educational environment respecting students with identified needs as well as all students.. However, Universal Design methods are not always a perfect 'inclusion tool' for all educational situations and a teacher may still need to ask for individual adjustments to ensure the overall accessibility of inquiry-based activities s/he plans to organise. Preparing educational content according to Universal Design principles will be much more effective and help to facilitate the application of individual adjustments if they are needed.

Let's give some examples of such accommodations. STEM subjects are very visual in their nature and this brings challenges for blind and partially sighted students. Ensuring the accessibility of information resources full of mathematical symbols and diagrammatic information requires a collaboration with experts. A teacher and/or author of such study materials can help to make this process easier and faster if s/he follows Universal Design principles and recommendations listed in Section 4.6.3. Accessibility of specialised software such as statistical software or computer algebra systems for users with severe visual impairment or physical/movement disability is another issue which should be discussed with experts in assistive technology.

Another individual adjustment is related to communication. Students with hearing impairment may need a sign language interpreter or speech-to-text transcriber in order to understand the spoken information and also to have the opportunity to interact with other participants in lectures/seminars. Students with visual impairment or upper limbs impairment may need an assistant to help them manipulate physical objects and instruments.

All these examples of accommodations and many more should be provided by organisations established inside or outside the university but the teacher should be prepared to ask for such help. We give several recommendations on planning for individual adjustments for students whose needs are not fully satisfied by Universal Design principles:

(1) know how to get in touch with institutions responsible for individual accommodations, or at least university organisations offering the first contact to students who need help because of their learning differences, behavioural problems, mental health issues, and so on;
(2) collaborate with experts to help them deliver the individual accommodation in time and properly; and
(3) share information about accommodations with students and explain them (at the syllabus) how to ask for its.

Additional resources: (ADCET, n.d.; Burgstahler, 2015, 2020a).

4.7. Conclusion and Discussion

This chapter is focused on students with identified needs and their active participation in inquiry-based mathematics education. With respect to the social model of disability we considered the issue of inclusive higher education as a problem to be primarily solved by universities. Universal Design is the methodological framework which sits well for such a purpose and we briefly explained the main principles of this inclusion tool. Inquiry in mathematics can be understood as a continuous interaction between teachers and students which consists of several pedagogical processes. We investigated how students with identified needs participate in these processes and in which cases they may display differences. Based on our detailed study we prepared a list of recommendations for a teacher to implement in order to guarantee active and equal participation of students with identified needs during inquiry-based activities.

Furthermore, teachers may be in the same boat as students. While students undertake inquiry-based instruction, teachers inquire how to implement some of the Universal Design ideas into their lectures and seminars. Such development is continuous and clearly needs the feedback not only from students but also from experts on inclusive education in order to evaluate the effectiveness of implemented recommendations and plan other modifications of the course. We can certainly consider this scenario as an example of the developmental research described in (Artigue & Blomhøj, 2013; Goodchild et al., 2013) and the three-layer model presented in Chapter 2.

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Part 2

PLATINUM: The Project

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CHAPTER 5

Origins and Implementation of the Project

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When it is obvious that the goals cannot be reached,
don't adjust the goals, adjust the action steps.
Confucius

5.1. Introduction

The main theme of the book about the PLATINUM project is inquiry-based mathematics education and, in general, inquiry as a form of exploration and discovery in the classroom. In this chapter, we inquire into the following important issue: How did it happen that the project became the reality? How did the project evolve and what was its impact on the individuals who worked together on the project? It is not very common to see mathematicians and mathematics educators collaborate productively in a large-scale joint educational project (see Mohn, 2018). Therefore, we would like to provide the reader with an honest insiders' account of what might bring mathematicians and mathematics educators to the collaboration in a joint project about teaching and learning mathematics. Our narrative might surprise those who believe that a project should always start with a concrete idea—for us it did not begin with an idea, our starting point was a heterogeneous group of university teachers and researchers interested in the collaboration in a project on university mathematics teaching. At the outset, let us briefly review the key characteristics of the PLATINUM project.

Nature. PLATINUM is an educational project within the Erasmus+ framework (Erasmus Strategic Partnerships). It is a developmental and innovative project in Mathematics Education with the focus on inquiry-based learning and teaching in higher education.

Purpose. The purpose of the project is to bring together academics with expertise in mathematics and mathematics education who wish to jointly explore the applicability of the main principles of Inquiry-Based Mathematics Education (IBME) to their own practice and share the experience with other academics expanding eventually the community beyond the project.

Goals. In the project proposal, we set the following concrete goals:

a) communicate sound inquiry-based principles of teaching and learning of mathematics;
b) develop our own teaching, and that of our colleagues, through communities of inquiry in local settings;
c) design inquiry-based tasks and mathematical units using digital media for blended learning providing resources for mathematics teaching and learning;

d) design induction workshops and seminars which introduce lecturers to inquiry-based practices in teaching and learning;
e) extend the model of inquiry-based learning to modelling activities which engage students with real problems in industry and society;
f) design assessment procedures and integrate them into the activity itself, to gain a realistic view of what the project is achieving.

Composition of the PLATINUM consortium. The project partners are eight universities in seven European countries:

- University of Agder (UiA) – the coordinator of the project
- University of Amsterdam (UvA)
- Masaryk University (MU)
- Leibniz University Hannover (LUH)
- Loughborough University (LU)
- Complutense University of Madrid (UCM)
- Brno University of Technology (BUT)
- Borys Grinchenko Kyiv University (BGKU)

As the PLATINUM project developed, several academic institutions supporting the ideas of IBME joined the consortium as external partners: The University of Auckland and Auckland University of Technology in New Zealand, Swinburne University of Technology in Australia, and Ternopil Volodymyr Hnatiuk National Pedagogical University in Ukraine. The new partners participated in PLATINUM events and promote inquiry in their daily teaching and research.

To support our narrative, in this chapter we occasionally quote the PLATINUM team members who kindly answered the questionnaire explaining the reasons why they joined the project and their expectations from it as well as the feedback from the surveys distributed during the project implementation period.

5.2. Formation of the Consortium

We start by describing how the initiative group came into being, from the very first contacts between the initiative group members and reflect about the motivation and common interests of individuals who played a key role in the formation of the project team. In mathematical terms, in the beginning the structure of the initiative group resembled an undirected simple graph with several disjoint and loosely connected components. Apparently, the idea of a joint project can be viewed as an attempt to connect these components with a common idea and goals. The overall essence and spirit of the initial stage can be described by the authors as an attempt to explore what do we all have in common and what can we do together. In simple words, we started with a group of enthusiastic and motivated people, rather than with a big unifying idea.

In the beginning not all the members of the group knew each other well enough, and some did not even meet. It was well visible and quite understandable that there were certain differences in perspectives of mathematicians and mathematics educators. On the one hand, it might be difficult for a mathematician to embrace at once the theory-grounded viewpoints of mathematics educators on teaching and learning of mathematics. On the other hand, it might be equally difficult for a mathematics educator to take seriously the intuitive views of professional mathematicians on the use of didactics and pedagogy in mathematics teaching.

In the summer 2020, halfway through the project, we conducted a brief survey collecting the PLATINUM team members' views on their participation in the project asking them to recall retrospectively what motivated them to remain in contact and bring forth the idea of a joint project in mathematics education, the reasons for joining the project preparation process and expectations from the project. The survey comprised of the following three questions:

(1) Why did I decide to join the initiative group which later has become the project consortium?
(2) When did I join the group?
(3) In the beginning, before writing the project proposal has started, what was originally my idea of what would/could be the project about?

Fifteen respondents from six partner institutions and one associate partner institution answered the questionnaire. In responses to the first question concerning motivation to join the initiative group, the following reasons were mentioned most frequently:

- I wanted to become a better teacher/improve my own teaching,
- I liked the idea of collaborating with the colleagues involved in the project,
- I wanted to do in the project something related to what I was already doing or planning to do,
- I wanted to explore principles and applicability of IBME in higher education.

In the responses regarding the preliminary idea of the project, the following issues were indicated:

- inquiry-based mathematics education;
- professional development of mathematics teachers;
- blended learning, ICT tools in mathematics teaching;
- learning approaches and teaching styles based on student's and teacher's personality.

Next, we briefly recall workshops, meetings, and other initiatives from the very early stages when the idea of joining efforts in a project was conceived to the day when the proposal has been awarded the EU funding. The purpose of this part is to give the reader an idea of how much time and effort did it take for the partners to prepare a successful project proposal. This part contains rather detailed information about all events where the future partners in the PLATINUM consortium discussed possibilities of a joint project proposal, including special project-dedicated meetings where much of the proposal has been built up and shaped. We also reflect on both processes of reshaping of the consortium and finalising of the proposal for submission.

For easy referencing, we list all preparatory meetings (PM1-PM9) including abbreviations for the institutions represented in the meetings.

PM1. Kristiansand, Grimstad, and Bergen – May/June 2015: UiA, LU, BUT, UvA, UCM
PM2. Loughborough – September 2015: LU, UiA, UvA, LUH, BUT, MU
PM3. Trondheim – November 2015: UiA, UvA, LU, BUT, MU
PM4. Brno – February 2016: BUT, MU, UiA, UvA, LU, UCM
PM5. Loughborough – September 2016: LU, UiA, BUT, MU
PM6. Prague – December 2016: UiA, BUT
PM7. Kristiansand – February 2017: UiA, LU, UvA, BUT, MU
PM8. Amsterdam – March 2017: UvA, UiA, LU, LUH, UCM, BUT, MU
PM9. Kristiansand – February 2018: UiA, LU, UvA, LUH, BUT, MU, BGKU

We gratefully acknowledge the crucial role played in the formation of PLATINUM consortium by the Centre for Research, Innovation and Coordination of Mathematics Teaching (MatRIC) at the University of Agder and its support of the project proposal from the very beginning. MatRIC organised many events where the discussions took

place and supported financially the process of the formation of the consortium and proposal shaping. MatRIC also contributed significantly to the writing and submission of the proposal (see Section 5.3) and the implementation of the project (see Sections 5.4 and 5.5).

The first contacts between the members of the initiative group were facilitated through their participation in a sequence of events organised by MatRIC starting with PM1. The idea of a joint project has been suggested for the first time by Professor Barbara Jaworski at PM2 where representatives of most of the project partners met on a joint MatRIC-MEC meeting. The first discussion about the joint project took place at PM3, with no specific outcome. The next meeting PM4, largely supported by the project METMAS funded by Norway Grants, brought more concrete ideas related to the call topic "Science Education Outside the Classroom" within the pillar "Science with and for Society" of Horizon 2020. Presentation of this call initiated discussion of *formal*, *non-formal* and *informal education* and possible topics for the joint project. Based on this discussion, the first outline of draft describing basic goals of the intended proposal has been prepared.

However, there was insufficient support for such proposal within the initiative group, perhaps because the ideas were too big and not focused enough. After the meeting in Loughborough (PM5) that did not bring any significant changes in this respect, a survey was launched to map the interests of the initiative group in the list that suggested 18 possible topics. The following three topics attracted most interest:

(1) Look for projects/initiatives/activities promoting mathematics, try to collect data and evaluate the outcomes (10 votes of 12);
(2) Identify, study, and evaluate already existing activities, routines and material elements that are used for mathematics education outside the formal educational systems (8/12);
(3) Consider what are the disadvantaged groups in mathematics education (7/12).

However, the results of the survey were not further discussed. It seemed difficult to fit the group's interests within the Horizon 2020 programme and it became obvious soon that there is too little interest within the initiative group to continue along this way.

The situation changed for better after the Closing conference of EEA and Norway Grants "Czech scholarship programme EEA and Norway Grants (CZ07)" that took place in Prague (PM6). Feedback to the joint presentation of the project PLATSUM by UiA and BUT was very positive, so the coordinator-to-be took the initiative and arranged two meetings (PM7 and PM8) where the future consortium partners were invited to work on the project proposal. One of the most significant outcomes was the change of the focus from Horizon 2020 to Erasmus+ programme. The latter seemed to be a better fit for initiative group's interests. During the meeting PM7 in Kristiansand, the initiative group had the possibility of a helpful online consultation with the Norwegian national agency, SIU (later renamed to DIKU[1] and lately merged with several state agencies into HK-dir,[2] The Norwegian Directorate for Higher Education and Skills).

Another interesting moment was a very useful Skype conversation with Professor Katja Maaß, the founding director of ICSE (International Centre for STEM Education) hosted by the University of Education in Freiburg, Germany, during PM8 in Amsterdam. Professor Maaß successfully coordinated many large-scale European projects fostering innovation in STEM education like PRIMAS, MaSciL, COMPASS,

[1] `https://diku.no/`

[2] `https://hkdir.no/norwegian-directorate-for-higher-education-and-skills`

MaSDiV, and her valuable recommendations were useful for the initiative group. The initiative group transformed into the core of the project consortium sometime between PM7 and PM8. We describe the process of shaping of the first proposal in more detail in Section 5.3.

Writing the proposal was a tough work, partly assisted by the engagement of the experienced external consultants hired by UiA. The first proposal was successfully submitted in March 2017. In the end of June, we received a message from SIU that the proposal scored 71 points from 100 possible and was not granted funding. However, the score was quite high, and we were encouraged to resubmit an improved and updated proposal the following year.

To address properly the critique, we first asked SIU for a more detailed feedback and started working on the improvement of the proposal along the lines indicated in the evaluation and further clarifications received from SIU. Unexpectedly, we had to reshape the consortium because one of the key persons in the project left for Australia. To complete the team in one of the IOs, a new partner—BGKU—was invited. Colleagues from BGKU met other partners at PM9 where the proposal has been modified in line with the detailed feedback from the reviewers. The proposal has been resubmitted and our hard work was finally rewarded: this time we scored a total of 82 points and the project was granted funding!

Together with the overall excitement about the success of the application there was an unpleasant surprise: the approved budget was by 5% lower, and no explanation was provided. We asked DIKU for the clarification and learned that apparently no expenditures in the suggested project budget were cut. Checking carefully the submitted proposal, we discovered that the amount asked for was exactly that allocated to the project, which confirmed what DIKU told us. Unfortunately, in the excitement of the last-minute submission of the proposal we did not check the budget details and the funding for some items of the total value of more than 30,000 EURO got lost! That was a very serious issue. However, with a kind help of the Director of the International Office at UiA, we negotiated additional financial support managing to convince the leadership of the Faculty of Engineering and Science and MatRIC to co-fund the project from their resources. What a happy start for the PLATINUM project!

5.3. Choosing the Project's Format and Focus

From the very beginning, it was very clear that the project would focus on the use of innovative approaches to teaching mathematics and on students' learning and understanding. This would have reflected the experience of the educators from seven European countries who were already engaged in the development of mathematics teaching at the university level and were keenly interested in methodologies aimed at students' more efficient learning of mathematics. One important reason for such focus was that the most frequently reasons for joining the project mentioned by the colleagues was related to their interest in becoming better teachers or in improving own teaching. We all genuinely believe that mathematics teaching is interesting, motivating, creative, adaptive to needs, fostering understanding, applicable to the real-world problems around. Teaching mathematics should be supported by the relevant educational experience and resources which many of us were looking for. As our team members recall, they joined the project led by their individual interests in teaching. Several reasons for joining the project mentioned included interest in inquiry-based approaches to teaching, general interest in university mathematics education, and interest in improving own teaching practices.

The initiative group was aware of the fact that despite many successful attempts to shift mathematics instruction from teacher- to student-centred, “a learning paradigm equalling instruction to content delivery, seems to dominate teaching practices in higher education” (Børte et al., 2020, p. 4). Our views on the changes in traditional teaching practice were well aligned with the idea of active learning introduced by Bonwell and Eison (1991, p. 2) as “anything that involves students in doing things and thinking about the things they are doing.” In case of mathematics teaching, active learning places more emphasis on the development of students’ skills and conceptual understanding; it engages students effectively into higher-order thinking which includes analysis, evaluation, and creation. This is usually achieved through instructional approaches that promote exploration, collaboration, and discussions rather than by passive transfer of new information from instructors (Lee et al., 2018), and this is not a simple task. As one of our team members pointed out “I recognise the complexity of teaching and learning mathematics on the university level from different perspectives. Especially teaching non-mathematician students I see as a big challenge.”

From our own practice, we knew about common barriers to instructional change identified in the report of Bonwell and Eison (1991)—the powerful influence of educational tradition, self-perceptions and self-definition of roles, the discomfort and anxiety that change creates, and limited incentives for faculty to change. A recent review paper (Børte et al., 2020) classifies the main difficulties with the engagement of students into active learning as related to (i) leadership and organisation, (ii) teaching competence, (iii) individual training and professional development needs, and (iv) the availability and use of technology all these factors. Those difficulties were experienced by many of us in different combinations to a larger or lesser extent. As a matter of fact, several chapters in Part 3 of this book provide examples of various challenges and difficulties that arose at partner institutions during the implementation of the project.

Partner universities in PLATINUM have different historical and cultural traditions, institutional structures, educational routines, as well as different academic priorities. Educational research suggests several perspectives on how teaching and research at university should be combined distinguishing four main categories: research-led, research-oriented, research-based and research-informed teaching (Griffiths, 2004). Our understanding of research in this chapter is as “an investigative method that teachers can use to make teaching the object of systematic inquiry to improve own teaching practice or teachers can use it in their teaching to promote student active learning” (Børte et al., 2020, p. 3). This is how the PLATINUM project was implemented, firmly based in recent educational research with the scope of improving students’ learning, own teaching, and contributing to the advancement of knowledge. For the preparation of the application and successful implementation of the project in the case the external funding were provided, it was necessary, on the one hand, to optimally use the experience of the partners and, on the other hand, to carefully match the interests, ambition and expectation of all individuals and academic institutions they represented. The task was very challenging, and it took almost two years to shape the proposal and finalise the first application for Erasmus+ funding in 2017. Nevertheless, partners-to-be were keenly interested in teaching in general and in gaining new collaborative experience, learning from each other how one can improve own teaching by applying relevant contemporary research-based educational methodologies. This motivation was strong during initial discussions, the whole process of the preparation of the application and during project implementation.

When we came to the choice of the funding framework for the proposal during PM7 (see Section 5.2), it was not a very difficult task. The initial idea to search for research funding within the Horizon 2020 programme was dismissed since none of the program options really suited our interests. Similarly, possible participation in the Marie Skłodowska-Curie actions did not fit well enough what we had in mind. The core project-to-be team quickly figured out that Erasmus+ was the best fit to the team's interests and the most suitable format for the first collaborative attempt to seek for external funding; this took the initiative group further to the discussions on the goals and context for the project.

After we decided to concentrate the efforts on the preparation of the proposal within the Erasmus+ programme, some time was spent on the analysis of most recent documents of the European Commission and European Council on education. One of the communications from the European Commission (2016) highlighted the quality of teaching as a key factor in improving the quality in higher education indicating that

> greater efforts are needed to invest in the pedagogical training of academic staff, which is an area that has traditionally been less valued than research output. In particular, the status and quality of teaching in higher education needs to be improved. This requires progress in developing, recognising, and rewarding high-quality teaching. In addition, the increasing diversity of the student population makes professional teaching ever more urgent. Teachers need to be well prepared and trained for being able to cater for students with diverse backgrounds, expectations and needs.
> (COM(2016) 941 final, p. 6)

One of the first ideas for a possible project supported also by both authors originated from the presentation *Students in academia are different. Who do we talk to?* by Solve Sæbø, professor of statistics at the Norwegian University of Life Sciences (NMBU), at the MatRIC modelling colloquium in May 2015. The results of an empirical study at NMBU reported at the MatRIC event[3] demonstrated that traditional organisation of teaching at universities with lecturing of large classes in big lecture halls with limited possibilities for collaboration and discussions, rigid curriculum structure and not very flexible assessment methods tend to disfavour students capable of 'out-of-the-box' thinking. Professor Sæbø conjectured that if similar tendencies were confirmed at universities worldwide, academia and society might have benefited much more from the implementation of the adaptive teaching methodologies.

The partners-to-be that attended this talk agreed with the speaker's message that "one size doesn't fit all students." Once again, we focused attention on traditionally disadvantaged categories of learners, this time, on students with various forms of learning difficulties, ranging from mathematical anxiety to visual and hearing impairment. The expectations included students' increased interest in mathematical disciplines, improved performance in mathematics and its applications, and testing of innovative learning techniques and digital solutions to teaching. The idea of a project about learning approaches and teaching styles that would take personality of a student into account looked attractive. It was based on an ongoing joint project between the Brno University of Technology and the University of Agder but did not receive sufficient support. However, the discussion of possible exploration of connections between personality type and mathematics learning left an important trace in the history of the current project which inherited a nice abbreviation PLATINUM. We kept it by rephrasing the working title *Personality, Learning And Teaching IN Undergraduate Mathematics* to *Partnership for Learning And Teaching IN University Mathematics.*

With the unchanged focus on students' more active engagement into mathematics learning and innovative approaches to teaching, the initiative group decided to follow the suggestion of Professor Jaworski to root our developmental activity in a three-layer

[3] `https://bit.ly/38m5P1P`

model of inquiry-based learning and practice with associated pedagogy and didactics and, wherever possible, with the educational research into key questions. The three-layer model had grown from previous developmental research projects at UiA and Loughborough University and was already reported and discussed in research literature (see Chapter 2 for more details) as well as at several preparatory meetings, see Section 5.2. Professor Jaworski was motivated to lead a developmental project in which all participants would learn more about inquiry-based teaching and learning in university mathematics by engaging practically in our own institutions, sharing our experiences communally and learning from each other.

The partners agreed that exploring how the three-layer model of inquiry activity can be implemented across different institutions in different countries would fit our developmental needs and experiences. However, it was necessary not only to discuss in depth how well the new idea meets the interests and needs of each partner institution but also how the project should be shaped in line with Erasmus+ criteria and expectations. All partners agreed to share the inquiry-based methodology and contribute to the project development by bringing own expertise and experience in tackling the needs and issues relevant to the project, as well as to work together exploring the differences in the ways of conceptualising learning and teaching from didactic and pedagogic perspectives. We expected that cultural and intellectual diversity of the consortium will provide a rich basis on which project's intellectual outputs will be built. It was very important that Professor Jaworski, the academic leader of the PLATINUM project, and other partners in the initiative group had substantial previous experience with mathematics education research and developmental projects.

From the very beginning, a distinctive feature of the PLATINUM community was its heterogeneity; this, on the one hand, created certain collaboration difficulties rooted in different educational, cultural, epistemological, and professional backgrounds but, on the other hand, it was setting a very rich and promising background for a large scale longitudinal case study. In fact, one of the team members described the team as "a heterogeneous group of people with extensive teaching experience, ready to design material and ponder their teaching techniques." During different stages of preparation and implementation, the project brought together more and less experienced mathematicians and mathematics educators whose primary intention was to reflect on own teaching and improve it on the basis of (assisted) self-analysis and discussions with colleagues. In the very beginning several colleagues were curious what IBME is and how it can be implemented in tertiary education and wanted to become more experienced in inquiry teaching approach. Nevertheless, the idea of inquiry was not completely new to several team members. One of them, for instance, related ideas of IBME to his previous experience matching the inquiry-based teaching and learning with the concept 'badatelsky orientované učení' used by Czech teachers at basic and secondary schools. Ukrainian and Dutch teams were familiar with other approaches to inquiry in science education and acquired relevant experience.

Although from the very beginning the partners were agreeing on many grounding principles, some discussions were emotional and rather heated. The first episode when the partners had quite divergent views on how the project should be shaped regards to the role of technology in the project and related terminology issues. Our Dutch colleagues had extensive experience with the use of technology in teaching related to the need to teach large groups of students and use computer tools for learning mathematics in order to enable support of individual students in large groups. Colleagues at UvA looked forward to the development of an introductory mathematics course for students in biomedical sciences following a blended learning approach, with the online environment SOWISO[4] for the online learning and teaching. This would have fitted the blended learning policy of the University of Amsterdam and the course development work at the Faculty of Science using an environment or online learning, teaching, and assessing of mathematics in large groups of students.

Although some mathematics educators clearly recognised the important role of technology in the project, they did not initially agree on the use of an unfamiliar term blended learning in the project description. Known already from the late 1990s, the term blended learning "could still be characterised as pre-paradigmatic, searching for generally acknowledged definitions and ways of conducting research and practice" (Hrastinski, 2019, p. 564) and thus might not have been accepted equally well by all team members, especially if blended learning does not become one of the corner stones for the institutional educational policy as at UvA. On the other hand, the 'three-layer model of inquiry-based mathematics education' and the concept of 'critical alignment' were new to most of the Dutch colleagues and opened a new dimension in the conceptual understanding of mathematics and a new perspective on learning communities. Considering that "the breadth of conceptualisations means that essentially all types of education that include some aspect of face-to-face learning and online learning are being described as blended learning in the literature" (Hrastinski, 2019, p. 568), many PLATINUM partners already used it in a variety of formats, perhaps without acknowledging, and this was one of the reasons that the use of the terminology in the project description was acceptable for all in the end. However, as fairly noticed by Hrastinski (ibid), "since blended learning seems to mean many things, it is important that researchers and practitioners carefully describe what blended learning means to them" and use "a more specific, descriptive term as a complement or replacement to blended learning when appropriate." This problem with the terminological ambiguity and epistemological differences was eventually overcome through several rounds of discussions and willingness of all parties to compromise in the end. The agreement was eventually reached and the tensions in the team eased as the parties recognised similarities with the ideas they were familiar with and became prepared to fully embrace the three-layer model, clearly acknowledging an important role of computers in teaching and learning in the second version of the project proposal.

Another issue where the partners' views significantly split was related to the suggestion of the Ukrainian team to develop stronger connections with business and industry through the development of the *university business incubators* (UBIs) similar to the one at the Borys Grinchenko Kyiv University (BGKU). The concept of UBI is widely recognised as a strategy for promoting the development of new research/technology-based companies and is believed to provide a nurturing environment for new business start-ups (Mian, 1996). UBIs positively impact the economy bringing "the non-financial resources of the university's infrastructure extensively in the form of tangible (research equipment and premises), and intangible assets (faculty time, scientific knowledge and contacts)" (Barbero et al., 2012, p. 893). Recently, BGKU decided to make their UBI a part of the educational process for several study programs including mathematics. Colleagues at BGKU wanted to connect the project participants with business structures through the collaboration on the analysis of market needs, design of authentic modelling tasks, organisation of student training at the incubator and formation of entrepreneurial competences. Unfortunately, the idea of developing UBIs at other partner universities during the project did not receive much support as not fitting well the goals of the project. Therefore, it remained in the project on

[4]https://sowiso.nl/en/

a smaller scale as a local initiative of the BGKU within the mathematical modelling work package (see Chapter 8).

Looking back to the discussions, we recognise that it was not easy to choose the methodology, describe expected intellectual outputs, and define the core activities in the project. Nevertheless, temporary difficulties were overcome, mainly due to partners' clear initial intention to collaborate on improving own teaching and, respectively, students' learning of mathematics. Although during the development stage there were many different possibilities, very intense and emotional discussions did not affect much the foundational ideas. We are also pleased to acknowledge that despite occasional divergence of opinions and approaches, management troubles and misunderstandings, the members of the PLATINUM consortium were not disappointed during the preparatory work with the application and later on during the implementation of the project. A member of the UvA team says:

> I expected that much of the work in PLATINUM case studies at UvA would be about course design, implementation and evaluation, and about the use of ICT in these processes. The project would provide me and my colleagues with extra food for thought, learning from what partners think and do. This also happened in reality.

Remarkably, every partner institution and every individual were primarily engaged in the activities of their choice and contributed to intellectual outputs sharing own expertise and experience. This led to a wide panorama of interesting deliverables.

Prior to submission, the proposal was shaped to better align with the concern of The European Commission regarding the quality of higher education by improving the quality of teaching in university mathematics. The three priorities in the project were chosen to match the project idea and the experience accumulated by the partners with the current educational needs of the EU.

Our intentions fitted well the initiative of the European Commission (2016) to "improve the interaction between research and teaching ensuring that teaching is based on state-of-the art knowledge and adequately recognised and that graduates have strong analytical and problem-solving skills" (COM(2016) 941 final, p. 7) and the first project's priority "Ensuring education and research are mutually reinforcing, and strengthening the role of institutions in their local and regional environments." The project as a whole was conceived with the goal to promote the inquiry-based approach to teaching and learning of mathematics in higher education through the design and use of inquiry-based tasks. We planned to improve mathematics teaching at partner institutions through a developmental process based on the state-of-the-art research in mathematics education. By informing initially mathematics lecturers about possibilities and challenges of inquiry in teaching and providing the relevant training, we wanted to involve lecturers in local communities into the design of inquiry-oriented tasks and their use in the class with the consequent self-reflections on teaching supported by the feedback from local communities of inquiry at other universities. Our ambition was that the spread of knowledge about inquiry-based learning approach and our good practice would positively influence the education at our universities. We were also willing to contribute to further improvement of professional competence of a wider community of university lecturers and students across Europe.

Matching the second priority "Developing, implementing and testing the effectiveness of approaches to promote creativity, entrepreneurial thinking and skills," our plan was to collect during the project time the evidence from our own practice that the inquiry-based approach to teaching stimulates students' creativity by strengthening their interest and curiosity in learning mathematics and fostering deeper understanding of the importance of mathematics in real-life applications. The reflection of this ambition is the PLATINUM volume you read right now. With the central role played by mathematics within STEM disciplines, the project also aimed to contribute to the development of entrepreneurial skills through interdisciplinary education approaches including design of authentic mathematical modelling tasks coming from industry, business, public sector.

We planned to contribute to the third priority "Supporting the use of digital technologies to improve pedagogies and assessment methods" by combining partners' expertise in inquiry oriented approaches to mathematics teaching and blended learning including the use of digital tools for active learning and professional development of STEM lecturers in the integration of novel technological and pedagogical tools for effective course design. The empirical research clearly points towards better integration of new technologies in the classroom and corresponding adjustment of teaching practices for a more efficient teaching. Regretfully, the current situation in the area was that "New technologies are adapted to the tradition instead of challenging the tradition. Hence, the *what* being taught in school and higher education is changing, but the *how* of teaching—the pedagogy—is remarkably stable" (Børte et al., 2020, p. 3, emphasis in original). Different aspects of the use of digital technology to improve the pedagogy and assessment in mathematics teaching are discussed throughout this volume.

5.4. PLATINUM Intellectual Outputs and Project Management

For convenient referencing, we list all Intellectual Outputs of the project. Abbreviations for the institutions are included, leading organisation being the first. We refer to relevant intellectual outputs as IO1-IO6.

- IO1. Framework for inquiry communities in mathematics teaching and learning through a reflexive developmental methodology: LU, LUH, UCM, UvA.
- IO2. Learning about teaching: Case studies for dissemination of community of inquiry developmental practices: LUH, UiA, UvA, MU, LU, UCM, BUT, BGKU.
- IO3. Teaching units for student inquiry: UvA, UiA, MU, LUH, LU, UCM, BUT, BGKU.
- IO4. Methods and materials for professional development of lecturers: UCM, UiA, LUH, LU, UvA.
- IO5. Mathematical modelling teaching resources from real-world problems in business, industry and society: BGKU, UiA, BUT.
- IO6. Guidelines and recommendations for quality assessment in Inquiry Based Learning environment: UCM, UiA, UvA, MU, LUH, LU, BGKU.

Already during the final stage of the preparation of the application, partners shared the inquiry-based aims of the project as the conceptual foundation for the project and agreed to bring relevant experience and know-how in tackling various related issues in mathematics and mathematics education. We have chosen the inquiry-based approach to mathematics teaching and learning as the main theme of our project because of its growing record of encouraging developments which foster students' conceptual understanding of mathematics and emphasise the applicable nature of mathematics that is learned. Our goal in the project was to explore both didactic and pedagogic processes and practices blending various methods and resources for the achievement of progress both in the teaching of and in the learning of mathematics. It was expected that the practices involved in the three layers of the model will vary according to the strengths and needs of the participating groups as well as the project as a whole. The main organisational idea was to share the responsibilities for IOs with most experienced members of the consortium taking the lead and other partners contributing to IOs in various locations. From the very beginning it was clear that all contributions to IOs were intended for partners' use across the project and beyond.

In view of the three chosen priorities and the preferences of the PLATINUM team members, the consortium partners distributed the responsibilities to achieve maximal efficiency and optimise the efforts. For example, in IO1, partner universities worked under the leadership of LU on defining the concepts of the project and inducting members of the project team and their colleagues. This is described in more detail in Chapter 2.

In IO2, LUH led the process of developing inquiry communities of practitioners in seven countries and monitored their developments in practice (see Chapter 3). All partners developed communities of inquiry among lecturers to explore their own practices and identify important issues for future practice. The insider's accounts of the evolution and activities of local communities of inquiry can be found in Part 3 of this book.

Important practical contributions of the PLATINUM project are made in IO3, by UvA. For this output, partners designed inquiry-based mathematical units and tasks that were piloted and tested within the project and prepared for dissemination and use beyond the project (for more details, we refer to Chapter 6). The design of many tasks assumes possibilities of using digital resources for their solution and/or assessment; partners explored how these tasks contribute to the learning and teaching of mathematics and discussed the ways of making tasks available to other partners in the seven countries and beyond. All consortium members were involved in the design of tasks and mathematical units related to specific programmes and needs. In addition, colleagues from the Teiresias Centre at MU focused specifically on the mathematics education of students with special needs or disadvantages, including students with visual and hearing disparity. They shared expertise in particular areas with the Eureka Centre at LU which offered complementing experience relating to certain special needs such as dyslexia and dyscalculia, see Chapter 4. MU and LU verify the accessibility of inquiry-based tasks designed in IO3 and IO5 and provide advice on the adaptation and adjustment of tasks for students with special needs. Furthermore, to assist the listeners with hearing disparity, speech-to-text captions to all presentations were provided by the specialists from the Teiresias Centre during the three PLATINUM webinars organised in spring 2021.

Parallel to the development of local communities of inquiry, UCM, LUH, and UiA were developing in IO4 the "support package" for setting up the professional development programme for lecturers/practitioners interested in establishing their own inquiry-based reflective practice. The three local communities started by piloting in their locations necessary steps to be made both by the consortium members in PLATINUM and by the eventual followers beyond the project. The work included the collection of data from local communities, consequent reflection on data and analysis of developing practices within collaborative groups. The main purpose was to identify important issues in learning and teaching and advance practical advice addressing these issues, see Chapter 7.

In IO5, the PLATINUM consortium pays special attention to real life applications of mathematics. For this reason, in addition to inquiry-based teaching units in IO3, partners at UiA, BGKU and BUT in collaboration with the Swinburne University in Australia, worked in a dedicated package IO5 on the development of tasks and activities which focus on mathematical modelling, including realistic tasks linked to local communities, businesses, and industry, see Chapter 8.

UCM, LUH, UiA, UvA, LU, and MU worked in IO6 on the provision of guidelines and recommendations for monitoring processes of implementing and advancing IBL and inquiry-oriented learning environments. The purpose of this IO is to ensure the quality of teaching units and materials for professional development of mathematics lecturers developed in the project. We refer to Chapter 9 for more details.

The organisation of the productive work on the deliverables in all six intellectual outputs required efficient management. The general project management and daily management routines were organised as follows. The project as a whole is coordinated by the UiA where the first author, the Project Coordinator (PC) is employed. The second author is employed at BUT and acts a Deputy Project Coordinator (DPC) assisting PC with all managerial issues. The Academic Coordinator (AC) Professor Barbara Jaworski is employed at LU and partly at UiA, which facilitates the management of the intellectual outputs of the project. Each of the partner universities has the Community Leader that coordinates the local activities and takes part in regular biweekly Project Management Leadership Meetings prepared and led by the PC, DPC, and AC. Each of the six IOs has at least one leader from the partner organisation responsible for the deliverable but may also have one or two Deputy Leaders if needed. For some partner institutions, more experienced individuals combine several leadership positions; this, on the one hand, puts more pressure on selected team members, but, on the other hand, facilitates daily management routines. The communication links within the PLATINUM consortium work reasonably well and community members receive the news in a timely manner by email and directly from Community Leaders.

Daily management routines in the project rely to a large extent on regular biweekly online management/leadership meetings which contribute greatly to the successful development of the project. Webex has been used as a communication platform for online meetings during the first two years of the project; we smoothly switched to Zoom afterwards. These meetings serve as a platform for discussion of all important issues related to the functioning of the PLATINUM consortium, organisation of events and continuous work towards the delivery of Intellectual Outputs. However, the most important decisions within the consortium are usually made at the Transnational Project Management Meetings (TPMMs) which, whenever possible, are combined with Learning/Teaching/Training Activities (LTTAs) and Multiplier Events (MEs). By the moment this chapter was written, we had the total of four regular face-to-face meetings running until January 2020. Due to the travel restrictions caused by the Covid-19 pandemics, the TPMM scheduled for May 2020 has been replaced with the first virtual two-days meeting on Zoom; all consecutive events are being currently rescheduled to the online format until the epidemic situation across Europe improves.

After each TPMM, participants were asked to answer a detailed questionnaire about the overall organisation of the meeting and about every particular session or activity. Usually questions were paired, one of a multiple-choice type and another with the open answer. In most cases the first question asked to evaluate the session/activity choosing one of the possible answers. For instance, "To what extent did the session succeed to meet the goals? Very well – Well – Satisfactory – Poorly - Very poorly - Not applicable (in case the respondent did not attend a particular session)." This information provided quick and detailed statistical evaluation of the event with a very high percentage for the first two answers. The second question sounded like "Please explain your answer to the above question and suggest possible improvements (optional)" and served well as a reliable source of criticism and suggestions. The feedback was initially critically assessed by the PC and DPC and then discussed in the next Project Management Leadership Meeting. In addition to live community discussion in the TPMMs, reflections on the meetings in the questionnaires and feedback were very important for the improvement of the project management. Critical feedback from the participants was especially important at the initial stages of the project; it helped to organise the meetings and associated events more efficiently, especially with respect to the content and time management. In the beginning, many requests were addressing the organisation of discussions, in particular, the need for efficient time management. Participant A: “The small groups were a very good idea. The discussions were inspiring, and we have done a lot of progress. The recap was a repeating of the discussion. This could be more effective.” Participant B: “I think we need more time for discussions in future. It happened several times that we started discussing something during the presentation and in the end the next presentation had to be slightly shorter.” Participant C: “We managed to answer some of the questions, but not all. Either more time or less questions would fit better. Anyway, the ideas discussed were important for PLATINUM.” As a rule, most participants were satisfied with the organisation of the meetings, yet the project coordinators were very attentive to constructive criticism. The reduction of the number of critical comments about the organisation of forthcoming events was clearly signalling the improvements in the project management and the organisation of the events.

The core of the PLATINUM community was composed of mathematicians and mathematics educators many of whom already knew each other, including the colleagues who had previous experience of collaboration in mathematics education research and educational projects. This also had a positive effect through their participation in preparation of the project (see Sections 5.2 and 5.3). However, not all PLATINUM team members and especially new colleagues joining the community during the project were well informed about inquiry based mathematics education. In line with the project proposal, the first three TPMMs were combined with LTTAs where the members of a larger PLATINUM community explored foundations of Inquiry-Based Mathematics Education (IBME); these events were very helpful for the formation of the PLATINUM Community of Inquiry. The feedback from the participants was very positive, as can be seen from the answers to the questionnaire distributed after the third LTTA/TPMM in Brno (see Section 5.5, list of meetings). Participant A: “Our activity in designing tasks and trying them out with students is one manifestation of inquiry in teaching. We need to reflect – to ask: What have I learned from this? What might I do differently?” Participant B: “This gave me the very nice first insight. In further discussions with our national group of inquiry we raised some questions and received answers so the presentation met the goals for sure.” Participant C: “Together with an example presented by Paola Iannone it gave me a good insight how the community of inquiry can be created in our university and mainly it doesn’t need to be too big in the beginning.”

Although the overall project management ran quite smoothly, we faced now and then certain difficulties. They were mostly related to setting the dates for events and online meetings, unfortunate delays with the responses to important letters from AC, PC and DPC, delays with response to the surveys conducted after the joint events, and with the delivery of requested documents and materials, including the preparation of the chapters for this book. Since partners at different universities had different academic calendars and teaching schedules, it has been always difficult to choose the day and time for biweekly project management meetings satisfying the expectations of 8-10 team members. Even minor details were important, as for instance, the time choice since the PLATINUM consortium works in three different time zones in Europe. Unevenly paced academic calendars at different institutions disturbed a bit the continuity of the workflow for the intellectual outputs because the teaching duties and exam periods were spreading over several months. Consortium partners experienced in the very beginning certain difficulties with the understanding of important budget and financial rules but there were no serious problems with the financial discipline. Orchestrating the work of eight local teams in seven European countries in three time zones was quite challenging, but we are very pleased to acknowledge that this turned out to be a realistic and doable task. It was especially useful to use a 'duet PC-DPC' rather than PC alone, as suggested by Erasmus+ regulations, for the strategic management and coordination of the entire project since the volume of the work to organise and monitor was very large.

With regard to the fulfilment of the work promised in the project proposal, one of the main challenges for the consortium was related to the organisation of the process of collection and submission of the project deliverables. Different partner institutions had different regulations with regard to what data should be collected and where could it be stored and, in particular, one of concerns was about the treatment of personal data that should be protected by the GDPR law. The second challenge regards the project's impact and creation of a global Community of Inquiry with the members outside the partnership. The first multiplier event in Madrid was planned in June 2020 and it became clear during the spring that it won't be realistic to have it because of Corona pandemic. Since then, the dissemination of project results required new formats we were not prepared for in the beginning.

The sequence of virtual events planned for 2021 was aimed at resolving this issue to some extent. By summer 2021, PLATINUM organised three webinars. The first Webinar "Inquiry in University Mathematics Teaching and Learning" took place on March 9, 2021. Founding MatRIC director Professor Simon Goodchild skilfully led the online event whose main goal was to present the PLATINUM team and the main intellectual outputs of the project. An invited keynote address on important aspects of Inquiry-Based Mathematics Education was delivered by the well-known mathematics education researcher Professor Michèle Artigue (Paris Diderot University). The academic leader of the project, Professor Barbara Jaworski (Loughborough University) introduced the inquiry basis of PLATINUM, and the team members outlined all six project's intellectual outputs. In the second part of the webinar five examples of inquiry-based mathematical tasks were presented. Several questions and answers sessions were organised during the event and participants were invited to ask questions and also leave feedback using the PADLET tool.[5] The event concluded with a panel discussion moderated by the MatRIC director Professor Thomas Gjesteland. For the second webinar "Bringing Inquiry into One's Mathematics Classroom" on May 7, 2021, we invited Professor John Mason (Open University, UK) whose important work on inquiry in mathematics is widely acknowledged. The PLATINUM community members Barbara Jaworski and André Heck started the webinar with the Introduction to inquiry-based mathematics tasks and teaching units. Then Professor Mason invited participants to engage in a sequence of challenging tasks designed to stimulate discussion about getting students to explore mathematical ideas. As in the first webinar, former and current MatRIC directors led the webinar and the panel discussion in the end. When this chapter is being written, the PLATINUM team is preparing for the third two-day webinar "Creating Communities of Inquiry: Focus on

[5] `https://padlet.com/`

Students with Special Needs and On Mathematical Modelling" organised on June 9-10, 2021. The webinar focuses on three main themes: (1) Communities of inquiry formed by university teachers of mathematics; (2) Design of inquiry tasks suitable for university students with special needs; and (3) Inquiry and mathematical modelling. Although online webinars allow to reach the audience from different continents (we had attendees from European countries, Middle East, North and South America, Australia, and Asia), proper organisation of such events is quite demanding and requires a very good coordination of the efforts of all participants and organisers. The first two webinars were hosted by the partners from Masaryk University in Brno and the third one by the partners from Borys Grinchenko Kyiv University. Expert team from Masaryk University helped us to assist the listeners with hearing disparity by providing speech-to-text captions to all presentations during the webinars. The programmes and recordings of all webinars are available on the PLATINUM website.[6]

Unexpectedly, we experienced certain difficulties with the maintenance of the project website and overall tracking of the promotion of the project. Many consortium members were taking part in various national and international events, both educational and research, and were talking about the project or at least mentioning it in formal presentations and informal discussions. Most of these occasions have found their reflection in project materials. However, it would have helped a lot to have one dedicated team member taking care of the dissemination and promotion through the web page and social media and coordinating relevant partners' efforts.

5.5. PLATINUM Community Meetings and Lessons Learned

We summarise now how the consortium evolved after the proposal has been accepted including the information about all physical meetings organised during the project implementation period.

For the purpose of easy referencing, we list all physical project meetings and events that will be referred to as E1-E7 in the text that follows:

- E1. Kristiansand, September 2018: TPMM followed by LTTA (co-funded by UiA) – 30 participants.
- E2. Madrid, February 2019: TPMM followed by LTTA - 33 participants.
- E3. Brno, June 2019: LTTA followed by TPMM - 28 participants
- E4. Madrid, November 2019: AC visit to local CoI (funded by MatRIC).
- E5. Amsterdam, December 2019: AC visit to local CoI (funded by MatRIC).
- E6. Brno, December 2019: AC visit to local CoIs (funded by Erasmus+).
- E7. Loughborough, January 2020: TPMM (22 participant + 2 guests).

Although the development of the project consortium and the shaping of the project idea resembled at times a roller-coaster ride (see Sections 5.2 and 5.3), there was something that remained constant all the time: comprehensive understanding of our needs and unfailing support from MatRIC and UiA. This unfailing support was made well visible already during the first kick-off event in Kristiansand (E1) where over thirty participants discussed the three-layer model of inquiry-based mathematics education and intellectual outputs (work packages) of the project and continued through the project implementation period. Discussions related to the fundamental model and IOs continued during the meeting E2 in Madrid. At this event a new important component of the workshop has been introduced: small groups discussions. This modification of the meetings format reflected the feedback received to the survey on the first meeting

[6]https://platinum.uia.no/

E1; small groups discussions also featured in the programme of the next event E3 in Brno.

The three workshops were crucial for the shaping of the PLATINUM community‘s idea of IBME. They also fostered development of new ideas and deliverables related to IOs. Despite overall success of the workshops, they did not bring opportunities to reveal the nature of the local CoIs, partially because not all PLATINUM CoI members could be present at the events. This was one of the reasons for the AC‘s visits to local CoIs in the end of 2019 (E4-E6). There were more visits planned for the beginning of 2020, but unfortunately, they could not be realised due to the unexpected COVID-19 outbreak.

Having at that time no idea about what would happen only a few weeks later, most of the PLATINUM CoI members participated in the last physical meeting in Loughborough (E7). Again, a new component has been introduced at this meeting: invited speakers external to the PLATINUM CoI. This new element was evaluated rather positively, but it also brought challenging questions: How could the PLATINUM CoI look like in the future? How could collaboration with the colleagues outside the project consortium be arranged? These questions still remain open, and we have no clear answer or recommendations yet.

There is no doubt that the epidemic outbreak affected the PLATINUM community and had significant impact on the future developments in the project. We continued to meet online with the same regular frequency, but the face-to-face meetings planned in the project proposal could not be arranged. This reduced possibility of direct interaction between the CoI members had unexpected consequences. The need for more interaction within the partnership community resulted in the organisation of a two-day virtual event in May 2020 replacing the physical meeting planned to be held in Hannover during the same period. This became first PLATINUM virtual event arranged on Zoom; it was attended by almost 30 participants from the 8 partner universities. The pandemics also brought changes to the management routine. Unlike previous year, the regular online meetings continued during the summer. This seemed to be a good practice as the community members kept working on the deliverables, in particular on this PLATINUM publication. All of this we see as a confirmation of the success of the project.

Intensive work on the PLATINUM book, which started in spring 2020, also had significant impact on the community. First, there was a need to decide on the editorial board of the book. That provided an opportunity for the inclusion of a few less experienced community members who could learn more about the editorial work from more experienced colleagues. Another positive effect of the PLATINUM book preparation is related to the formation of cross-disciplinary trans-institutional teams that worked on the book chapters featuring project's deliverables and the functioning of the local CoIs. Again, we see this as a big success of the project.

Another activity that influenced the PLATINUM CoI development was the organisation of the PLATINUM webinars. Three webinars have been organised before submission of this chapter:

(1) March 9, 2021: "Inquiry in University Mathematics Teaching and Learning" hosted by MU – 193 registered participants.
(2) May 7, 2021: "Bringing Inquiry into One's Mathematics Classroom" hosted by MU – 90 registered participants.
(3) June 9-10, 2021: "Creating Communities of Inquiry: Focus on Students with Special Needs and on Mathematical Modelling" hosted by BGKU – 111 registered participants.

The success of these events was ensured by the presentations of the PLATINUM consortium members in combination with keynote addresses of invited speakers external to the PLATINUM consortium. Webinars brought another dimension into the PLATINUM CoI: Interaction with the events' participants. They also had a positive effect similar to the PLATINUM book preparation in formation of yet another cross-disciplinary trans-institutional teams that worked on the events' arrangement.

We conclude this section summarising the lessons learned from the development of the PLATINUM project and consortium.

Lesson No. 1. It is important to pay attention to the interests of people involved in the preparation and implementation of the project. When someone suggests what might be done in the project, this may very often indicate explicitly what the person is interested to do or intends to do within the project. The PLATINUM experience shows that many team members prefer to do in the project what they intended to do before the project started (see the results of the survey conducted in 2020 in Section 5.2), at least in the beginning. It is good that the local project activities match as much as possible with the local needs and expectations of the participating universities because this will support the implementation of the project results into the teaching practice and enable the sustainability of the project results at partner institutions.

Lesson No. 2. It is good to have a back-up plan for the case when any of the key persons unexpectedly leaves the consortium for any reason (see the final part of Section 5.2). The same applies on a smaller scale to local communities of inquiry.

Lesson No. 3. It is wise to prepare the proposal on time and double/triple check the budget before submitting (compare with the last paragraph in Section 5.2). Nowadays, Erasmus+ proposals are submitted through an online platform—a website maintained by European Commission. It is recommended to submit the proposal and check the details afterwards. If there is anything to modify, correct or improve, do it and resubmit before the deadline because the last successful submission is the one that will be evaluated.

Lesson No. 4. It is important to keep the management effort reasonably balanced to steer the project efficiently. If the project management is too active and authoritative, much less space is left for the initiative of other consortium members. On the other hand, inactive management leads to reduced activity at the project level and delays in the project implementation might occur (see the final part of Section 5.4). It is reasonable to steer the project with the management effort placed somewhere in the middle between doing almost nothing and nearly everything. The initiative of the team members should be always encouraged and supported with the management monitoring the workflow in the background. The authors adopted the following strategy for taking important managerial decisions: first the opinions of the team members are sought. They are discussed in the meetings or through the email correspondence and then the decisions are made by the project leadership.

Lesson No. 5. There is a persistent risk of delays. Usually, this issue is considered in the Risk Management Plan submitted with the proposal. Even though people tend to neglect this issue ("we plan well and there will be no risks of delay"), it is necessary to take delays into account. There are always people who respond with a certain delay, and there always are people who deliver with a delay. If something is needed from the partners, it is better to ask them well in advance and remind them regularly (compare with the final part of Section 5.4).

Lesson No. 6. If a message is sent to the community and a response is expected from the community members, it is necessary to set a fixed deadline (see the final part of Section 5.4). Statements like "please respond at your earliest convenience" should always be combined with "but no later than ..."

Lesson No. 7. It is good to arrange the system of "emergency communication" (SMS, a messenger app). It can be used in situations when a response is needed urgently, and people do not have immediate access to e-mails or cannot promptly process large documents (compare to the experience with running webinars in Section 5.4).

Lesson No. 8. It is challenging to schedule regular online meetings for large groups of people (see the final part of Section 5.4). The larger is the group, the more difficult it is to find the time slot when everybody is available. On top of all that, it is very likely that regular meetings need to be re-scheduled every semester/term. Polling services like DOODLE are helpful in arranging regular meetings. However, the general advice/solution could be: run the project with a smaller consortium, if possible.

Lesson No. 9. Sometimes, financial rules and regulations of the funding programmes have certain peculiarities. However, we need to deal with it. It is a big advantage to have someone in the team who is skilled in accounting and financial issues and can support other partners when necessary. This helps to maintain the financial discipline (compare with the final part of Section 5.4).

Lesson No. 10. All people have their working habits. This also applies to the work with digital technology. It is good to know in advance how are the project participants used to work with digital tools. Based on that knowledge it is easier to predict or design the means for collecting and storing project documents and deliverables (see the final part of Section 5.4).

Lesson No. 11. It is a big advantage to have someone in the team who is sufficiently knowledgeable and particularly enthusiastic about the topic of the project and also well skilled in social media, communication, and public relations (here we refer to the last paragraph in Section 5.4).

Lesson No. 12. The idea of building the global community is a big thing. The implementation of this idea might be challenging (we refer to the open questions introduced in Section 5.5). Even if there is a plan of how to achieve this goal, circumstances can change, and the plan becomes no longer realistic. Fortunately, the internet opens a new universe of possibilities for reaching out to people and building networks or communities. Organising virtual events might be one of the possibilities, but there are many more: discussion forums, hackathons, etc. It is necessary to use the imagination and creativity, two very important human attributes, then only the sky is the limit!

5.6. Reflection About Collaboration Between Mathematicians and Mathematics Educators

We acknowledge that the "lessons learned" in Section 5.5 are rather general and may be applied to many groups of people and many projects. So, what is special about the collaboration between mathematicians and mathematics educators in the PLATINUM project and what have we learned from it?

The first lesson we learned is that the collaboration between mathematicians and mathematics educators is fruitful, though it takes a lot of time and effort. As described in Sections 5.2 and 5.3, it took us three years to arrive from the very first idea of a kind of collaboration to the start of the project. To date, we have almost three years of collaboration developing together common understanding of IBME principles, designing own teaching and learning resources in line with the main principles of

IBME, reading and writing about local CoIs, disseminating project's ideas, promoting intellectual outputs, and organising online events (see Sections 5.4 and 5.5). However, we acknowledge that the outcomes described in this book are worth all the effort and fully agree with Professor Barbara Jaworski, who stressed after the third PLATINUM webinar that "we need more collaboration between mathematicians and mathematics educators."

Another observation is related to the question "How to bring such a collaboration into life?" PLATINUM experience suggests that the key word for the start and realisation of a collaboration could be the development. Even though this term might be perceived differently by the team members, it encompasses sufficiently well the interests of both mathematics educators and mathematicians. In our experience, developmental educational project blended harmonically the research experience and interests of mathematics educators with the teaching experience of mathematicians and their interest in the improvement of own teaching and students' learning. There might be other possible points of common interest, but this one—development—worked for PLATINUM.

We learned a lot in the project, and we are grateful for this opportunity. It has been a great experience and a pleasure to be part of the PLATINUM Community of Inquiry. We hope very much that the book produced by the team will provide the reader with useful information and many valuable ideas.

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Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project
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CHAPTER 6

Creating Teaching Units for Student Inquiry

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6.1. Introduction

Tasks determine to a large extent how students develop mathematical thinking abilities and become fluent in applying mathematical methods and techniques. As Stein et al. (1996, p. 459) put it: "The mathematical tasks with which students become engaged determine not only what substance they learn but also how they come to think about, develop, use, and make sense of mathematics." They distinguish (p. 466) in mathematical tasks four increasing cognitive demands: (1) memorisation; (2) use of formulas, algorithms, or procedures without attention to concepts, understanding, or meaning; (3) use of formulas, algorithms, or procedures with connection to concepts, contexts, understanding, or meaning; and (4) "doing mathematics," including complex mathematical thinking and reasoning activities such as making and testing conjectures, framing problems, looking for patterns, and so on. Tasks at the highest level of cognitive demand are complex, possibly ill-structured, and require students to make strategic decisions, make connections between concepts and contexts, reason in a mathematical way, and explain their thinking. In other words, student are invited to work as a mathematician or as a professional using mathematics in her/his field.

A good and varied selection of academic tasks is especially important at university level, where the expectation is for students to spend considerable time outside of class studying course material and doing homework. We refer to familiar or routine tasks that aim to increase student fluency with mathematical content and techniques as tasks that promote procedural understanding of mathematics. When procedures are used with connection to concepts, contexts, understanding, or meaning, or when tasks encourage doing mathematics, then we speak of tasks focusing on conceptual understanding of mathematics. In mathematics education at university level, especially in service teaching,[1] student tasks and activities are in practice more often directed towards procedural understanding than to conceptual understanding of mathematics and use of higher-order thinking skills (see, for example, Artigue et al., 2007). The main goal of the PLATINUM project was to explore possibilities to shift the balance in student learning towards conceptual understanding of mathematics. As part of their inquiry at all levels of the three-layer model explained in Chapter 2, PLATINUM partners formed communities of inquiry to develop teaching units for student inquiry, to try them out in their own practice, to evaluate the use of these units, and to document their work. In this chapter we report on this work (Intellectual Output 3 of the project; see Section 2.5), put it in a broader perspective of inquiry

[1]Service teaching of mathematics is an umbrella term for teaching mathematics in higher education outside mathematics programmes, e.g., teaching mathematics to engineers, students in life sciences, etc.

into mathematics education, discuss the possible role of ICT in student inquiry, identify from the developed teaching units some general task features and guiding design principles (including those for students with identified needs), and discuss what a community of inquiry can achieve in practice.

6.2. Frameworks Used in PLATINUM for Designing Student Inquiry

Much research has been done focusing on task design in mathematics education. The ICMI study 22 (Watson & Ohtani, 2015) is a very good source of information. We distinguish three types of frameworks that are used to underpin the task design for student inquiry:

- a grand theoretical frame (e.g., constructivism) or an intermediate-level frame such as Realistic Mathematics Education (RME) (van den Heuvel & van Zanten, 2020), the Theory of Didactical Situations (TDS) (Brousseau, 2002), the Anthropological Theory of the Didactic (ATD) (Bosch et al., 2019), and Commognitive Theory (Sfard, 2008), to name a few;
- a learning cycle instructional model for mathematics and science built upon general notions about how people learn (Bransford et al., 2000; Donovan & Bransford, 2005), such as the 5E-instructional model of Bybee et al. (2006);
- a model of processes and actions in professional practice. Examples of this type of framework are a list of words denoting processes and actions when mathematicians pose and solve problems (Mason, 2008), a categorisation of tasks that encourage concept development, an identification of design principles that make teaching for conceptual understanding more effective (Swan, 2008), a mathematical questions taxonomy (Smith et al., 1996; Pointon & Sangwin, 2003), and a modelling cycle (e.g., Blum & Leiß, 2007; Heck, 2012).

We elaborate on some of these frameworks and discuss how they played a role in the design of teaching units by PLATINUM partners. Most of our attention is on the third type of frameworks because they seem, in our view, closer to the world of university lecturers and more appealing to them.

6.2.1. Use of Intermediate-Level Frames in PLATINUM. Many mathematics education researchers use an intermediate-level frame such as ATD or RME to position their developmental work. For example, partners from the Leibniz University Hannover (LUH) use concepts from ATD, TDS, and Critical Psychology (Holzkamp, 1995, 2013) in their case study presented in Chapter 14 to analyse their teaching and learning practice. Partners from the Borys Grinchenko Kyiv University (BGKU) refer to RME principles when they describe in Section 8.4.1 their view on mathematical modelling. Partners from the University of Amsterdam (UvA) also discuss the attractiveness of RME principles in their case study on teaching Systems Biology (see Chapter 12), in particular the idea to routinely invite students to explain and justify their mathematical thinking, their solution strategies, and actions in open-ended activities. Yet they did not adopt (and explain why not) the RME-based inquiry oriented instructional approach of Rasmussen and collegues (Rasmussen & Kwon, 2007), in which emphasis is on student reinvention of mathematical concepts, on student inscriptions and their role in the development of the mathematics, and on instructor inquiry into student thinking.

Kuster et al. (2018) identified the following most important principles of inquiry-oriented mathematics instruction: (1) generating student ways of reasoning, (2) building on student contributions, (3) developing a shared understanding, and (4) connecting to standard mathematical language and notation. Kuster et al. (2019) developed an instrument for scoring a lesson along seven inquiry-oriented instructional practices. Similarly, spidercharts have been developed as instruments within the PLATINUM project (see Chapter 3) to facilitate project-wide thinking and communication about activities in local communities and to promote reflection on and further elaboration of a common vision on inquiry-based mathematics teaching and learning from the student perspective, the teacher perspective, and the community of inquiry perspective. The spidercharts in PLATINUM are not a scoring tool, but a reflection tool for a community of inquiry. For example, while working on a basic mathematics module for first-year students in biomedical sciences, the UvA community of inquiry was supported by the spidercharts in the pedagogical decision-making processes and in discussions about suitability of RME for its design of student inquiry. At first sight it seems attractive to use instructional activities in which students reinvent the concept of direction field (Rasmussen & Marrongelle, 2006), Euler's method (Kwon, 2003), solution of linear systems of ODEs (Rasmussen & Blumenfeld, 2007), or bifurcation diagram (Rasmussen et al., 2019; Goodchild et al., 2021), and in which students more or less constitute the formal mathematics. But in the reality of university teaching, the UvA partners had pragmatic reasons for rejecting the principles of guided reinvention and emergent modelling in their mathematics module: limited student-teacher contact time, insufficient availability of suitable learning spaces for small-group work, large number of students and their differing mathematics background that would complicate the reinvention and emergent modelling processes, and a mismatch with the dominant teaching and learning culture in the discipline. Another obstacle foreseen by UvA partners in their course design was the extent to which lecturers could elicit and inquire into student thinking in practice, which is considered a crucial aspect of design research and inquiry-oriented education. Guidance and monitoring small-group work and utilising student work to promote a shared and more sophisticated understanding of mathematics commensurate with the important mathematics concepts and conventions addressed in the module was cumbersome given the layout of the tutorial rooms and the number of students present during practice sessions.

The above objections and hesitations toward the inquiry-oriented instructional approach can also be found amongst university lecturers toward other inquiry-based approaches that are based on a grand theoretical or intermediate-level frame, for example grounded on (socio-)constructivism or cultural-historical activity theory, especially amongst lecturers involved in service teaching of mathematics. Often these lecturers feel uncomfortable with a constructivist perspective on mathematical representations and acts of representing. From a constructivist point of view (see, for example, Cobb et al., 1992; Davis et al., 1990; von Glaserfeld, 1995; Jaworski, 1994), the learning of mathematical representations should not take place in a transmission approach of instruction, in which lecturers explain for their students the meaning of mathematical and scientific representations and how they are to be used. Instead, informal representations created and used by individual learners during the learning process should play a role in the route towards conventional mathematical notations. In other words, in a constructivist perspective, both acts of representing and representations are a means of constructing mathematical knowledge and understanding by students. University lecturers involved in service teaching often feel that there is too little space in the already overladen mathematics courses for a constructivist approach, which is more time-consuming than traditional instruction. Often they do not have the power to reduce the mathematical content of the courses in order to make space for student inquiry activities, or they lack personal experience with a constructivist approach. They may even have a limited view of grand theoretical and intermediate-level frames, and are unaware that these frames leave space for various approaches to student inquiry with respect to intellectual sophistication and to student participation or locus of control (cf., Wenning, 2005, in the context of science education).

In PLATINUM we distinguish student inquiry on the following levels, ranging from rather closed to completely open inquiry work:

- *limited inquiry*, in which students follow directions and make sure that their results match the requirements set in advance;
- *structured inquiry*, with no predetermined answers but conclusions solely based on students' investigation;
- *guided inquiry*, with no predetermined method but students having to determine how to investigate the problem;
- *open inquiry*, with no predetermined questions but students proposing and pursuing their own questions;

Under the assumption that the sum of the levels of teacher and student participation in each of the above inquiry types is roughly the same, the above types of student inquiry are ordered with increasing student participation and independence (locus of control) and with decreasing degree of teacher's guidance. Many lecturers are willing to move students in a course from a teacher-dependent to a more teacher-free and independent role, i.e., to shift the locus of control gradually from teacher to student, but are afraid that a course is too short for doing this. Promotion of inquiry-based mathematics education often boils down to breaking barriers like the ones mentioned.

6.2.2. Use of Learning Cycle Models in PLATINUM. PLATINUM partners have also used learning cycle models, not only to design their student activities but also to analyse how these activities actually went in classroom practice. For example, to compare the design of their inquiry task with the student actions in class, partners from the Brno University of Technology (BUT) refer in Section 8.4.2 to the model of Pedaste et al. (2015) for IBME activities, consisting of the phases Orientation—Conceptualisation—Investigation—Conclusion—Discussion, and to a simple 4-stage modelling cycle, consisting of Understanding task—Establishing model—Using mathematics—Explaining results.

Quite popular in the PLATINUM project has also been the 5E-instructional model of Bybee et al. (2006) and the 7E-model of Eisenkraft (2003) for characterising tasks in a teaching unit for student inquiry.[2] The 5E-model requires instruction to include the following phases: engage, explore, explain, elaborate, and evaluate. The UvA partners have, for example, used the 5E-model to characterise the task sequence about enzymatic kinetics developed in their case study presented in Chapter 12 (see Table 12.1, p. 224). The 7E-model expands the Engage phase into two components——Elicit and Engage. Similarly, the 7E model expands the two phases of Elaborate and Evaluate into three components——Elaborate, Evaluate, and Extend. Partners of the Complutense University of Madrid (UCM) have used the 7E model in their documentation of the teaching unit about matrix factorisation, which is part of the case study presented in Chapter 16, to typify the student activities (see Table 6.1).

The above examples illustrate that a learning cycle model not only provides lecturers and educators with a documentation and assessment tool that they could use over time to both tell the story of their teaching and the learning of their students in a particular setting, but also supports lecturers and educators in developing learning experiences for their students. The latter use of a learning cycle model fits very well

[2] Maybe the popularity of the 5E- and 7E-model originates from the inclusion of these models in a working document used in the PLATINUM project to help partners document their work.

Part	*Activity*	*E-Emphasis*
Task 1	Apply the Gaussian elimination method to compute the LU factorisation of matrices in a special class of matrices	Elicit
Task 1	Propose a conjecture	Engage
Task 2	Check the conjecture: does it hold for other classes of matrices	Explore
Task 2	Modify the conjecture	Engage
Task 3	Find a general result	Explore
Discussion	Each group makes their results public	Explain
Discussion	Groups assess, accept or criticise the results and opinions of other groups	Elaborate, Extend

TABLE 6.1. Characterisation of student activities in the matrix factorisation teaching unit of the UCM partners, presented in Chapter 16, via the 7E-instruction model of Eisenkraft (2003).

with the design of structured or guided inquiry. Then an activity sequence contains all phases in the learning cycle model, which are (repeatedly) divided over the activities. Use of a *cyclic* instructional model is intentional because it emphasises the role of the model as a formative documentation and assessment instrument that supports lecturers in designing learning experiences for their students by reflecting on where their students have been, what they have learned, and what they might do next. It also reflects that student inquiry is ideally a cyclic process with more than one iteration.

6.2.3. Using Models of Processes and Actions in Professional Practice. There are many different definitions and interpretations of the term inquiry-based mathematics education (IBME). All university lecturers have an intuitive feel for what is meant by this term and whether a clear definition is given or not, they probably recognise inquiry-based teaching and learning of mathematics. This is especially true for the following conceptualisation of IBME formulated by Dorier and Maaß (2020), in which active engagement of students with mathematics is central:

> Inquiry-based mathematics education (IBME) often refers to a student-centred paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions (like carrying out experiments, systematically controlling variables, drawing diagrams, calculating, looking for patterns and relationships, and making conjectures and generalisations), interpret and evaluate their solutions, and communicate and discuss their solutions effectively.

What does *active engagement* with mathematics at university level mean? Mason (2002) provides a list of words he believes to denote processes and actions when mathematicians pose and solve mathematical problems. He distinguishes the following innate powers employed by practitioners when they work on mathematics: "exemplifying, specialising, completing, deleting, correcting, comparing, sorting, organising, changing, varying, reversing, altering, generalising, conjecturing, explaining, justifying, verifying, convincing, refuting, and depicting" (p. 125). Mason (2002), Mason et al. (2010), and Mason and Johnston-Wilder (2006) discuss in detail how the design of student tasks might benefit from using these words to give students a richer experience of aspects of mathematical thinking. The cited authors are of opinion that mathematical thinking tasks should be rich tasks that encourage students to be assertive and active rather than taking a passive approach to learning: tasks should enable students to encounter significant mathematical ideas and concepts and to discuss them with peers, allow them to use their 'natural' thinking powers to work on mathematics, involve various ways of thinking, engage them in the use of precise mathematical language, and be appropriately challenging. The listed processes and actions taken from mathematical practice are expected to help designers of rich mathematical tasks. Mason and colleagues provide guidance to underpin this expectation in the form of a variety of tactics. An example of how this guidance helps in practice can be found in the paper of Breen and O'Shea (2018) in which they select the following six types of tasks that would engage students in the practices and habits of minds of research mathematicians: (1) generating examples, (2) analysing reasoning, (3) evaluating mathematical statements, (4) conjecturing and/or generalising, (5) visualising, and (6) using definitions. PLATINUM partners have also used identified tactics, albeit sometimes implicitly, to (re)design student tasks and activities that foster conceptual understanding of students and promote an inquiring atmosphere. Some examples of tasks developed and used are shown below; more examples will be discussed in Section 6.4.

But before going to examples we draw attention to two other frameworks that may help lecturers create tasks that foster and assess aspects of mathematical thinking. Swan (2008) lists the following five task types that encourage concept development at secondary school level: (1) classifying mathematical objects, (2) interpreting multiple representations, (3) evaluating mathematical statements, (4) creating problems, and (5) analysing reasoning and solutions. These task types encourage students to use their innate powers of organising, classifying, characterising, examining, comparing, verifying, interpreting, evaluating, creating, expressing, analysing, and reflecting. There is no reason to believe that these task types would not serve the same purpose at undergraduate level. In addition, Swan (2008) lists the following design principles to make teaching for conceptual understanding more effective in a classroom setting:

- use rich, collaborative tasks;
- develop mathematical language through communicative activities;
- build on the knowledge learners already have;
- confront difficulties rather than seek to avoid or pre-empt them;
- expose and discuss common misconceptions and other surprising phenomena;
- use higher-order questions;
- make appropriate use of whole class interactive teaching, individual work and cooperative small group work;
- encourage reasoning rather than 'answer getting;'
- create connections between topics both within and beyond mathematics;
- recognise both what has been learned and also how it has been learned.

Many of these principles seem valuable in a university setting as well (cf., Breen & O'Shea, 2018), but some of them may be difficult to realise in lectures to large groups of students. However, the principles seem applicable for tutorials with smaller groups of student and for the design of homework tasks.

Based on an analysis of what undergraduate students are in reality asked to do in course work and examination questions, Pointon and Sangwin (2003) identify eight classes of mathematical questions and tasks, listed in Table 6.2. In the four classes on the left-hand side, students are asked to apply knowledge in bounded situations. The classes on the right-hand side require higher-order mathematical thinking skills. The authors notice that the latter tasks are hardly asked in reality. In PLATINUM, we regret this because these types of tasks probably promote conceptual understanding of mathematics more than the other ones. Adding more questions of this type is expected to improve the balance between procedural and conceptional learning of mathematics. This also positions the frameworks discussed in this section: whereas Pointon and Sangwin (2003) categorise questions and tasks that are actually used in school and university practice, Swan (2008) and Mason (2002) describe processes and actions to which students should be encouraged in their opinion. There are also similarities between the frameworks: ‘construct example/instance’ in Pointon and Sangwin’s taxonomy is more or less the same as ‘exemplifing’ and ‘specialising’ in Mason’s framework.

1.	Factual recall	5.	Prove, show, justify (general argument)
2.	Carry out a routine calculation or algorithm	6.	Extend a concept
3.	Classify some mathematical object	7.	Construct example/instance
4.	Interpret situation or answer	8.	Criticize a fallacy

TABLE 6.2. Pointon and Sangwin (2003) task classification scheme.

Let us continue now with some tasks and activities that have been developed by PLATINUM partners to foster conceptual understanding of students and promote an inquiring atmosphere. These examples have also been used at PLATINUM project meetings to discuss what inquiry-based mathematics education could mean and how student inquiry could be promoted by mathematical tasks.

The first two problems (Figure 6.1) come from partners at the University of Agder (UiA) and is about the use of nonstandard problems in an ordinary differential equations course (see also Rogovchenko et al., 2018, and Chapter 11). These are unusual problems for which “students have no algorithm, well-rehearsed procedure, or previously demonstrated process to follow.”

Sample problem 1

a) Verify that $y(x) = \dfrac{2}{x} + \dfrac{C_1}{x^2}$ is the general solution of a differential equation

$$x^2\, y' + 2xy = 0$$

b) Show that both initial equations $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solutions. Does this fact violate the Existence and Uniqueness Theorem? Explain your answer.

Sample problem 2

a) Verify that $y(x) = C_1 + C_2\, x^2$ is the general solution of a differential equation

$$x\, y'' - y' = 0$$

b) Explain why there exists no particular solution of the above equation satisfying initial conditions $y(0) = 0$; $y'(0) = 1$.

c) Suggest different initial conditions for this differential equation so that there will exist exactly one particular solution of a new initial value problem. Motivate your choice.

FIGURE 6.1. UiA examples of nonstandard ODE tasks.

The following words from Mason's framework (2002) can be recognised: verify (1a, 1b, 2a), explain (1b, 2b), and specialise (verbalised in problem 2b as "Does this fact violate ..." and in problem 1c as "Suggest different initial conditions such that ..."). In terms of Swan's framework (2008), students are in these two problems mainly invited to evaluate mathematical statements. In terms of the taxonomy of Pointon and Sangwin (2003), students are asked to interpret a situation or answer (1a, 2a, 2b), to show/justify (1b), and to construct an example/instance (2c).

Partners from the University of Amsterdam (UvA) have used the tactic of turning an existing textbook question into a more inquiry-based question (cf., Dorée, 2017) for several problems in an analysis course for first-year mathematics students. Here we only discuss the following original problem (Ross, 2013, Exercise 14.7):

> Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

Past experience of tutorial lecturers is that this is a fairly difficult exercise for students unfamiliar with the subject: you need to treat small and large values of n separately. Many students do not get this idea and are already lost at the start of the proof. In order to guide students, the new exercise (Figure 6.2) starts with the special case $p = 2$ and students are asked to consider the magnitude of the squares compared to the original sequence. Once they understand this case, they can use it to prove the specialised statement, and hereafter generalise towards arbitrary p, including the case $0 < p < 1$.

Sample problem 3

Let $\sum a_n$ is a convergent series of nonnegative numbers.

a) For how many values of n can we have $a_n^2 > a_n$?
b) Show that $\sum a_n^2$ converges as well.
c) What can you say about a_n^p for $p \in (0, \infty)$?

FIGURE 6.2. UvA example from an elementary analysis course.

In terms of Mason's framework (2002), the task designers first specialise the original statement to the case $p = 2$ in the hope and expectation that students can hereafter see the general approach to proof from the particular case (specialising to help generalising). Instead of asking to prove a theorem, they ask students in the third subtask to make a conjecture for the general case with $p \in (0, \infty)$. Of course students must justify their statement. In terms of Swan's framework (2008), students are invited in the revised exercise to evaluate mathematical statements and to analyse reasoning and solutions (actually analysing their own reasoning in the special case). In terms of the taxonomy of Pointon and Sangwin (2003), students interpret a situation or answers (3a) in the special case $p = 2$ and prove the statement in this special situation (3b) before they extend this to the general case (3c).

The last two examples of inquiry-based tasks (Figure 6.3), which illustrate the use of models of processs and actions, are taken from instructional materials of partners at Loughborough University (LU) for first-year materials engineering students. The first subtask of Problem 4 is designed for use in a lecture, but all other subtasks are considered more appropriate for tutorials, preferably in the form of small group work. The computer environment GEOGEBRA[3] allows students to explore mathematical situations, in particular to explore functions using multiple representations.

Because many competencies are addressed in Problem 4 it comes to no surprise that, in terms of Mason's framework (2002), many powers of students are triggered in

[3] www.geogebra.org

Sample problem 4

a) Consider the function $f(x) = x^2 + 2x$ (x is real). Give an equation of a line that intersects the graph of this function (i) Twice (ii) Once (iii) Never.

b) If we have the function $f(x) = a\,x^2 + b\,x + c$, what can you say about lines which intersect this function twice?

c) Write down equations for three straight lines and draw them in GeoGebra. Find a (quadratic) function such that the graph of the function cuts one of your lines *twice*, one of them *only once*, and the third *not at all* and show the result in GeoGebra.

d) Repeat for three different lines (what does it mean to be different?)

Sample problem 5

Use sliders in GeoGebra to determine which of the graphs on the right could represent the function

$$y = a\,x^4 + b\,x^3 + c\,x^2 + d\,x + e$$

Here a, b, c, d and e are real numbers, and $a \neq 0$. Explain your thinking.

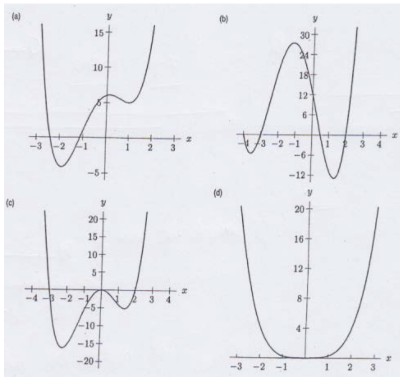


FIGURE 6.3. LU examples of inquiry-based tasks using GeoGebra.

these subtasks: exemplifying, specialising, generalising, comparing, organising, varying, conjecturing, explaining, justifying, verifying, imagining, and depicting. This is typical for an inquiry-based task. Mason and Johnston-Wilder (2006) actually recommend the use of a 'mixed economy' of tasks in order to realise as many goals as possible because no single strategy or task type has proved to be universally successful in developing mathematical thinking. In terms of Swan's framework (2008), students are invited to classify mathematical objects (4b), to interpret multiple representations (4a, 4c, 4d), and to evaluate (their own) mathematical statements (4b). In terms of the taxonomy of Pointon and Sangwin (2003), students classify some mathematical object(4b), interpret a situation or answer (4a), show and justify outcomes(4c, 4d), and construct instances/examples (4a,4c,4d).

6.3. Documentation of Inquiry Tasks in PLATINUM

Problem 5, taken from (Hughes-Hallett et al., 2005, p. 47), is a guided inquiry task designed with the intention that students use GeoGebra to experiment with coefficients in equations and scales on axes to gain insights into mathematical relationships and that lecturers/teaching assistants circulate among groups observing activity, encouraging work on tasks, probing students' mathematical thinking, and discussing students' ideas. In Mason's framework (2002), the tactic 'say what you see' is expected to help students make progress while they are (hopefully intentionally) manipulating sliders in order to get a sense of what is going on then in terms of the graphic representation of the polynomial and over time be able to articulate this sense in a mathematical way. This task is not meant to be a random exploration because students are asked to explain their thinking during the classification process of which graph can be constructed from a fourth degree polynomial function. Explanation works best, especially for the person who explains, if there is someone else to explain to. This is why this task is actually meant for group work, even though the task itself does not demand it, and that students give feedback to their peers, lecturers and/or teaching assistants. This makes a student task distinct from a student activity: like Mason and Johnston-Wilder (2006) we consider in this chapter a task as being what students are asked to do, whereas an activity means what students actually do in their interaction with peers, lecturers, resources, environment, and so on around the task.

A mathematical task initiates mathematical activity of a student: it sets the direction of the student thinking and acting, influences the level of student engagement, and determines to a large extent what a student learns. However, a task is actually no more than a means to steer a student toward meaningful learning and practising of mathematics. There is no guarantee that a student will work as planned by the task designer and achieve the intended learning outcomes. Mason (2002, p.105) uses the following words to express the importance of careful task design and that the task itself does not automatically lead to the intended mathematical activity of students and/or the realisation of the set pedagogic purpose:

> In a sense, all teaching comes down to constructing tasks for students, because most students believe (however implicitly) that their job as a student is to complete the tasks they are set, including attending sessions and sitting examinations. This puts a considerable burden on the lecturer to construct tasks from which students actually learn.

Rephrasing Watson et al. (2013, p. 10), tasks generate student activity which affords opportunity to encounter mathematical concepts, ideas, strategies, methods and techniques, to use and develop mathematical thinking and modes of inquiry, and to form a view of mathematics. In the PLATINUM project we are in particular interested in the design and use of tasks and activities that promote conceptual understanding of mathematics through student inquiry. The objective of Intellectual Output 3 in this Erasmus+ project is to

- develop a collection of teaching units that promote mathematics conceptual learning through an inquiry approach;
- synthesise working models from the designs of teaching units;
- use teaching units in specific regular courses and to collect data about their use;
- explore possibilities to make teaching units accessible for students with identified needs; and
- package and present teaching units for a wide international audience of teachers, teacher trainers, and educators with an interest in IBME.

What is the meaning of *teaching unit* within the PLATINUM project? First we note that student inquiry is not necessarily restricted to a single event with a single task, perhaps divided in subtasks. Just like substasks in a single task, the earlier tasks in a task sequence are meant to provide students experiences that scaffold them in the solution of later tasks, allowing them to engage in more sophisticated mathematics that would otherwise not have been possible. This is certainly important for students who are not yet well trained in mathematical fundamentals, still need to learn mathematical concepts relevant for a student inquiry, and can benefit from support of lecturers and task designers to establish this mathematical grounding. Being aware that bachelor students most likely do not have the mathematical experience to ask the questions and follow the directions that lecturers of mathematics spontaneously engage with, PLATINUM partners have been trying to stimulate inquiry for students while they learn the basics of mathematics in calculus, linear algebra, and so on. This cannot be achieved in a single task, but requires a teaching and learning path with multiple tasks. Herein not only the task sequence matters, but also the intended mathematical activity, and the pedagogic purpose. In a PLATINUM teaching unit these three aspects come together and are documented to inform others interested in the student inquiry or lecturers who use the teaching unit, learn from this use, and try to improve it. For the task design phase this resembles the notion of a hypothetical learning trajectory introduced by Simon (1995), which is "made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning processes–a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136). In developmental research, a hypothetical learning trajectory is cyclically adapted and improved on the basis of experiences with the trajectory in teaching practice. This is the approach that most communities of inquiry at the PLATINUM partner universities have chosen and that they describe in more detail in their case studies in Part 3 of this book.

The work done by lecturers in the PLATINUM project also illustrates the important role they play in the design of teaching units as inquirers who explore

- the kinds of tasks that engage students and promote mathematical inquiry;
- ways of organising the learning situation that enable inquiry activity; and
- the many issues and tensions that arise related to the discipline, classroom, colleagues, and educational system;

and who reflect on what occurs in practice with feedback to future action. An inquiry cycle of teaching adopted from (Jaworski, 2015) in PLATINUM to characterise the work of lecturers-as-inquirers in the design of teaching units is shown in Figure 6.4.

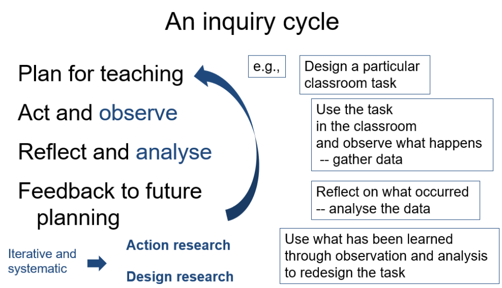


FIGURE 6.4. An inquiry cycle used in PLATINUM for the design of teaching units.

The three-layer model of inquiry outlined in Chapter 2 (see also Jaworski, 2019) distinguishes the following three forms of inquiry practice that involve students, lecturers and educators:

- *Inquiry in mathematics*: university students learning mathematics through exploration in tasks and problems in classrooms, lectures and tutorials; lecturers using inquiry as a tool to promote student learning of mathematics;
- *Inquiry in mathematics teaching*: lecturers using inquiry to explore the design and implementation of tasks, problems and activity in classrooms; educators using inquiry as a tool to help lecturers develop teaching;
- *Developmental research inquiry*: lecturers and educators researching the processes of using inquiry in mathematics and in the teaching of mathematics.

This is too much inquiry work for a single person to do professionally and too hard to maintain under pressure of other job obligations. This is why the notion of community of inquiry (CoI) has been adopted in the PLATINUM project, as discussed in

Part 1. By working together in a community, each might learn something about the world of the others, and equally important might learn something more about his or her own world. Belonging to a CoI also motivates lecturers to document their inquiry activity, and in particular the design and use of teaching units. Documentation is not only useful for the lecturer, but also for the communities of inquiry to which s/he belongs. No demands or constraints have been set within the PLATINUM project for the documentation of teaching units. Communities of inquiry were only given a template with aspects to which they could pay attention in the documentation and some concrete examples of documentation written at an early stage by the UvA CoI. The main reason for this freedom in documenting the teaching units, but still providing a template for guidance has been that many similarities and differences were identified amongst the communities of inquiry concerning ambitions/scope, study programmes and target groups, mathematical concepts/contents, envisioned use of digital technology, and planning of tasks and timeline. Shared interests and goals of partners were in improving the learning of mathematics in relevant contexts, increasing authenticity in student activities (includes use of digital technology), improving students' understanding of mathematical concepts, methods and techniques and their roles in applications, introducing inquiry-based activities in mathematics courses, and in innovating instruction (e.g., to increase student motivation and engagement). Some partners were thinking of modifying existing courses (UiA, multivariable calculus; LUH, discrete mathematics) or starting from scratch new courses (UvA, basic mathematics for biomedical sciences, analysis of neural signals), while others were planning to modify units/topics within existing courses to make them more inquiry-based (e.g., BGKU, sequences and series; BUT, complex functions; LU, complex numbers; MU, optimisation; UCM, special forms of matrices). Most plans were made for bachelor study programmes with quite often large numbers of students participating in the pedagogic cases (typical for programmes in engineering, economy, and life sciences), but there were also plans presented for courses with a small number of master students (e.g., in pre-service teaching training). It turned out that there was a large variety in mathematical concepts treated in the pedagogic case ranging from complex functions (BUT), complex numbers (LU), basics of discrete mathematics (LUH), differential equations (MU, UCM, UiA, UvA), logic (UCM), mathematical modelling (BGKU, UiA, BUT), matrix theory (UvA, UCM), multivariable calculus (UiA, UvA), sequence, series and limit (BGKU, UvA) to statistics/regression (UCM, BGKU). Besides virtual learning environments (Moodle, Canvas, …) and smartboards, beamers, voting systems, and so on, many different mathematical software environments (mostly mainstream software for higher education) were envisioned to be used by students in their work with the developed teaching units, ranging from Autograph[4] (LU), Excel (BGKU), GeoGebra[5] (LU, MU, UvA), Maple (UCM, UiA, BUT), Mathcad (BGKU), Mathematica/Wolfram Alpha (BUT, BGKU), MATLAB[6] (UCM, UvA), Maude[7] (UCM), Rstudio[8] (UCM, UvA) to SOWISO[9] (UvA). We actually consider the variety of pedagogic cases as a strong point of the PLATINUM project because in this way inquiry-based mathematics education could be explored in various university teaching practices.

[4] https://completemaths.com/autograph
[5] www.geogebra.org
[6] www.mathworks.com
[7] doi.org/10.1016/j.jlamp.2019.100497
[8] www.rstudio.com
[9] www.sowiso.com

Each PLATINUM teaching unit has been built around one mathematical topic, is designed for student inquiry, and is used in higher education classroom practice. These teaching units serve as exemplary materials for mathematics lecturers and for instructors in professionalisation programmes to experience inquiry-based mathematics education (IBME) at university level and to inspire further development of IBME. The documentation of each teaching unit consists of (1) information for lecturers, (2) information about the learning activities, and (3) the worksheets and files used in the classroom, plus supplementary material. Some items in the information for lecturers are:

- *Unit description*: a short description of the unit about its subject matter and organisation, the student level, expected prior knowledge, the significant mathematical concepts and essential questions addressed, the course and context in which it has been used in HE practice, and the estimated duration;
- *IBME character of the teaching unit*: the kind of student inquiry that is applied and the addressed inquiry abilities;
- *Technological Pedagogical Content Knowledge (TPACK)*: the common students' difficulties and alternative conceptions that have been identified by mathematics education research and/or by lecturers in higher educational practice, and the role of ICT in the teaching unit;
- *Lecturers' experiences in the teaching practice*: a short reflection about its use within HE classroom practice (which expectations were met or not, challenges encountered in the implementation, students' reactions, ...).

Information about the learning activities in a teaching unit consists of short descriptions of learning objectives, main concepts and essential questions, envisioned student engagement in the construction of conceptual understanding, and of tool use for each learning activity. The third part of the documentation of a teaching unit consists of

- *student tasks and worksheets*, in source format (WORD, LaTeX, ...) and in PDF format;
- *auxiliary files* such as data files, software-specific files, simulation files, assessment sheets, reference materials, and so on; and
- *supplementary files*, for example, more detailed notes about the design of the unit and the activities, classroom experiences, related narratives, etc.

The template for documenting a teaching unit for student inquiry and all documented PLATINUM teaching units can be found in the website of this Erasmus+ project.

6.4. Examples of Inquiry Tasks Developed and Used in PLATINUM

In this section we present in detail three examples of inquiry-based tasks developed and used in PLATINUM. They are selected to represent typical designs and approaches of student inquiry that can be used when teaching mathematics to first-year undergraduates.

6.4.1. Exploring Data-Driven Numerical Differentiation. This example is taken from the Basic Mathematics Module for Biomedical Sciences developed by the UvA partners. The entire module, discussed in more detail in Chapter 12, can be seen as a learning trajectory to introduce Systems Biology to first-year students of biomedical sciences. In Systems Biology, biological processes of change are modelled by differential equations and values of parameters in these models are estimated by comparing modelling results with measured data. But in order to be able to do this estimation one must be able to compute values of derivatives of the modelled quantity. Students investigate early in module how to compute the numerical data. First they are challenged in a lecture to form small groups of two or three students and come up themselves with ideas how to do this (they had to suggest at least two possibilities). The task shown in Figure 6.5 is used for that purpose.

Given are the following values of a function $y(t)$ in the neighbourhood of $t = 1$:

t	0.7	0.8	1.0	1.1	1.2
$y(t)$	0.741	0.819	1.000	1.105	1.221

What is the best approximation of $y'(1)$?
(exact answer = 1 because the used function is $y(t) = e^{t-1}$)
Try several methods and compare the results with each other.

FIGURE 6.5. An inquiry task used in a lecture.

The lecture part of the teaching unit, with the invitation to compute a derivative at a point on the basis of few surrounding data points, can be characterised as guided inquiry, meaning that there is no predetermined method, but that students must determine how to investigate the problem and find answers to the question raised by the lecturer. By raising the question in a classroom discussion, the students are expected to be intrigued and tuned in on the exploration of mathematical methods. Preferably, they do not do this individually but with peers. The goal is that students experience that by talking about mathematics with each other, their own thinking becomes deeper and fruitful.

Numerical differentiation is a subject that is suitable for a more open inquiry approach when students are familiar with the concepts of a derivative at a point, tangent line, and difference quotient as approximation of a derivative at a point in the domain of some mathematical function. One might expect that they can then indeed come up with the forward finite difference as a numerical approximation of a derivative at a point. This seems a good starting point to let students discover other ways to numerically approximate a slope at some point. Students are invited to discuss for about 20 minutes possible approaches with peers in small groups. Methods and results students come up with are then discussed in classroom: it is expected that they can propose a backward finite difference method and a combination of the forward and backward difference method. The discussion offers the opportunity to pay attention to what underpins mathematical methods and why it is common in mathematics to look for alternative methods and techniques for solving the same problem and to explore what works best and under what conditions. It is important that there are many possible methods because inquiry means asking questions and seeking answers, raising follow-up questions and seeking more answers, recognising possibilities, explore options, discuss pros and cons, and so on. There should not be an early end point in student inquiry and in the discussion about mathematical methods.

After the lecture, students implement their methods in RSTUDIO during a practice session in order to further explore the numerical methods regarding accuracy, efficiency, coping with noise in real data, and so on. The tutorial in which students implement standard finite difference methods for numerical differentiation and explore the advantages and limitations of the methods is an example of structured inquiry, meaning that students follow more or less directions to implement 2-point and 3-point difference methods and set up a numerical experiment to explore by example which method gives better results with data that are noisy. In Table 6.3 we typify these student activities in terms of the 7E learning cycle of student inquiry.

Assignment	*Activity*	*E-Emphasis*
1.	Plotting a function	Elicit
2.	Plotting a function and its derivative in one diagram	Elicit
3.	Implementing the 2-points and 3-points numerical derivative	Engage
4.	Exploring the effect of step size and data noise on numerical differentiation	Explore

TABLE 6.3. Characterisation of student activities in the numerical differentiation practice session in the teaching unit of UvA partners via the 7E-instruction model of Eisenkraft (2003).

At completion of the teaching unit, students are expected to have strengthened their abilities to

- talk about and work with the concept of function when it is merely presented in the form of function values;
- understand why one would be interested in a numerical derivative;
- compute numerically the rate of change of a quantity when only data are given instead of a formula;
- carry out computations of numerical derivatives in RSTUDIO;
- develop investigations (numerical experiments) in order to inspect and explain the accuracy and efficiency of numerical differentiation methods; and
- think more critically about mathematical methods and techniques.

These abilities contribute to what Goodchild et al. (2021) call a ‘critical stance’ toward learning and teaching of mathematics, which is complementary to critical alignment. The notion of critical stance is according to these authors distilled into three components: awareness, self-evaluation, and agency:

> Stance, we assert, is a mode of ‘being’ an attitude, perspective or disposition. Critical stance is dependent upon the student’s awareness, the information and experience they possess to reach an informed judgment about an issue, and recognition of their agency to make a difference. Critical alignment to a practice relates to a person’s relationship with the practice. On the other hand, critical stance also relates to the personal characteristics and attributes that the person brings to their participation.

In the student activities described in this example the designers try to give students opportunities for critical awareness and reflection on one’s own experience, meanings, and knowing. By letting students come up themselves with various methods for computing a numerical derivative and explore the effectiveness of various methods they can recognise that one method is from mathematical point of view more sophisticated and effective than another, and that one can be on the one hand critical about mathematical methods but on the other hand have agency to change or try-out things in investigations on the basis of own reflection and evaluation of experiences.

6.4.2. Exploring Properties and Rules of Probability. The following example is again a small teaching unit for use in a lecture and aimed at steering students away from passive listening to the lecturer toward active learning via hands-on/brains-on activities. It comes from partners at Masaryk University (MU), who developed it for a statistics course in the first-year study programme of Business and Economics.

Students work for about half an hour in small groups during a lecture. They use an A4 sheet with all possible outcomes of a roll with two dice (actually four copies of Figure 6.6 are used in the worksheet shown in Figure 6.7) to carry out short inquiry tasks and they formulate their findings and conclusions.

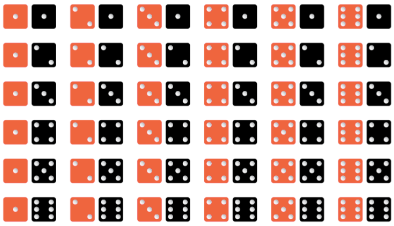


FIGURE 6.6. A dice sheet showing all outcomes of a roll with two dice. The sheet is used by MU partners in a teaching unit about properties and rules of probability.

Based on the first task sequence with the solution sheet to the right, try to replace the question mark symbol in the following relationships.

(i) $P(A) + P(A') = ?$

(ii) If $A_1 \subset A_2$, then $P(A_1)\ ?\ P(A_2)$ Take inspiration, for example, from events A and D.

(iii) If $A_1) \cap A_2 = \emptyset$, then $P(A_1 \cup A_2) = ?$

(iv) If $A_1) \cap A_2 \neq \emptyset$, then $P(A_1 \cup A_2) = ?$ Take inspiration, for example, from events A and C.

(v) $P(A_1 \cup A_2 \cup A3) = P(A_1) + P(A_2) + P(A_3) - ? - ? - ? + ?$ Take inspiration, for example, from events C, E and H.

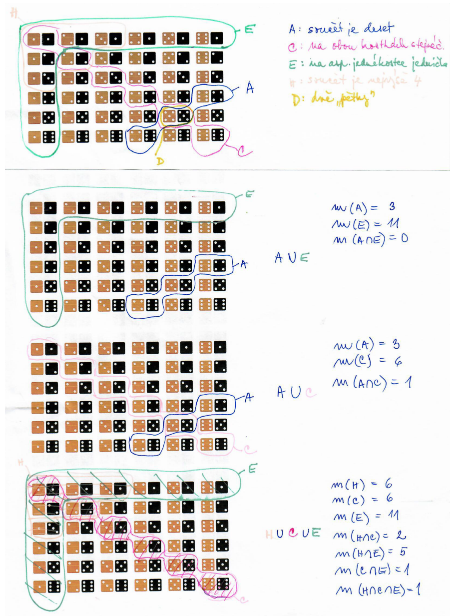


FIGURE 6.7. Task sequences for students to conjecture rules of probability on the basis of results of a sample problem situation.

The tasks introduce properties and rules of probability. But instead of stating the rules and using them in an application, the MU partners chose to have a set of introductory tasks that help students conjecture rules of probability. Although these conjectures are made on the basis of one concrete situation, the drawing of two dice, it is hoped and expected that students start to understand that such examples are common in mathematical investigations to understand problem situations and come up with solutions that work in other situations as well. According to Mason's framework (2002) this process means that specialisation is often needed to make generalisation possible.

The student activity consists of two parts: firstly, students determine the sample space of all possible outcomes for the following events in rolling two dice.

(A) The sum of dots in a roll equals 10;
(B) The sum of dots in a roll differs from 10;
(C) Each dice rolled has the same number of dots;
(D) Each dice rolled is 5;
(E) At least the roll of one of the dice is 1;
(F) The sum of the dots in a roll equals 10 or at least one dice rolled is 1;

(G) The sum of the dots in a roll equals 10 or each dice rolled has the same number of dots;
(H) The sum of the dots in a roll is less than 5;
(I) The sum of the dots in a roll is less than 5, or each dice rolled has the same number of dots, or at least one dice rolled is 1.

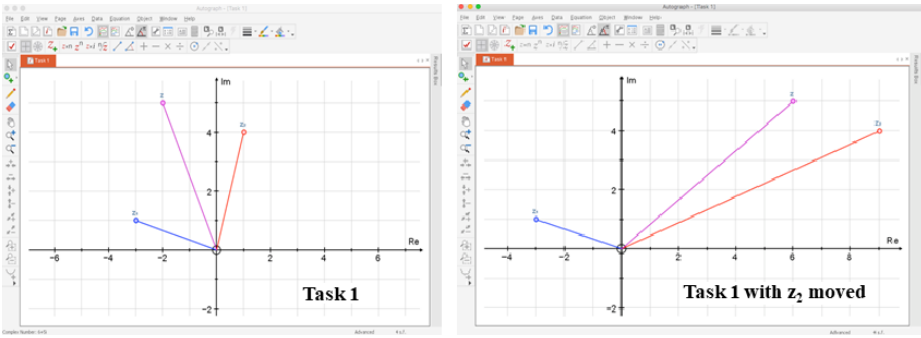
Hereafter students get the task sequence shown in Figure 6.7, in which they must conjecture basic probability formulas and underpin their conjectures.

This student work is finished with a whole classroom discussion of the proposed conjectures. The social aspect of learning and doing mathematics is considered important for students to adjust their view on mathematical inquiry.

6.4.3. Complex Number Arithmetic. The following example is part of a teaching unit for small-group work on complex numbers, which comes from a mathematics module in the Foundation Studies programme developed by partners at Loughborough University (LU) and is described in more detail in the case study of Chapter 15. Whereas traditional instruction often starts with specifying the calculation rules of complex numbers and illustrates this with examples using algebraic representations, the designers of this task have chosen to apply *reverse-engineering* of such questions and use the mathematics software tool AUTOGRAPH[10] for helping students in tutorial sessions to explore complex number arithmetic in a geometric perspective and connect geometric insights with algebraic manipulation. The whole teaching unit, created together with student partners (Treffert-Thomas et al., 2019), consists of 6 tasks: (1) addition, (2) subtraction, and (4) multiplication of complex numbers, (4) complex conjugate of a complex number, and (5) squaring and (6) cubing a complex number. Here we use the original Task 1, shown in Figure 6.8, to exemplify the more general ideas of the task designers. The adaptation of this task to make it more suitable for students with identified needs will be discussed in Section 6.7.

In this task students see three complex numbers on the computer screen, labelled z_1, z_2 and z, and one of the complex numbers (z_1) is specified in the question text. They must figure out what happens when they move z_2 and in this way try to give a geometric interpretation of the relationships between the shown complex numbers. No reference is made here to calculations or algebraic manipulations. AUTOGRAPH is used as a tool to visualise the mathematical relationship, but it is left up to students to make the link. A *reverse engineering* approach is used in this task, meaning that instead of asking the straightforward question "What is the sum of z_1 and z_2?" with only a correct or wrong answer and no scope for investigation, students are asked to move z_2 to the position so that the sum with z_1 reaches a particular position in the complex plane. Only in a later subtask (c) are students invited to undertake some associated calculation by hand in the hope and expectation that they relate movements on the computer screen to the written work and the theory involved. In subtasks (d) to (f), students are explicitly invited to reflect on specific results to develop more general awareness of complex number concepts related to addition. In terms of Mason's framework (2002), students are asked in Task 1 to apply the tactic *say what you see* in a special case, to explore more special cases to see a pattern, and then to generalise their findings. For the explorative phase, no suggestions are made in the tasks; students work independently and follow their own strategy. Tutorial lecturers circulate in the classroom, listen to what goes on in group work, encourage students, and lead whole-classroom discussions. This collaborative aspect is part of the pedagogic use of the task and not explicitly stated in the task itself.

[10] `https://completemaths.com.autograph`



Task 1

There are three complex numbers labelled z_1, z_2 and z.
z_1 is to be kept fixed while z_2 and z can be moved.
Select z_2 and move it until z reaches the position $6 + 5j$.

(a) What complex number is z_2? Right click and "Unhide All" to check your answer. The correct answer appears in green.
(b) What is the relationship between z_1, z_2 and z?
(c) Now calculate by hand:
With $z_1 = -3 + j$ and $z = 6 + 5j$, find z_2 such that $z_1 + z_2 = z$.
(d) Re-load *Task 1*. Move z_2 around the screen and notice how z changes as a consequence. What is the geometric connection between z_2, z and the complex number z_1 (which has stayed the same during your movements)?
(e) Now you are allowed to move both z_1 and z_2. Move these to different locations but make sure that z still ends up being $6 + 5j$. Make note of the positions of z_1 and z_2. Does your geometric connect from (d) still hold?
(f) Repeat another four times so that you have five different pairs of values for z_1 and z_2 with each of them making z to be at $6+5j$. For all of these, what is the relationship between z_1, z_2 and z and does your geometric relationship still hold for each of them?

FIGURE 6.8. Screen shot of AUTOGRAPH files and instructions for the original Task 1 in the complex number arithmetic teaching unit, used in the Loughborough Foundation programme.

6.5. Use of ICT in Student Inquiry

Much research has been done about the use of ICT in mathematics education, especially at primary and secondary school level, and it has offered a range of theoretical perspectives. Two volumes of the National Council of Teachers of Mathematics (Blume & Heid, 2008; Heid & Blume, 2008), the 17th ICMI study (Hoyles & Lagrange, 2010), books in the Springer series called 'Mathematics Education in the Digital Era' (e.g., Leung & Baccaglini-Frank, 2017), and the proceeedings of the International Conference on Technology in Mathematics Teaching (ICTMT) are good sources of information. Many lessons have been learned; the most important ones are that

- use of ICT for improvement of the depth and quality of mathematics learning is much more complicated than initially anticipated by proponents of tool use;
- ICT tools serve at a more fine-grained level many different goals in teaching and learning of mathematics;
- task design of ICT-enhanced mathematical activities is a delicate, multi-faceted issue; and
- the terrain of technology-supported education is rapidly changing and offering new ways of engaging with mathematical thinking, but with didactic theory development hardly keeping up with technological progress.

ICT use in inquiry activities for students is even more complex because of the twofold nature of inquiry learning, which can be described as *inquiry as ends* and *inquiry as means*. The first of these sees inquiry as a set of instructional outcomes for students that involve understanding of inquiry and abilities to do inquiry. In the perspective that university study programmes should enable their students to become literate in mathematics and ICT at the level that their discipline requires, the dominant idea is that students should learn to use ICT tools that are commonly used in their profession for doing mathematics. The use of R and RStudio in the basic mathematics and statistics course for biomedical sciences students, described in Chapter 12, is a typical example in which this perspective plays an important role.

The second aspect of inquiry, inquiry as means, is related to inquiry as an instructional approach or pedagogy. The PLATINUM objective to promote conceptual understanding through student inquiry is an example of this perspective. Teaching units designed for this purpose use ICT tools as means to realise instructional goals as best as possible. Task design emphasises in this case the mediating role of the tools. In this section we look in detail at a PLATINUM example of this type of use of ICT, namely the teaching unit about isometries and tessellations of the Euclidean plane which has been developed for first-year mathematics programmes by the UCM partners (Sáiz, 2020) and uses the dynamic mathematics environment GeoGebra. But before doing this, we would like to stress that, despite the apparent distinction between tool use in inquiry learning at university level, the two modes of tool use are better not treated as opposite modes because one cannot do without the other: without mathematical knowledge and skills and without inquiry abilities students will not learn much from ICT-enhanced inquiry and, conversely, a scientific context is always needed as a practice arena for inquiry abilities. For example, in Chapter 12, UvA partners describe how the use of R and RStudio enables their students to learn basic concepts of Systems Biology in ways that would otherwise not be possible.

In this section we adopt the model of Pedaste et al. (2015) for IBME activities, consisting of the phases Orientation, Conceptualisation, Investigation, Conclusion, and Discussion, to discuss the use of ICT in student inquiry in these phases and in particular in the teaching unit about isometries and tessellations of the Euclidean plane developed by UCM partner as this may serve as a prototypical example. This teaching unit, which takes about 5 hours of student work,[11] consists of two parts: thinking and learning about (1) planar isometries and (2) crystallographic groups and tesselations in the plane. For details we refer to the documentation of this teaching unit, which is available in the PLATINUM website (`https://platinum.uia.no`).

In the orientation phase, students are introduced to a domain of knowledge or a subject of study. Tasks in this phase are designed to activate students' prior mathematical and disciplinary knowledge, raise interest in the subject (relevance to the discipline), and relate to the students' background (e.g., skills, culture, and language). Their main aims are to enable students to explore and analyse a given problem situation. ICT is in this phase typically used to practise prior skills and to provide microworlds or simulations for initial exploration of the subject. The first part of the UCM teaching unit about isometries, lasting about one hour, serves this purposes. The dynamic geometry environment GeoGebra is in the first activities used to visualise the effect of transformations on points and triangles so that students can draw their own conclusions. Students are not given full access to the GeoGebra environment, but instead get tailormade GeoGebra applets to explore properties; see for

[11] Duration of work on the teaching unit about tesselations depends on whether students also create their own tesselations and/or explore work of the Dutch artist M.C. Escher.

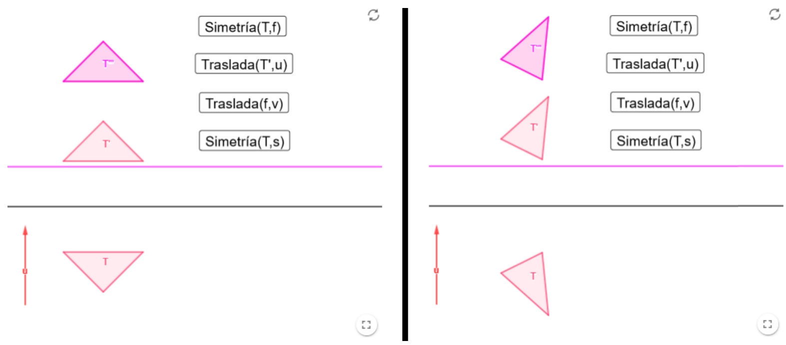


FIGURE 6.9. GEOGEBRA applet created by UCM partners for the composition of a reflection and a translation in a direction perpendicular to the mirror line. The result is a reflection in the translated mirror line. Students can press dedicated buttons to carry out transformation on objects in the plane, observe what happens, and formulate a hypothesis. On the left-hand side results of the initial settings of the applet are shown; on the right-hand side results are shown for a more general triangle obtained by dragging the original triangle.

example Figure 6.9 for two screen shots of an applet for composition of a reflection and a translation in a direction perpendicular to the mirror line. The task designers provide the students in this way with a microworld that (hopefully) helps them focus in their work on the mathematical properties instead of the technicalities of the computer environment. In terms of the framework of Kaput (1992) on computer use in education, ICT is in this case for the task designers a toolmaker/mediummaker and for the students an educational medium. The introductory activities about isometries also prepare the students for using GEOGEBRA in their prospective inquiry work in the second part of the teaching unit.

In the second part of the teaching unit, students explore planar tesselations, also known as wallpaper patterns. A wallpaper pattern is a way to cover a flat surface with a repeating pattern of shapes such that there are no overlaps or gaps and a translational symmetry in two independent directions can be identified. Its symmetries can be viewed as planar isometries and together they form a group, the symmetry group of the pattern. Seventeen symmetry groups of planar patterns can be distinguished (see, for example, Schattschneider, 1986). In the teaching unit seventeen GEOGEBRA applets have been created, one for each symmetry group, and most of these activities are inspired by the work of the Dutch artist M. C. Escher (1958)[12] The task designers connect mathematics with art in the hope and expectation that this motivates students in their inquiry and let them study the underlying mathematics in an attractive way. This is important for students as it helps them persevere as they engage in studying the wallpaper patterns.

The first three GEOGEBRA applets allow students to visualise in a detailed way how two of Escher's tilings (Seahorse, No 88; Beetle, No 91) can be created from a single tile by repeated application of generators of a matching group of isometries. Figure 6.10 shows two screen shots to construct from an initial geometric shape (a parallelogram) containing some black lines via rotations and translations a basic tile

[12]It is funny to see that Spanish task designers are inspired by a Dutch artist who himself got inspired by islamic geometrical art during his visit to the Alhambra in Granada, Spain.

(a seahorse) that can then be used to create a wallpaper pattern of seahorses with symmetry group labelled p2 in the notation of the International Tables for X-ray Crystallography.

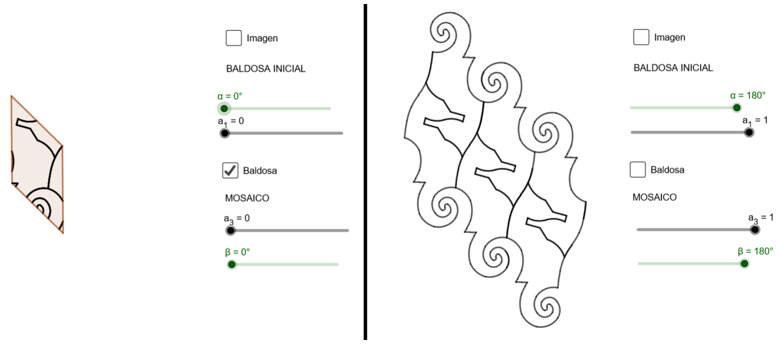


FIGURE 6.10. GEOGEBRA applet created by UCM partners to help students visualise in a detailed way how a seahorse wallpaper pattern with symmetry group labelled p2 can be constructed from an initial geometric shape containing lines. On the left-hand side is a screen shot with the initial shape and on the right-hand side is a screen shot with created seahorse shapes.

The construction of the basic tile is not explained in the applets; students have to find this out by dragging the sliders acting on the initial shape and observing what goes on. The upper slider rotates every black line that intersects the inner part of the upper edge of the parallelogram about the midpoint of the upper edge, and at the same time rotates every black line that intersects the inner parts of the left or lower edge of the parallelogram about the midpoint of the lower edge. Hereafter the lower slider acting on the initial shape translates all black lines that intersect the right edge of the parallelogram and its imaginary extension along the vector from the lower right vertex to the lower left vertex of the parallelogram, and at the same time translates all black lines that intersect the left edge of the parallelogram and its imaginary extension along the vector from the lower left vertex to the lower right vertex. The end result of this whole process, shown in Figure 6.11, is the creation of the basic tile for the wallpaper pattern, namely, the seahorse.

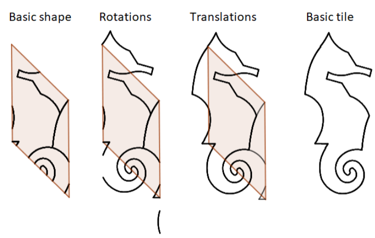


FIGURE 6.11. Screen shots that illustrate the creation of the basic tile (a seahorse) for the wallpaper pattern from an initial shape through a two-step procedure involving rotation and translation.

The inquiry is directed toward understanding the basic tile construction used by Escher in his designs (cf., Schattschneider, 2010). It is followed by the generation of parts of the wallpaper pattern by repeatedly applying generators of the matching symmetry group. In both phases of the inquiry, the dynamic nature of GEOGEBRA helps students discover what the movement of the sliders actually means in terms of geometrical changes in the plane. Focus is in both phases on conceptualisation.

For better understanding of techniques to generate a basic tile and a wallpaper pattern from this tile students need to investigate more examples of wallpaper designs. For this purpose, the designers of the tasks have created for each remaining symmetry group a dedicated microworld that allows students to go through various stages of this process by checking options in the applet. Figure 6.12 shows the applet for investigating the symmetry group labelled p6, connected to Escher's tiling Flying Fish, No 99.

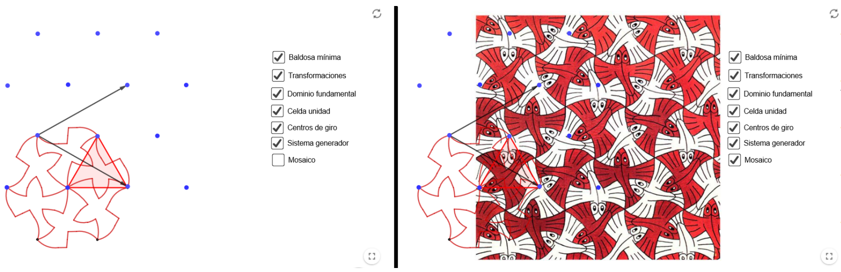


FIGURE 6.12. Screen shots illustrating the creation of the wallpaper pattern of type cmm connected to Escher's tiling Dragonfiles, No 13.

The conclusion and discussion phase of the teaching unit is a guided inquiry activity in which students can use the full toolbar to complete a wallpaper pattern of type p6m with all elements for creation of this tiling (reflection axes, rotation centres, translation vectors, ...) already present in the applet; see Figure 6.13,

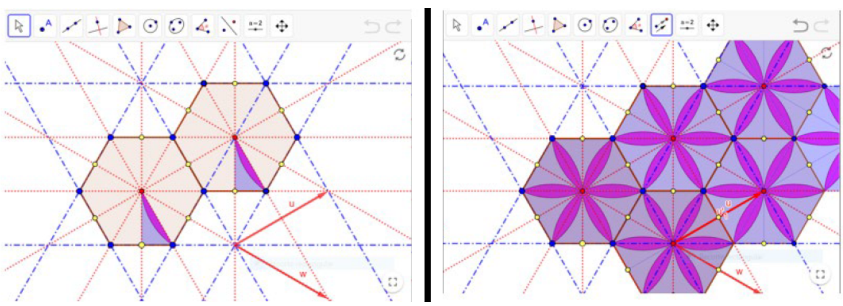


FIGURE 6.13. Screen shots illustrating the creation of the wallpaper pattern of type p6m using the complete functionality of GEOGEBRA given all elements needed for the creation of the tiling.

This progressive introduction to the use of GEOGEBRA from dedicated microworld to a dynamic mathematics environment with all tools available is a deliberate choice of the task designers. They do this because in past research studies they have experienced that the wider diversity of approaches among students to explore configurations and discover new geometric properties via GEOGEBRA is accompanied by an increase in complexity of integrating technology into the classroom. Lecturer should take into account the conditions of learning mathematics with GEOGEBRA and pay attention to a genesis of and transition between figural, instrumental, and discursive reasoning (Gómez-Chacón & Kuzniak, 2015; Gómez-Chacón et al., 2016). Task designers could help lecturers by designing effective teaching and learning paths with tasks that promote, support and sustain student inquiry. The PLATINUM project is built on the idea that the best way for lecturers and task designers to achieve these goals is to work together in a community of inquiry.

Step by step exposure of students to the full power of a dynamic mathematics environment is one of strategies that have been found effective in promoting inquiry. Another one is the inclusion of opportunities for exploration of mathematical ideas by students in learning paths, that is, by inclusion of student activities in which students pursue conceptual understanding of mathematics by posing and answering questions as they do mathematical experiments, develop strategies, make conjectures, and try to find evidence. The teaching unit of UCM partners contains plenty of such tasks and follows the strategy of gradually exploring more complex situations through GeoGebra applets, ending with a more open inquiry. The task sequence could still be extended with activities in which students create their own basic tiles for own designs of wallpaper patterns. This would be a fun challenge for students with artistic talents.

Dynamic mathematics environments such as GeoGebra have also been found effective in promoting student inquiry by dynamically linking multiple representations of mathematics objects. Part of mathematics literacy, and more generally scientific literacy, is that one has developed representational fluency. Sandoval et al. (2000, p. 6) provide the following comprehensive definition of representational fluency:

> We view *representational fluency* as being able to interpret and construct various disciplinary representations, and to be able to move between representations appropriately. This includes knowing what particular representations are able to illustrate or explain, and to be able to use representations as justifications for other claims. This also includes an ability to link multiple representations in meaningful ways.

Mathematicians and scientists often use multiple representations because

- different kinds of information can be conveyed with specific types of representations (e.g., phenomena with simulations, animations, or video clips);
- interaction with multiple representations supports various ideas, strategies, and processes in problem solving;
- different representations of a problem are seldom equivalent computationally, even when they contain equivalent information; and
- use of multiple representations promotes deeper and general understanding.

We concur with Kaput (1992, pp. 533–543) that computer technology, through the dynamic linking of representations and immediate feedback, can assist students in their learning process from concrete experiences to ever more abstract objects and relationships of more advanced mathematics and science, and can support visualisation and experimentation with aspects of investigated phenomena. Ainsworth (2008) summarises a number of heuristics that could be used to guide design of effective multi-representational systems:[13]

- minimise the number of representations employed and avoid too similar representations (the coherence and redundancy principle);
- carefully assess the skills and experiences of the intended learners in order to decide on support of constraining representations to stop misinterpretation of unfamiliar representations, and to avoid unnecessary constraining representations (pre-training principle);

[13] Between brackets we place labels of the connected principle(s) of multimedia learning listed by Mayer (2020).

- select an ordering and sequencing of representations that maximises their benefits by allowing learners to gain knowledge and confidence with fewer representations before introducing more (segmenting principle);
- consider extra support like help files, instructional movies, exercises, and placement of related representations close to one another on the computer screen, to help learners overcome the cognitive tasks associated with learning with multiple representations (guided activity principle, worked-out example principle, segmenting principle, modality principles, navigation principles, spatial and temporal contiguity principle).

Several PLATINUM partners have done their best to use these design principles; you may recognise them in the described teaching unit about isometries and tesselation of the UCM partners or in the teaching unit on complex number arithmetic of the LU partners with AUTOGRAPH files that are kept as simple as possible. Figure 6.14 shows a GEOGEBRA applet used by UvA partners to illustrate how the phase plot of a parametrised differential equation depends on the value of the bifurcation parameter and what information is actually presented in the bifurcation diagram. Students (or the teacher in a lecture) drag the triangle along the axis for the bifurcation parameter, observe what happens on both sides of the applet, and draw conclusions (perhaps after first using Mason's 'say what you see' tactic). The GEOGEBRA applet is designed to be as simple as possible, with no redundant information present, and with the multiple representations close to each other to make it easier to observe changes in linked representations. In other words, principles of multimedia learning are applied.

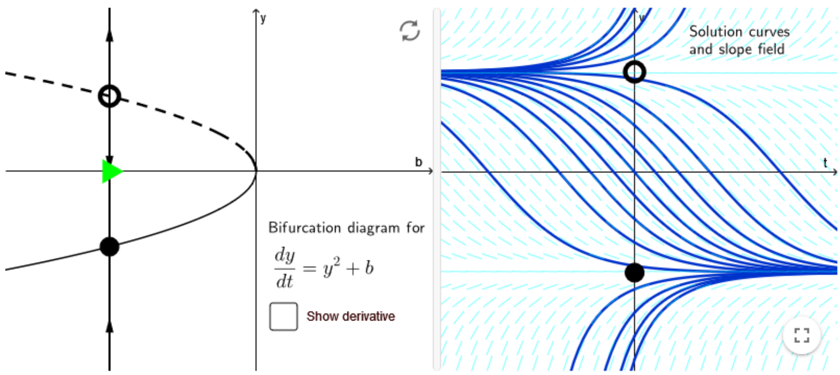


FIGURE 6.14. Screen shot of a GEOGEBRA applet used by UvA partners to connect a bifurcation diagram with changes in phase plots of a differential equation with a bifurcation parameter. Dragging the icon for the bifurcation parameter changes both sides of the applet

6.6. Guiding Design Principles Identified in PLATINUM

As we have noted before in Section 6.3 and can also be read in Chapter 2 of the book, there exist many views on inquiry-based mathematics education. Therefore it comes to no surprise that there also exist many views on what makes a good inquiry-based task for students. The examples shown in this chapter and the case studies in Part 3 of this book illustrate a great variety of inquiry-based tasks. Yet some common characteristics can be distinguished in the PLATINUM inquiry-based tasks (see also Jaworski, 2015). They

- provide easy access to mathematical ideas;
- are inclusive in the sense that they enable everyone to make a start and inspire engagement by all;
- provide opportunity to ask questions, solve problems, imagine, and explore;
- encourage discussion and reasoning;
- encourage student centrality/ownership in/of the mathematics; and
- promote mathematical thinking.

The role of lecturers and teaching assistants while a small number of students work on inquiry-based tasks can be characterised by words like

- circulating and listening;
- asking and encouraging students to ask questions;
- encouraging dialogue and/or debate;
- fostering reasoning; and
- prompting and challenging.

These ways of working are a big challenge for lecturers with large numbers of students. PLATINUM partners have in these cases often used lectures to plant seeds for student inquiry into a mathematical concepts by whole classroom discussions in which students were invited to express their ideas developed in small group work with neighbours in the lecture room. Hereafter students could dive more into the inquiry in tutorials with smaller number of students. The teaching unit about data-driven numerical differentiation presented in Section 6.5 is a good example of this approach.

The value of using ICT in mathematics education and in particular in student inquiry is manyfold. Like van Joolingen and Zacharia (2009) we distinguish the following ingredients of computer-based inquiry activities:

- *a mission for inquiry* that introduces students to a domain of knowledge or subject of inquiry;
- *a source of information for inquiry* that allows students to extract relevant data needed for cognitive growth;
- *tools for expressing knowledge* in external forms;
- *cognitive and social scaffolds* to overcome the paradox that in order to learn through inquiry, one needs the abilities that are acquired through the learning itself.

In the design and implementation of ICT-enhanced inquiry activities goes much thinking and trying-out to the above ingredients.

Tasks in the orientation phase of student inquiry are designed to activate students' prior mathematical and disciplinary knowledge, to raise interest in the subject or show the relevance for the discipline, and to relate the mission for inquiry to the students' background (e.g., skills, culture, and language).

The subject of an inquiry activity is a source of information that allows students to extract relevant data. Students can obtain data from microworlds (like the GeoGebra applets of the UCM partners or the Autograph files of the LU partners discussed in the previous section), data logging tools (Heck, 2012), and from modelling and simulation environments (like the use of Rstudio to explore dynamic systems in the UvA case study presented in Chapter 12), to mention a few. Information sources play a role in three inquiry phases of the framework of Pedaste et al. (2015), which is used by PLATINUM partners to describe modelling activities (see Chapter 8): orientation, conceptualisation, and investigation. In the orientation and conceptualisation phases data are needed to shape one's initial ideas. In the investigation phase data are needed to test and deepen ideas.

In the conceptualisation, investigation, conclusion and discussion phases of inquiry-based learning (Pedaste et al., 2015) one needs tools to provide the means for representing, processing, and analysing new data or information. How could students otherwise explain and justify their mathematical thinking, their solution strategies, and actions in open-ended activities. These tools can be mathematical representations invented or co-created by students as in inquiry-oriented mathematics education (Kuster et al., 2018) or more widely-used standard mathematical representations. They can also be mathematical constructions created in dynamic mathematics software environments like GeoGebra, results of computer-based modelling and simulations, a spreadsheet, report or presentation written with office tools, a computer-aided form of evidence, and so on. Thus, ICT offers opportunities for mediating the learning activities in which students engage (cf., Sfard & McClain, 2002).

The paradox that in order to learn through inquiry, one needs skills that are acquired through the learning itself, is similar to what is called the *learning paradox* (Bereiter, 1985). In ICT-enhanced teaching and learning of mathematics it means that tools enable, mediate and shape mathematical thinking, while being themselves, at least to some extent, a product of these processes. An instrumental approach to digital tool use in mathematics education (Trouche, 2020a,b) is one of the theoretical frameworks developed to address the problems that may arise when one starts to use a ready-made computer tool and explains the importance of aligning techniques that emerge in problem situation with the techniques available in the computer tool. The UvA partners have used this framework to understand the difficulties with programming in R and working with RStudio of their students, and to make improvements in their instructional materials (see Chapter 12). They use cognitive scaffolds to structure R-based tasks, and they give hints and supporting information for these tasks. But such cognitive scaffolds can also be provided to students in other computer-based inquiry activities. In addition, social scaffolds can provide students with means for coordinating and streamlining collaboration with others, such as tools to visualise contributions to a shared knowledge building process, concept maps in the orientation phase of a student inquiry, a shared use of a glossary, a teacher-led classroom discussion of mathematics with a digital whiteboard for notes, figures, or mathematical representations. In case studies described in Part 3 of this book one can find accounts of classroom discussions with students during inquiry activities.

6.7. Accessibility of Teaching Units for Students With Identified Needs

One of the goals of Intellectual Output 3 of the PLATINUM project is the exploration of possibilities to make teaching units accessible for students with needs. We refer to Chapter 4 for an introduction to teaching and learning of students with identified needs. It also introduces the principles of Universal Design, a methodology adopted by PLATINUM partners to strive for an inclusive learning environment reaching the needs of as many students as possible. These principles have been worked out for an educational context as Universal Design for Learning (UDL) and general UDL guidelines are presented in Section 4.6. Below we look at how the UDL principles have guided PLATINUM partners in the design of inquiry tasks.

The first UDL principle is the use of multiple means of representation (not to be mixed up with the notion of multiple representation). Students differ in the ways that they perceive and comprehend information presented to them. At the extreme are students with impairments (e.g., those who are blind or deaf), for whom some forms of presentation are completely inaccessible. In task design one could spend time and thought on how to adapt an inquiry task for students with such identified needs. For example, the dice sheet in the teaching unit of the MU partners cannot be used by visually strongly impaired students, but the information on the sheet could also be given in the form of a table with pairs of numbers that represent the number of dots on each dice. Such a table can be processed by a screen reader and transformed to speech output or brailled. More prevalent are students who, because of their particular profile of perceptual or cognitive strengths and deficits, find information in some formats much more accessible than others (e.g., students with dyslexia, aphasia, or mental retardation). Students coming from different cultural backgrounds and with native languages different from the instructional language used can have difficulty accessing information when words and symbols are not clearly defined. To best support all students, teaching units should include definitions of all requisite variables, symbols, and vocabulary. Certainly in the field of mathematics there is beside convention also much ambiguity in mathematical representations, and one can be best be open to students about this and emphasise that this is also a strong point of mathematical language. Anyway, the first principle reflects the fact that there is no one way of presenting information or transferring knowledge that is optimal for all students. Multiple means of representation are key. UvA partners (see Chapter 12) have for example provided several options for perception and comprehension in their instructional materials: all video clips taken from the UK Mathcentre and used in the online instructional materials offer closed captioning; GeoGebra applets can be reset and maximised to fill the entire screen; chapters with background knowledge such as expected prior mathematical knowledge are online available in the course material and students can practise herein skills that they were supposed to possess already; page layout includes highlighting of key words, framing of important statements and randomised examples, and hiding/opening of extra information. But in the end, multiple means of representation is not just a matter of design of instructional materials. Lecturers also play a role herein by the way they highlight critical features, emphasise big ideas, connect new information to prior knowledge, and so forth. They can lead a whole class discussion before students work through an inquiry activity to activate prior knowledge

The second useful UDL principle is the use of multiple means of action and expression. Students differ in the ways they can navigate a learning environment and express what they know. Students do not share the same capacities for action within or across domains of knowledge. Some students have specific motor disabilities (e.g., cerebral palsy) that limit the kinds of physical actions they can take, as well as the kinds of tools that they can use to respond to or construct knowledge. Other students lack the strategic and organisational abilities required to achieve long-term goals in an inquiry (e.g., students with executive function disorders or ADD/ADHD). Moreover, many students can express themselves much more skilfully in one medium than in another (using drawing tools as opposed to writing and reading print, for example). Therefore, in task design one has to make sure that there are alternatives for students' means of expression or that one maximises the accessibility of tools. For example, the UvA partners explain in their case study in Chapter 12 how they pay attention to these aspects in the design of their ICT tools. But scaffolds and supports at university level can also include optional readings, i.e., readings providing either background information or more advanced discussion of course topics, to address students with different levels of prior knowledge. Support of student planning and strategy development can be incorporated in tasks by adding questions like "Stop and think," "Make a guess," "Verify your answers," "Look for another possibility." "Give an example," and "Explain your reasoning." In terms of Mason's framework (2002) one adds questions that trigger students innate powers of mathematical thinking and doing. But like before, multiple means of expression is not just a matter of design of instructional materials. Lecturers also play a role herein by the way they select multiple media for communication, how they guide appropriate goal-setting in an inquiry activity, how they manage information and resources, what tools they select for students to use, and so forth.

The third UDL principle is the use of multiple means of engagement. Students differ markedly in the ways in which they are engaged or motivated to learn. Some students are engaged by risk and challenge, while others seek safety and support. Some are attracted to dynamic social forms of learning and to collaboration with peers, and others shy away and prefer to work on their own. There is no single means of engaging students that will be optimal across the diversity that exists. Moreover, not all students are engaged by the same extrinsic rewards or conditions, nor do they develop intrinsic motivation along the same path. Therefore, alternative means of engagement are critical. In the design of an inquiry task, one can provide options for sustaining effort and persistence such as varying demands and resources to optimise challenge, fostering collaboration and community, clarifying expectations and structuring of group work, and increasing mastery-oriented feedback. In the design of UvA courses that use SOWISO as environment for learning, practising and assessing mathematics (Heck, 2017) increased mastery-oriented feedback is realised by providing students always randomised exercises with automated feedback. But often it is also an option to make a task more engagement-neutral. For example, in the numerical differentiation task of UvA partners shown in Figure 6.5, in the task sequence about rules of probability of MU partners shown in Figure 6.7, and in the task sequence about complex number arithmetic of LU partner shown in Figure 6.8, no words are spent on whether these are individual tasks or small group tasks. Although the task designers in PLATINUM may have thoughts about and suggestions for learning arrangements and may have specified these in documentation of the teaching unit, the decision on how to engage students is in the discussed case left to the lecturer who wants to use these tasks with her/his students.

Because more and more instructional materials become web-based and contain digital contents, task designers better look at the basic principles of web accessibility made up by the World Wide Web Consortium. This consortium organises a wide variety of recommendations for making web-content more accessible for people with disabilities (World Wide Web Consortium, 2018). Although these guidelines are made for design of web pages, they can also be generally applied to the design of any digital content (e.g., GeoGebra applets, simulation environments, etc.).

Multiple studies (cf., Scanlon et al., 2021, plus references herein) show that there is a world to win because many webpages used in higher education still have numerous accessibility errors and are not compliant with current Web Content Accessibility Guidelines. We expect the same for digital content in general. This is not because of unwillingness of authors to make their webpages or digital content more accessible, but is caused by lack of knowledge, unfamiliarity with principles of multimedia learning, and/or insufficient time or effort to pay enough attention to accessibility. The situation is not different for the use of Universal Design for Learning: multiple studies (cf., Schreffler et al., 2019, plus references herein) show that Universal Design for Learning is still not widely used in postsecondary STEM education after the Center for Applied Special Technology[14] (CAST) introduced its first UDL Institute for educators in 1998.

We give an example from the PLATINUM project to illustrate how UDL principles can help task designers change an existing inquiry task and make it more accessible

[14] `www.cast.org`

for students with identified needs. Figure 6.15 shows a new version of the first task in the complex number arithmetic teaching unit developed by LU partners after applying some UDL principles. Looking back at the original task in Figure 6.8, there are many instructions divided over six subtasks; they are very detailed and use a lot of words. This was also reflected in the feedback from students using the task. Dyslexic students told that they lost track during their work. UDL principles and guidelines are of help here. The first thing one could do is reduce the number of instructions and length of text so that it is shorter, there is less to read, and it seems that there is less to do, even though the task overall has not changed (only the task presentation has changed).

The next improvement is the addition of approximate times, for each subtask and the task overall. This helps students with autism spectrum disorder including Asperger syndrome, who need a bit more structure and like to know some boundaries in the time to spend on tasks; otherwise they might end up spending too much time. But it also helps students in general, because managing the time in an activity is something many first-year students still have to get used to. Giving a time limit for a task one helps students better understand how far to take the task. But setting time limits for the subtasks and the task overall also helps a task designer or lecturer think about how realistic the demands on students are given the time constraints of study.

A further improvement is adding colours to the variables and mathematical formulas in the instructions and letting them match to the colours in the AUTOGRAPH files. It is often helpful for dyslexic students to keep track of their work. But the same holds for students in general: adding colouring may help reduce cognitive load while extracting information from multiple linked representations (cf., Ainsworth, 2008).

New Task 1: less wordy, with times and colours

Task 1: (Total time 15-20 mins.)

Open the AUTOGRAPH file *Task 1*.

There are three complex numbers labelled z_1, z_2 and z.
z_1 is to be kept fixed while z_2 and z can be moved.
Select z_2 and move it until z reaches the position $6 + 5j$.

(a) What complex number is z_2?
Right click and "Unhide All" to check your answer. (2–3 mins.)

(b) What is the geometrical relationship between z_1, z_2 and z?
(2–3 mins.)

(c) Now calculate by hand: With $z_1 = -3 + j$ and $z = 6 + 5j$, find z_2 such that $z_1 + z_2 = z$. (2–3 mins.)

(d) Re-load *Task 1*. Move z_2 around the screen and notice how z changes. Describe the position of z in relation to z_1 and z_2. (5 mins.)

(e) Explore this relationship. Move z_1 and z_2 to different locations but make sure that z still ends up being $6 + 5j$. Does what you thought in (d) still hold? (5 mins.)

FIGURE 6.15. Instructions in Task 1 about addition of complex numbers, with the same AUTOGRAPH files as in Figure 6.8, after application of some UDL principles (coloured version in the ebook).

6.8. Concluding Remarks

As was noted before and also becomes clear when reading the second chapter of this book and the case studies in Part 3, there is no unique view on inquiry-based mathematics education (IBME). But the following broad conceptualisation of IBME of Artigue and Blomhøj (2013, p. 808) covers the perspectives of PLATINUM partners:

> An educational perspective which aims to offer students the opportunity to experience how mathematical knowledge can be meaningfully developed. Thus, IBME becomes a powerful means of action, through personal and collective attempts at answering significant questions, making these experiences not just anecdotal but inspiring and structuring for the entire educational enterprise. As for IBSE,[15] inquiry-based practices in mathematics involve diverse forms of activities combined in inquiry processes: elaborating questions; problem solving; modelling and mathematising; searching for resources and ideas; exploring; analysing documents and data; experimenting; conjecturing; testing, explaining, reasoning, arguing and proving; defining and structuring; connecting, representing and communicating. These actions contribute to the students' knowledge and competences, but also to the formation of habits of mind for inquiry.

Artigue and Blomhøj (2013, p. 797) relate these actions to processes of inquiry of mathematicians and scientist:

> Inquiry-based pedagogy can be defined loosely as a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work.

As we have seen, various theoretical frameworks support the conceptualisation of IBME and its implementation in practice. This diversity in the conceptualisation of IBME and supportive framework explains the diversity in the teaching units developed by PLATINUM partners. But they have one thing in common: all have been designed to promote conceptual understanding of mathematics through student inquiry. This means that in all teaching units the purpose of inquiry is to engage students deeply with concepts that they should learn or develop, in contrast with procedural learning or learning by rote. The concepts with which the students engage are already well-known and valued in mathematics and science, and have become essential ingredients of mathematical literacy. This contrasts with the purpose of inquiry for research mathematicians and scientists: they engage deeply with concepts to create *new* knowledge in their field of interest. Levy and Petrulis (2012) also distinguish between these purposes of inquiry and refer to them as *inquiry for learning*, when one explores what is already known, and *inquiry for knowledge building*, when the purpose is to build new knowledge. Most PLATINUM teaching units are aligned with inquiry for learning, in the form of guided or structured inquiry activities in which the lecturer acts as a facilitator of learning rather than as a source of information.

In addition, many PLATINUM teaching units have in common the use of ICT in inquiry activities. This, at first sight, is not surprising: mathematicians and scientists use ICT in inquiry and thus, if the goal is to let students work in ways similar as these professionals do, it is natural to let students use ICT as well. But there is an important difference: mathematicians and scientists use very sophisticated ICT tools that require deep knowledge of mathematics and their scientific discipline in order to use the tools successfully; most students lack the required mathematical and scientific knowledge and therefore need simpler ICT tools or a learning path for using the more sophisticated tools. The designers of tasks and teaching units in PLATINUM often use dynamic mathematics environments like GeoGebra and Autograph to create for their students more dedicated and simpler tools for inquiry-based learning.

Important to the design of effective inquiry tasks and teaching units are the *three-layer model of inquiry* outlined in Chapter 2 (cf., Jaworski, 2019) and the notion of *community of inquiry* (CoI). Designs of mathematical activities for student inquiry improve when those involved have inquired into

[15]IBSE is an acronym of inquiry-based science education.

- the mathematical concepts with which students are supposed to engage in an inquiry way,
- IBME approaches to teaching and learning of the mathematical concepts, and into
- research findings of educators about IBME.

These three types of inquiry are often too much work for a single person and a community of inquiry is needed. Ideally such a community of inquiry comprised of different members in the field (e.g., discipline-based and/or general educational researchers, specialists in supporting students with identified needs, educational technologists, experts in mathematics and/or the field of application, students, etc.) so that shaping and implementing ideas for inquiry tasks can be taken to a higher level through collaboration of members of a CoI. Effort in task design is more sustainable when working in a team.

Sustainability of task design is fostered by documenting the work. Not only is documentation important for designers to keep track of discussions within the team and of design and implementation choices made, but it is important also for other lecturers who want to use or adapt tasks, or who simply want to be informed or inspired. For this reason we have included in this chapter tasks or task sequences developed by PLATINUM partners that exemplify design processes. The case studies presented in Part 3 are more detailed accounts of the partners' explorations of IBME at university level, and of their creation and use of teaching units for inquiry by their students. We hope that the case studies and this chapter on the design of inquiry activities inspire university lecturers to undertake similar explorations of IBME in their own practice.

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CHAPTER 7

Methods and Materials for Professional Development of Lecturers

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7.1. Introduction

Traditional lecturing in universities is still a common teaching practice, although research has shown that lecturing on its own is often not sufficient and leads under the existing examination conditions to surface learning (Biggs, 2003; Freeman et al., 2014). In many science and engineering programs, mathematics is still learned mainly procedurally instead of having a purposeful balance between procedural and conceptual learning (Mason et al., 2010). Occasionally there is the belief that being able to teach mathematics is automatically acquired along with years of teaching (Chalmers & Gardiner, 2015) and the unsuccessful learning results are more or less the necessary consequence of untalented students. In the professional development programmes that are organised together for university lecturers from different disciplines, there is usually no specific focus on mathematics education at university level. Pritchard (2010) argues that lecturing in mathematics has three functions that should be considered: (1) communicating information, (2) modelling problem solving including heuristic reasoning, and (3) motivating students. Mason and Johnston-Wilder (2006) point to the great importance of students' active learning involvement in the learning of mathematics and the important role of learning tasks that initiate mathematically fruitful activities, stimulate student involvement and support the development of mathematical thinking. Inquiry could be understood as a form of collective intellectual engagement. It intends to help students to gain a deeper understanding by recognising problems, searching for answers on their own, applying different heuristics and discussing them with their peers.

For collaborative inquiry with the aim of achieving a deeper understanding of mathematics, the lecturer might use appropriate students' learning tasks called IBME (Inquiry-Based Mathematics Education) tasks. Learning what a good IBME task is and how a lecturer should design and apply it in his or her class is therefore very important. But there is still practically very little or no opportunity to learn how to do this. Reflection is an indispensable element for good education also specific in university mathematics teaching practice. Supporting lecturers in their reflection on IBME tasks is very important as the explorations and discussions possibly develop their critical thinking. Moreover, reflection and a collegiate approach can support lecturers in their professional development towards IBME.

In the following section, we will first outline how PLATINUM Professional Development is presented and framed from the project's point of view. In doing so, we refer to the three-layer model that is introduced and explained in Chapter 2. Against this background, we then describe the concept and implementation of three professionalisation workshops organised in the project. These took place in Hannover, Agder, and Madrid, and pursued partly common goals, but also specific goals adapted to local conditions. Finally, we summarise the respective experiences and conclude the chapter with a discussion of some consequences.

7.2. Professional Development in IBME

Co-learning in Communities of Inquiry effectively supports lecturers in IBME and fosters their professional development in teaching mathematics (Goodchild et al., 2013). This way teaching of mathematics at university level supports the aim to achieve students' conceptual learning of mathematics. The theoretical model of IBME in higher education by (Jaworski, 2006, 2019) introduces three levels which all approach teaching and learning through the developmental principles and the interaction (see Figure 7.1).

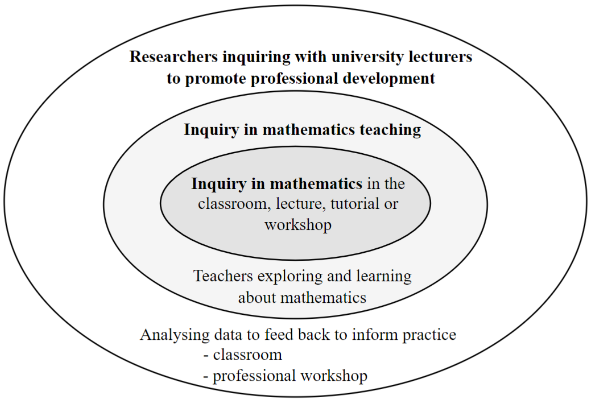

FIGURE 7.1. Three-layer model of inquiry regarding professional development.

The first layer describes the inquiry in mathematics that is carried out by students and their teacher in the classroom. Here interactions student–teacher and student–material (task) are essential. In the second layer, teachers reflect on the process in the first layer and the learning tasks are designed and adapted based on the experiences in the classroom (first layer). The lecturers discuss teaching and learning in a safe/intimate environment, give and receive feedback on the design of the learning tasks and their implementation. In the second layer, the development of the lecturers takes place in a co-learning process often together with co-creation of IBME tasks.

Next to the lecturers, in this layer more experienced members and invited experts or didacticians can promote learning and support professional development of lecturers. Indirectly these activities could improve the quality of teaching and learning in the first layer. The third layer is the evidence based layer. In this layer the didacticians and educational researchers reflect together with the lecturers on the developmental process that takes place in the second level, which performs the developmental research. From this level, they also support and intend to stimulate the reflective and evidence-informed teaching attitude of the lecturers and promote different categories of reflections on teaching and learning process. The boundaries of the first and second and second and third layers are crucial nodes in the development process because they are the communication and critical reflection / feedback nodes connecting a teacher and students, and teachers and support staff and peers, respectively. Participation in the IBME Community of Inquiry supports lecturers in their professional development.

One of the goals of the PLATINUM project is to develop and pilot a platform for the professional developments of mathematics lectures on a regular basis in the format of a "hands-on" workshop. The need for such training platform reflects the current situation in university mathematics education: mathematics lecturers often have limited or no access to the information about contemporary pedagogic and didactic methods which, in turn, contributes to the lack of motivation to use them. The research confirms that the knowledge of teaching methods, that is, a command of various teaching methods and the apprehension of when and how to apply each method has a positive impact on students' achievements (Voss et al., 2011). Looking for the possibilities of introducing a larger community of university mathematics teachers to IBME, the PLATINUM project has included in its structure the development of this topics Intellectional Output by offering methods and materials for professions development of lecturers (see the description of IO4 in Section 2.5), and PLATINUM team members piloted workshops for local communities in three countries, namely Germany, Norway, and Spain.

Three professional development workshops on Inquiry Based Mathematics Education for university lecturers of mathematics are described in this chapter. The workshops were organised at different universities in different countries. Interestingly, all three workshops focused on task development, albeit in some different ways. This is not really surprising, because tasks are the central activity drivers for students. Tasks can also be changed without fundamentally modifying the rest of a course, such as the lecture and tutorial structure, or its basic pedagogy. The change of the course organisation is hardly possible under the legal framework conditions, especially the stipulations in examination and study regulations. And the change of the basic pedagogy would require an effort that most teachers cannot or do not want to cope with due to the already heavy workload. In view of this, the (re-)designing of tasks represents a rather local change in a course. Anyway, this raises the question of how or in which directions tasks should be developed or modified, in essence the question of what makes a task an IBME task.

The workshops dealt with this question in different ways. In Hannover, dimensions and qualities of IBME tasks were introduced in advance and illustrated with examples (see, for example, Table 7.1). In Agder, on the other hand, IBME was presented as a teaching strategy together with its goals and, against this background, a reflection on suitable tasks was initiated. The Madrid workshop similarly focused on the inquiry-based task means in mathematics and the design of inquiry pathways with complex tasks. Thus, in all workshops, university mathematics lecturers were learning how to develop IBME tasks. In the final part of this chapter we analyse the workshops from the perspective of the three-layer developmental model and in view of the local contexts.

7.3. IBME Workshops

This section describes the workshops in Hannover, Agder and Madrid. In each case, we first go into the local institutional context, goals and the workshop concept based on them. Then the organisation and central contents are described by way of example. In a concluding section, the respective experiences and results are discussed.

7.3.1. IBME Workshop in Hannover, Germany.

Institutional context and goals. In Germany, as in many other countries, the content and examination requirements of mathematics teaching are largely fixed, both in the mathematics courses for majors and in service lectures for engineering courses

(Bosch et al., 2021). Introductory courses and their organisation are often characterised by high numbers of students, usually 200 to 600, and by numerous parallel exercise groups led by research assistants or tutors, each generally with more than 30 students, also because resources are limited. In addition, substantial modifications to the content, such as a reduction in breadth in favour of depth, are difficult, because subsequent courses in all degree programmes, such as advanced courses in mathematics or theory-based engineering courses for example, require knowledge of the topics dealt with. Altogether, this considerably restricts the possibilities of making changes to the methodological-didactical and content-related organisation of teaching. Given that the scope for design changes is considered to be limited, it is not surprising that general training opportunities in the didactics of higher education are hardly ever used by mathematics lecturers, and are often experienced as not very appropriate for introductory courses. Another reason is that the general lecturers' trainings do not really address mathematics teaching, which is different in many aspects, also because the nature of the content and the teaching objectives differ.

The concept of the PLATINUM workshop held in Hannover was developed with these boundary conditions in mind. Possibilities for the further development of teaching were seen particularly at the level of tasks. Modified tasks can easily replace previous tasks in regular teaching if they meet the curricular requirements. This should make it possible to address IBME aspects within the context of existing course structures, i.e., without extensive and usually hardly realisable new concepts for courses and their contents. Without neglecting the importance of pedagogical and institutional-societal constraints for the implementation of IBME, our approach was based on the assumption that "...the deliberate neglecting of topic-related aspects can lead to shortening, but above all to an underestimation of the teachers' methodological freedom to act and their subject-specific didactic decisions" (Reichel, 1995, p. 180).[1] Thus, the aim of the workshop was to provide teachers with resources to further develop existing materials, especially tasks, with regard to IBME on the basis of subject-specific analyses.

With a view to the workshop to be developed, a relevant representative of the Centre for Higher Education Didactics at the University of Hannover was invited to give a lecture in advance, in which, on the one hand, existing formats of higher education didactic training at the University Hannover were presented and, on the other hand, subject-specific possibilities for expansions were discussed. As a result, it was found that units in which tasks, their objectives and design, are discussed could be integrated into existing training programmes without any problems, whereby one could even consider that lecturers bring along their own tasks.

The workshop in Hannover was therefore aimed at providing lecturers with an offer to support them in developing tasks in the perspective of IBME or in modifying existing tasks with a view to this. The reflection of tasks, their potentials and goals was anyway a current issue in the department, since several projects were already developing digital courses to provide extra support for students in their first year of study, and these consist to a large extent of STACK tasks.[2]

[1] "...dass das bewußte Ausblenden stoffbezogener Aspekte zu *Verkürzungen* führen kann, vor allem aber zu einer *Unterschätzung* fachmethodischer Spiel- und Handlungsfreiräume der Lehrer und ihrer fachdidaktischen Entscheidungen." [Emphases as in the original]

[2] STACK is an open-source online assessment system for mathematics and STEM, which is available for Moodle, ILIAS and as an integration through LTI. It was introduced by Chris Sangwin (University of Edinburgh), see also `https://stack-assessment.org/`.

Organisation and contents. In accordance with the institutional context and mentioned goals the workshop was structured as follows:

(1) In an introduction, central ideas of IBME were first presented. These were made concrete with regard to dimensions of tasks. A tool in the form of a table (see Table7.1) was developed and made available to encourage reflections on the IBME character of a task and to guide the further process. Of course, it must always be taken into account that a task is not or is not IBME per se, but that this must always be assessed in the overall context of a course, its contents and objectives.
(2) In a subsequent second part, concrete example tasks were discussed. The starting point in each case was formed by conventional tasks from courses, which were further developed with IBME in mind. The table presented in the introduction was used to round off each of the tasks and to reflect on their IBME character.
(3) In the run-up to the workshop, the participants were asked to bring along their own tasks which they would like to discuss and, if necessary, modify. In the third part of the workshop, the tasks brought along were first presented and curricularly embedded in their respective courses. Against this background, they were then discussed with regard to further possibilities for development in the sense of IBME and, in particular, classified by means of the aforementioned table.
(4) In a final round, all participants exchanged their experiences in the context of the workshop. This naturally also involved criticism and the possibility of further development, both with regard to the organisation of the workshop and with a view to the tools provided for further development of tasks.

In the following we will go into a few details of (1) and (2). Regarding (1) we describe characteristics of inquiry-based tasks represented in the table mentioned. With regard to (2) we outline two examples that were presented at the workshop.

Characteristics of inquiry-based tasks

To identify characteristics of inquiry-based tasks we considered the following dimensions:

(a) Openness of tasks,
(b) Enabling specific inquiry strategies,
(c) Enabling discourses on techniques,
(d) Enabling inner-mathematical knowledge linking,
(e) Enabling interdisciplinary knowledge linking.

On (a): A task can be set rather open or closed. We call a task process open if it allows for different solution strategies, approaches or techniques, open-ended if it does not have a clear solution and content-open if knowledge from different areas can be profitably brought into the processing of the task.

On (b): It is addressed whether a task is (only) about applying techniques or also about conveying mathematical heuristics and developing solution strategies. Following Schoenfeld and Sloane (2016) frequently used heuristics are: draw a diagram if at all possible; examine special cases; try to simplify the problem; consider essentially equivalent problems. Regarding problem-solving cycles Mason et al. (2010) highlighted the activities: specialise, generalise, conjecture, convince.

On (c): Discourse on techniques is understood to refer to the fact that a task is not (only) about the correct application of techniques or calculations, but also about describing, discussing and questioning these techniques or calculations: What possibilities does a certain technique offer in comparison to another? When is the use of a certain technique appropriate? On (d): Inner-mathematical knowledge linking aims at connecting different mathematical concepts and overcoming the compartmentalisation of knowledge.

On (e): Interdisciplinary knowledge linking aims at connecting mathematical and extra-mathematical concepts.

We have compiled the dimensions and their aspects in a table (Table 7.1) and in this way made them available to the participants of the workshop. In view of a particular task, the table could be used to generate questions about its IBME characteristics and further development possibilities. In the next section we exemplify how we arrive at a kind of IBME profile for tasks.

Content dimension / Task format	***inquiry*-strategies** are to be learned	a **discourse on techniques** should be stimulated	**inner-mathematical knowledge linking** is to take place	**interdisciplinary knowledge** is to take place
Closed				
Process-open				
Open-ended				
Content-open				

TABLE 7.1. Table of dimensions for inquiry-based tasks.

Examples

The first example is a task developed for first-year student teachers. Its development is based on the assumption that the students know 'curve discussion' from school essentially as a computational and procedural application of criteria. Accordingly, the idea of the task is that a purely computational and procedural approach does not always lead to desired results. Instead, definitions must be consulted and applied. Although such procedures are addressed in school textbooks, they are rarely used in the classroom. Thus, the aim of this task is to problematise the fixed focus the procedural and to draw attention to definitions and concepts. The task is discussed in detail in Chapter 3 of this book.

The second example presents a somewhat more advanced task that might be given in an Analysis I course for weekly work. The basis for the development of this task was an explorative subject-specific analysis (Hochmuth, 2020) of a classical theorem by Kahane (1961). This theorem answers in the one-dimensional case for continuous functions on the interval with regard to the maximum norm the question about necessary and sufficient conditions for the convergence order $O(n^{-1})$ if one allows piecewise constant functions with at most n subdivision points, which otherwise may be chosen arbitrarily in the interval. The answer was that this exactly applies for functions of bounded variation. Obviously, Kahane's Theorem as such is too complex for an introductory course in Analysis 1. However, this does not apply to the idea and questions underlying the theorem. Accordingly, in the workshop the potential of the context was discussed in view of the objective to make basic courses more meaningful and relevant for students of mathematics. Rationales for the treatment of content in basic mathematical courses, here for example the notions of bounded variation or approximation order, are often not clear to students, but can be exemplarily explained against the background of Kahane's theorem.

With regard to functions of bounded variation typical tasks in Analysis 1 are limited to questions of the following type (Heuser, 2013): Show, that the function $g(x) = x\cos(\pi/x)$ when $x \neq 0$ and $g(0) = 0$ is continuous on $[0, 1]$, but not of bounded variation. Or: Show that the variational norm possesses the norm properties. These tasks are in no way 'bad.' They address basic techniques that students should have at one's disposal. However, the tasks are limited in terms of understanding the concepts and their meaning. So the point made here is that all tasks in Analysis 1 about bounded variation are more or less of this type.

Based on Kahane's Theorem and with a view to the intended openness of the task format various tasks can be formulated. A rather open version is represented by the following:

> Consider continuous functions on the interval $[0, 1]$. To approximate such functions, you can use piecewise constant functions on $[0, 1]$ with respect to any subdivisions. Consider the error regarding the maximum norm. How is the convergence order related to the smoothness of the function? Characterise differences or extensions to the approximation ideas known to you so far. Find contexts in which such ideas play a role (inner- and/or extra-mathematical).

The IBME profile of this task is represented in Table 7.2.

Content dimension / Task format	***inquiry*-strategies** are to be learned	a **discourse on techniques** should be stimulated	**inner-mathematical knowledge linking** is to take place	**interdisciplinary knowledge** is to take place
Closed				
Process-open		✓		✓
Open-ended	✓	✓	✓	✓
Content-open	✓	✓	✓	✓

TABLE 7.2. Table of dimensions for the IBME task related to Kahane's Theorem.

The appropriate formulation of a task is, of course, dependent on the students' level of knowledge. In (Barquero et al., 2016) problems with tasks that are to some extent too open are discussed. A task which is much more closed but still has IBME character is for example as follows:

> Given $f \in C[0, 1]$ with $f(x) = x^\alpha$ for $0 < \alpha < 1$. How well can these functions be approximated in terms of piecewise constant functions and uniform subdivisions? What is a good choice for the respective constants in this case? Why? Is there a better choice of subdivision points? Is there a fixed sequence of subdivision points with the convergence order $O(n^{-1})$? Is there a sequence of subdivisions with even faster convergence? If necessary, how must the choice of subdivision points be adapted? The respective choice of subdivision points assumes that you know $0 < \alpha < 1$. Is it possible to find a procedure that provides $O(n^{-1})$ for all $0 < \alpha < 1$? Characterise differences or extensions to the approximation ideas known to you so far. Find contexts in which such ideas play a role (inner- and/or extra-mathematical).

After the presentation of the table and the explanation of its use with examples, the participants of the workshop presented the tasks they had brought with them. These were first discussed together with regard to their IBME character and then against the background of the respective teaching-learning situation with regard to possible changes or extensions. In these lively discussions, it became apparent that the tasks brought with them, in terms of their content, already aimed at conceptual learning. However, it also turned out that the potentials of the tasks for conceptual learning could be more precisely specified in the course of the discussion about their IBME character and that various proposals for their expansion could be worked out.

In the final round, the participants rated the workshop as overall positive, especially the joint discussion based on the tasks they had brought with them. The common work on these enabled the non-didacticians to see a value in the terminology and the table. However, it also became clear that it would definitely be a problem in everyday life to think about tasks in such detail and for such a long time. Another important finding was that it is not a problem for non-didacticians to discuss and reflect on tasks from a subject-specific-didactic point of view.

Discussion of results. As a result of the workshop, a prototypical methodical procedure could be described which has proven to be successful: At the beginning there is a kind of praxeological analysis (in the sense of the Anthropological Theory of Didactics (ATD) (Chevallard, 1999)) of the task and the related mathematical domain. This focuses especially on the dialectic between technique (How?) and technology (Why?). This does not necessarily require the use of terms from the ATD. The central point here is to orient the didactic analysis of the material to the underlying questions that are answered by the specific piece of mathematics of the material. Of course, the necessary preknowledge (prerequisites), possible embeddings and references, possibilities of linkage as well as the desired knowledge should be taken into account, also with regard to its relevance. At this stage, the table for the task at hand could also be filled in. It can then be used to look for potential development needs or opportunities. In this respect it is useful to further differentiate questions which correspond to the task and the field of knowledge in a specific way. With regard to paths along which answers can potentially be developed by students, central techniques and technologies, as well as appropriate subtasks and partial solutions and the respective previous knowledge to be updated has to be worked out. In this process, it can be particularly useful to identify the semiotic and instrumental valence of used mathematical symbols and objects as possible activators of activities with regard to other objects, practices and concepts (Bosch & Chevallard, 1999). On this basis, the task can then be modified as desired and the table can be filled in again. The extension of the IBME profile can then be documented by comparing the initial with the modified table. With regard to support measures and feedback by lecturers, potential obstacles in the treatment of tasks by students should be considered with regard to necessary or possible solution steps and respective adequate support and its focus (content-related, strategic, methodological, material, impulse-giving). This leads over to didactic considerations in the narrower sense regarding the design of sequences of questions (possibly concept-maps or similar), suitable materials and media, teaching steps including their social forms as well as possibilities for diagnosis and feedback. Last but not least, expectations regarding the quality of the intended learning processes and their products has to be considered.

A final, albeit brief, look at the professionalisation workshop from the perspective of the three-layer model allows us to note the following observations: The planning and design of the workshop was done from the perspective of the third layer. Both the table and the examples presented for explanation have a theoretical background (here especially ATD, as indicated above), but this is not explicitly presented in the workshop itself. The material created against this background and the proposed procedure for modifying the tasks should be presented in such a way that they should not only be understandable and insightful for the participants of the workshop, but should also stimulate new processes of reflection regarding tasks brought along and their use in their teaching. The participants should thus be provided with new tools for processes on layer two. For this to work, it was helpful to develop an atmosphere of trust and a mutual recognition as experts. In this way, a community of inquiry could be created locally for the duration of the workshop and development could take place among both the participants and the organisers.

7.3.2. IBME Workshop in Agder, Norway.

Institutional context and goals. All newly appointed university lecturers in Norway are supposed to take a UNIPED (UNIversitets- og hogskole-PEDagogikk) course as a professional development course. The main objective of this course is to support a number of pedagogic skills required from a university lecturer (`www.uhr.no`):

- Plan and carry out teaching and supervision, both individually and in collaboration with colleagues, in a way that promotes student learning and professional development.
- Plan and implement R&D (research and development) based teaching and involve students in R&D-based learning processes.
- Select, motivate, and further develop appropriate learning activities and teaching and assessment methods in relation to academic goals and educational programs.
- Contribute to academic and pedagogical innovation through the choice of varied teaching methods that include the use of digital tools.
- Motivate personal views on learning and teaching reflecting the teacher's role.
- Analyse, prepare, and further develop course and program plans within lecturers' subject areas.
- Assess and document results from own teaching and supervision based on expectations in curricula and national frameworks for higher education.
- Collect and use feedback from students, colleagues, and society to further develop teaching and learning processes.
- Be familiar with relevant management documents related to teaching in higher education.

Norwegian universities offer UNIPED courses in different ways: for instance, the duration may vary from 100 hours to 150 hours extended over one or two semesters, the content depends on the priorities and resources of each university, but a typical UNIPED course focuses on innovative methods of teaching at university level. Usually these courses are not specific to mathematics teaching but encompass rather general topics of university education. Mathematics plays an important role in many students' study curricula and future careers; therefore, many university undergraduate and graduate programs contain mathematics courses. Teaching of mathematics focuses not only on the computational aspects but also on very important conceptual aspects which influence the choice of mathematical content, design of mathematical tasks and ways of communicating them. This brings the need to address the issues arising in relation to teaching and learning mathematics. In 2015, MatRIC, The Centre for Research, Innovation and Coordination of Mathematics Teaching at the University of Agder (UiA) in collaboration with the Norwegian University of Science and Technology (NTNU) launched a University Level Mathematics Teaching Course at NTNU complementing pedagogy courses offered by universities and university colleges. The course was designed to address the problems that university mathematics lecturers and students face: large classes (especially for engineering students), students' lack of motivation, diversity of students coming from different educational programs etc. Positive feedback from the participants indicated that such courses offer an opportunity for mathematics lecturers to grow professionally.

Organisation and contents. In order to introduce a wider audience to main ideas of inquiry-based mathematics education (IBME), PLATINUM organised a one-day workshop at the University of Agder in association with MatRIC. The workshop was organised with the following goals: to

- discuss the concept of IBME and describe the differences and commonalities in participants' views on what the IBME problem is and what it is not,
- present the IBME problems and their solutions,
- understand the ways how the non-IBME problem can become such, and to
- support and extend the network of IBME community of the University of Agder and include the participants from other Norwegian universities.

It was the second in a series of three workshops organised by PLATINUM as part of activities contributing to the Intellectual Output IO4 "Methods and Materials for Professional Development of Lecturers." The main activity of the workshop was the design, development, and piloting of activities through which (new) university teachers may be introduced to inquiry-based approach to teaching mathematics and gain insight into an inquiry-based task design, tasks structure, and their characteristics. The associated pedagogical and didactical ideas were also another focus of the event. Prospective participants who already used inquiry-based approach and tasks in their teaching were invited to bring own examples for the discussion at the workshop along with any tasks they could suggest for collaborative group discussions aimed at turning of "standard" tasks into "inquiry-oriented" ones.

In addition to local participants from UiA, Campus Kristiansand and Grimstad, colleagues from other Norwegian universities: NTNU, University of Stavanger, University of Trømso, Norwegian University of Life Sciences participated in the workshop. The event provided an opportunity to university lecturers who teach different mathematics courses in various study programs (engineering, teacher education etc.) to discuss their teaching practices together and share the experiences. The main activities of the workshop were:

- discussing the foundations of inquiry-based teaching and learning of mathematics,
- working together on mathematical problems selected from teaching units and tasks for student inquiry developed within the PLATINUM community, and
- discussing the foundations of inquiry-based teaching and learning of mathematics.

Reflecting on what has been experienced and how it can be implemented in own teaching practices, the activities in the workshop were organised in three main parts. For the first part, participants were split into small groups and offered a set of eight (proposed) inquiry-based mathematical tasks that were extracted from the contributions from several PLATINUM partners. The groups were asked to read and discuss some or all of the tasks. The purpose of the suggested discussion was to consider what makes a mathematics task an inquiry-based task. The following question were asked: What do we mean by an inquiry-based task? In what ways are they similar or different? How could you describe the inquiry nature of the task? What are the characteristics of an inquiry-based mathematical task? After the discussions in small groups all participants were invited to the general discussion in which the characteristics of an inquiry-based mathematical task were suggested. The characteristics reflected the discussion about the proposed tasks and included accessibility, openness, openness to multiple strategies, opportunity to iterate, motivation to investigate and explore, evaluation, necessity of reasoning, possibility to generalisation and specialisation, suitability for group discussion. The moderator gave the comments on the discussion pointing out that in a mathematical course it is necessary to consider the place for the following: exposition, investigation, exploration, focusing, evoking, stimulating, motivating, and problem solving.

In the second part of the workshop participants worked on inquiry-based tasks. They were asked to choose one of three options: (1) try to design your own tasks; (2) talk with others about the tasks you have brought yourselves; (3) work on the Linear Algebra or Analysis (or other) tasks, modifying them to be more inquiry-based. For Option (3), a folder was provided containing sample sets of tasks from regular mathematics courses taught at the University of Agder and at Loughborough University. These were tasks used by colleagues with their students; some were intended as inquiry-based tasks and some were not designed to be inquiry-based. Four groups were formed, one to work on Option (1), two to work on Option (2) and one to work on Option (3). The groups worked together for 45 minutes and then each group presented their work to all participants. One of the groups suggested the design of the task that could be offered to the students of a teacher training program, namely asking the students to compose an inquiry-based task.

Example of task (1). Working in groups of two or three, develop a series of realistic mathematical problems that you can meet in real life either in collective life or in your personal life. These problems should require one to create and arrange a series of quantitative information into rectangular arrangements (RA) in $n \times m$ dimensions. Both n and m should be equal to or greater than 3. Then, pose some realistic problems that can be answered by at least two of the following operations on these RAs: multiplication, finding an inverse, addition, subtraction, and multiples of the RAs. Find the answers. Another group discussed the problem brought to the workshop by one of the participants who reported that it had been tested at school.

Example of task (2) (unfolding a three-dimensional cube) Suppose that you have a cube. How many unique ways are there to unfold it? (Or how many different nets of a three-dimensional cube can be obtained?) The answer to this combinatorial problem is eleven, but this problem can be modified in various ways. The group discussed a possible modification: ask for the minimal number of adhesive flaps to be included in a net so that it can be folded and glued together to make a waterproof cube. Another possible modification is the transition from a cube to a dice. One can ask the question how many different possibilities there are for the distribution of the natural numbers ranging from one to six on the sides of a cube, that make it a dice. In a classical setting, a dice possesses the property that the sum of the natural numbers on two opposite sides is always equal to seven, but this limitation still leaves two different possibilities or orientations. This difference becomes indeed visible when comparing a European dice with an Asian one. However, this classical constraint can also be omitted. Some tasks using simulations (generated by programming tool SimReal[3] were presented. Examples included Pythagoras Theorem, exponential functions, Green's theorem, and complex numbers. Finally, in the third part, participants reflected on their experience and were asked to rate the interest and usefulness of the workshop, commenting on what was good about the day and suggesting the ways for possible improvement of future events. The survey was distributed and the responses were analysed. The reflections showed that there were discussions on

- the degree of openness on IBME-problems;
- balance of the learning goals and syllabus;
- the interplay between the problems and other classroom factors such as students' group discussions;

[3] https://grimstad.uia.no/perhh/phh/MatRIC/SimReal/Menu/Science.htm

- teacher's role in guiding and supporting students' discussions, explorations, and reasoning;
- and teacher's role in bringing students' small group work into a whole-class discussion.

One of the participants wrote: "It was very nice with these discussions. The only thing I missed, are more examples of good inquiry-based tasks. I especially need more examples of good tasks at undergraduate level." According to the PLATINUM proposal, the goal of the project is the creation of community of inquiry, and the activities on the IBME Workshop in Agder contribute to this goal in a significant way.

Discussion of results. The experiences gained by the workshop participants were reflected in the discussions of the characteristics of IBME tasks and the ways how good IBME tasks can be designed. Participants focused their attention on the goals of such tasks that should address a number of important issues: to motivate students to foster curiosity, to ask questions, to explore topics, to be engaged in active learning. The participants argued that since the IBME is based on problem-solving, such tasks must be specifically designed to support the development of deep understanding of the material and intellectual problem-solving skills. Students must be able to build up on their previous knowledge which requires an accurate assessment of such knowledge by the teacher. Indeed, the problem that can be classified as an IBME task for one group of students may not be such for another group. The level of inquiry should be also taken into consideration: there are different levels of inquiry with respect to openness, from very structured to fully open. For example, the problem in which the result is known and students need to choose and apply the correct rule or method would be of limited inquiry while the problem which is set up by students themselves and solved by using the knowledge from different areas to develop the procedure would be of high level of inquiry. To achieve the goal the teacher should be able to make the right choice of the level of inquiry with respect to the level of mathematical content and to the level of students' prior knowledge. The question was asked: How much can be removed from a student's 'agency' before a task ceases to be an inquiry task? Along with the task design the participants indicated the importance of class management in the IBME process. They named some important topics that can contribute to successful learning, such as students' collaborative study, the crucial role of students' freedom to choose the tools and direction of solving the task. The participants also attempted to define the relationship between exploration tasks, modelling tasks and inquiry tasks. They pointed out that not every modelling task is an exploration task and not every exploration task is an inquiry task. But in general there were different suggestions to define what each of these tasks is. It seems that there is no agreement in such definitions and it depends not only on the content but also on the background and preparation level of students.

7.3.3. IBME Workshop in Madrid, Spain.

Institutional context and goals. The Faculty of Mathematical Sciences has been contributing to teacher professional development for more than 60 years. Since 2007, the Extraordinary Chair UCM Miguel de Guzmán, taking into account this trajectory, has contributed with programs of continuous development of university lecturers, developing formative actions of post-graduates in Mathematical Education at local, national, and international level (Corrales & Gómez-Chacón, 2011).[4] Concerning the professional development of the mathematics lecturers, this is a great milestone in the

[4]`http://blogs.mat.ucm.es/catedramdeguzman/proyecto-novelmat/`

country. It is an exceptional case because professional development for university lecturers is usually considered as a generalised pedagogical training and it is not mediated by specific mathematical knowledge.

In our tradition in the implementation of the teacher's professional development, research and development are inseparable. In the PLATINUM Project, the development of this perspective is based on the three-layer model presented in Chapter 2 of this book (see also, Jaworski, 2019). When we want to promote teaching through inquiry-based pedagogy we consider mathematics as "a process and an experience." Knowing mathematics is equated with doing mathematics. In our group, research in mathematics education has focused on examining the characteristics of the context in which this "doing" is fostered. Designing research-based tasks requires deep analysis of the "mathematical experience" that is generated in both students and lecturers. Thus, the creation of inquiry-based activity for the students is itself an inquiry process: lecturers learn from the practices resulting from their teaching designs. In the UCM case study presented in Chapter 16, it is suggested that the research strategy followed in the professional development for the teacher is framed within Design-Based Research. The design research is characterised mainly in terms of simultaneously developing theory and designing instruction; engaging in iterative cycles of design, enactment, analysis, and revision; and performing fine-grained analyses to link processes of enactment with outcomes of interest.

Organisation and contents. In what follows we will focus on the Course proposal developed within the framework of PLATINUM. This was entitled: "Inquiry-based education in mathematics at university level: Examples for the classroom." Aims of the course are the following:

- initiation in the Inquiry-based learning approach applied to teaching and learning situations at university level;
- development of methodological skills to design inquiry tasks, and
- knowledge of resources,i.e., examples of inquiry-based projects in university mathematics teaching.

It lasted ten hours distributed in two specific and complementary parts. Each part forms a unit by itself.

First part: A workshop for five hours addressed to senior and novice lecturers. Forty-five participants attended. The group was mixed from five universities: university lecturers teaching in mathematics in the faculties of Science (Mathematics, Physics, Computer Science), engineering faculties, and faculty of Education; research assistants (PhD students in the last year who collaborate in teaching or planning to teach soon); and students in Master's degree programmes in teacher training. The contents were the following:

(1) Inquiry approach talk: Teaching and learning mathematics through inquiry.
(2) What do we mean by an inquiry-based task?
(3) Inquiry projects and tasks implemented in the classroom in Spain by the Spanish PLATINUM Group at UCM.
   (a) Escape room. Elements of ordinary differential equations;
   (b) Teaching linear algebra and video games. Affine transformations and rigid motion;
   (c) Matrix factorisation. Numerical methods (see Chapter 16).

Second part: Only addressed to those participants of the first part who are research assistant in the UCM (PhD students in the last year who collaborate in teaching or planning to teach soon). For five hours, 6 participants attended. The aim and contents of this part are to deepen different aspects of the first part and design a task or project using the inquiry approach. Taking into account the contents of the first part, this part focuses on:

- Encourage self-reflection on the teaching of the novice lecturers after receiving training in Inquiry based approach.
- Improve the knowledge of professional development reflection about level of inquiry and progressive movement to abstraction and symbolisation in affine transformations and rigid motions.

Discussion of results. Two experiences are highlighted about the characteristics of an inquiry-based mathematical task: meaning of an inquiry-based task and example of complex tasks through an inquiry project.
1. What do we mean by an inquiry-based task?
For this aim a set of eight inquiry-based mathematical tasks was prepared. This sample sets of tasks come from regular mathematics courses taught by several PLATINUM colleagues with their students; some were intended as inquiry-based tasks and some were not designed to be inquiry-based. These tasks were discussed at the workshop in Brno for the PLATINUM team and also at the workshop in Norway. Our purpose in discussing these tasks is to consider what makes any mathematics task an inquiry-based task. At the UCM Workshop the groups (composed of 5 participants) worked together for 45 minutes and then each group presented their work to all participants. After the discussions in small groups all participants were invited to the general discussion in which the characteristics of an inquiry-based mathematical task were suggested (45 minutes). At the beginning of the analysis the proposed tasks were characterised. Characteristics such as openness, motivation to investigate and explore, autonomy of students, evaluation, necessity of advance reasoning, possibility to generalisation and specialisation, suitability for group discussion. In relation to the recognition of tasks and their use in the Spanish context there were tasks whose level of knowledge was very elementary, not corresponding to the university level. This raised discussion about transferability, learning outcomes and the kind of purposes and criteria according the local syllabus. The discussion allowed us to ascertain the range of inquiries according to the profiles of students and curricula in each country and in relation to the phases of inquiry were focused on regulation and inquiry pathways in the sequences of actions by the lecturers.

Also, and not less important to consider is the conceptual model of the inquiry process. We see inquiry as having both epistemological and strategic aspects, with developments on the two fronts reinforcing one another. The epistemological point of view of mathematical knowledge is the key. The inquiry approach to mathematics is not only a pedagogical strategy. It seems pertinent to take into account some of the tensions that seem inherent and at the same time can provide us with tools for teaching. These are tensions between the development of the mind's research habits and the progression of mathematical knowledge with the necessary attention to curricular progression, the tension between the internal and external sources of mathematical activities and differences between inquiry paradigms according the mathematical field (Geometry, Calculus, etc.). In the next section we describe an example of how to make this articulation developed together with the new lecturers.

2. Design of inquiry pathways with complex tasks.
We describe the Inquiry project "Teaching linear algebra and video games." Affine transformations and rigid motions" as a professional development reference situation developed together with the participants in the workshop (novice teachers), i.e., as

a reference situation that can be developed with their future students. Below we present the project's tasks, key elements in the process of inquiry, and didactical and mathematical moments.

We take the game Silent Hunter III as a starting point. Silent Hunter III is a submarine combat simulator for PC developed by the company Ubisoft and published in March 2005, set in World War II. If you have never played the video game, you can watch the YouTube video `www.youtube.com/watch?v=2Xa4gWCHlFU` to familiarise yourself with the game. Some of the missions that take place in the video game use one of the most significant elements of a submarine, the periscope (see Figure 7.2). Using this tool, players are able to inspect a large part of the map, view enemy positions or establish safe routes for the submarine's travel.

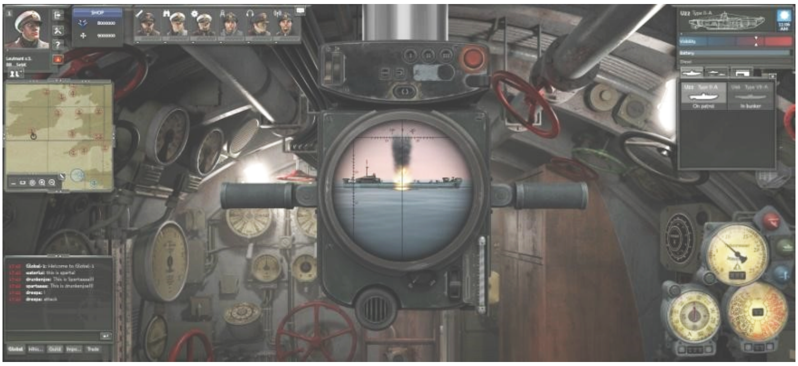

Figure 7.2. Periscope of a submarine of the video game Silent Hunter III

Task 1. What kind of mathematical tasks/activities would you propose to a first-year student of the degree in Mathematical Engineering for teaching the contents of Mathematical Elements or Linear Algebra? Describe some of them.

Task 2. With the help of dynamic software GeoGebra, create a 'hypothetically ideal' periscope that can move up and down and rotate a maximum of 45 degrees, from a starting point to any of the cardinal points. Prior to using GeoGebra to solve the problem, write down your thought process, noting steps you would take, necessary mathematical knowledge, and places where you struggled.

Task 3. If you had difficulties completing the GeoGebra construction, specify the causes: mathematical knowledge, instrumental construction with GeoGebra, etc.

Task 4. Based on the activities carried out, answer the following questions:

(a) Provide the matrix representations for the following transformations:
   (i) The z-axis rotation with angle $\alpha = \frac{\pi}{4}$;
   (ii) the translation of vector $\vec{v}$;
   (iii) the axis rotation the line r and angle β, $\beta = \frac{\pi}{6}$ (the axis rotation by angle β around the line r)
   What effect does each of them have on the periscope?

(b) Is it the same to turn the periscope 45° to the west and then to raise it 1m than to raise it 1m and then rotate it 45° to the west? How did you work this out? What is the name of the movement resulting from applying both?

(c) Can you establish the set of fixed points for each of the transformations in the first question? Did you work this out intuitively by using the GeoGebra construction, or by applying the definition?

(d) In what position could an enemy find a place to hide from the radar of the periscope, assuming that the periscope is located in the starting position and can only turn in an east-west direction?

Key elements in the process of inquiry. Participant lecturers worked with these specific linear algebra tasks, identified key elements in the process of inquiry, and paid attention to how to formulate questions such that they encourage students to move between the embodied, symbolic and formal worlds of mathematical thinking.

Multiple modes of thinking which result in richer conceptual understanding of the concept of affine transformation (linear maps) and rigid motion are explored. In order to acquire experience in inquiry-based teaching, it will often require moving between the levels in the same lesson. It might give more student responsibility in open inquiry, while, at the same time, it could be required support through structured inquiry. For lecturers, these challenges included: (a) the grading of inquiry levels of concept use, technique use, and teacher regulation of activity; and (b) the relationship of students' intuitive, informal, or flowering ideas to conventional and more formal mathematics. Figure 7.3 shows a template that helped the participants discuss the determination of sequences of inquiry pathways, concept visualisation, formalisation, and symbolisation.

Level of inquiry Linear Algebra	**Profile of mathematical thinking difficulties**	**Phases of inquiry**				
		Questions and observations	Regulation	Inquiry pathways	Results of inquiry: concept visualisation	Results of inquiry: formalisation and symbolisation
Structured inquiry						
Guided inquiry						
Open inquiry						

FIGURE 7.3. Levels of inquiry and intuitive and formal thinking

Finally, the implementations developed in the classroom were shared. As indicated above we conceived the inquiry as having both epistemological and strategic aspects. For a specific characterisation of the area of knowledge in which the inquiry takes place the inquiry project was described in didactic terms following different moments of mathematical organisation: initial, exploratory, work of the technique and concepts, technological-theoretical, institutional, of engagement, of critical alignment and of motivation and, articulated around tasks associated to concrete objectives.

Some challenges in order to support professional development of lecturers were raised: what do we mean by an inquiry-based task in mathematics, what is specific to inquiry in mathematical work and differentiates it from another knowledge. The development of rich tasks contributes to mathematicians' feelings of effectiveness in their discipline. The workshop (with its two parts) addressed a number of other ways in which lecturers can integrate IBME approach. These include activities such as choosing problems, predicting student reasoning, generating and directing discussion, asking for questions that extend student knowledge, guiding students for high-quality explanations and mathematical justification.

We see the epistemological point of view of mathematical knowledge and relation of epistemological and cognitive dimension of inquiry process are the key attributes. Two aspects were highlighted in order to make progress in the professional development of participants in the determination of sequences of inquiry pathways of the tasks:

- The relationship of students' intuitive, informal, or flowering ideas to conventional and more formal mathematics
- The nature of the question (extra-mathematical system and mathematical system).

The nature of the question, obviously, has consequences for the inquiry process. In the case of questions that come from an external source, such as is the example of video games project, their transformation into questions of a mathematical nature is an important part of the inquiry process, which involves a modelling process. Mathematical inquiry, when it comes from external situations, necessarily includes a modelling process and combines several logics. In the shared experience, a greater depth in the relations and interactions between the two systems is open: an extra-mathematical system and a mathematical system. Each one has its own logic and, consequently, it is necessary to differentiate epistemological aspects as well as strategy aspects of the inquiry process.

7.4. Summary and Looking Ahead

The experience of the three workshops provided us with valuable information and allowed us to highlight milestones and challenges in addressing an international workshop offered to academic staff. Here we highlight two challenges by answering the following questions:

- What do we mean by an inquiry-based task?
- How can I support teachers' growth on inquiry-based teaching approach?

Regarding the meaning of inquiry tasks, the shared experience of analysis of the same set of tasks in two different workshops (Norway, Spain) showed that there exist varied conceptions of 'inquiry' by the lecturer's participants depending on their mathematics views and according to the content of the discipline they teach. Therefore, it seems pertinent to pay attention to the conceptual model of the inquiry process that we bring to this work and the role of this model in the sequence of activities that we employ. We see inquiry as having both epistemological and strategic aspects, with developments on the two fronts reinforcing one another. A lecturer should think about the forms of knowledge and procedures in mathematics when structuring questions of the inquiry tasks in order to let his/her students engage in authentic and productive inquiry. When undertaking inquiry students are motivated in the sense that the process is driven by an explicit intention to find out. In each of the three different workshops (Germany, Norway and Spain) examples have been worked on that favour this epistemological analysis of knowledge.

This epistemological analysis of knowledge helped to characterise the knowledge domain in which the inquiry is developed (it is different if we are working on geometry or algebra or analysis contents). The knowledge domains have their specificity and address determined aspects of mathematical thinking. This will require a learning accompaniment with adequate didactic elements.

The respective specification of these aspects cannot be made a priori in advance of a workshop, but instead results in particular from the exchange between the participants. This is not only due to the great epistemological diversity of mathematical content or pedagogical-strategic possibilities for action, but also to the specificity of each IBME development process. This will be briefly illustrated again in the following against the background of our experiences using the three-layer model.

Certainly, the students' learning should be optimised and deepened. Therefore, the first layer orients itself towards a common goal. However, what this means in the respective context, thus also with regard to the possibilities and limits of learning activities, can be very different. The same learning activity can mean a big step towards a greater taking of responsibility and independence in one context, but in another context it can essentially mean the unreflective reproduction of something previously trained. Explicating and thus disclosing the respective didactic contract (Brousseau, 2002) is therefore also of great importance. Decisions based on this contract significantly determine the activities and their assessment at layer 2 of the three-layer model presented in Chapter 2. Here, too, it is not possible to say absolutely and without taking into account the teaching-learning culture prevailing at a university what teaching activities promote or hinder IBME in the respective subject context, etc. (see layer 1 of the three-layer model).

The respective workshops intended to establish a community in the sense of Layer 3 of the three-layer model. This means in particular to enable a trusting exchange. It seems trivial, but without a personal commitment, without questioning one's own assessments and experiences (see also Layer 2), an intentional development in the sense of IBME is impossible. The introduction of didactic terms and concepts and the respective focus on mathematical aspects at all three workshops allowed for an open discussion and reflection by objectifying the exchange to a certain extent. The degree to which prior clarification is necessary or useful depends, of course, on the context of the workshop and the previous knowledge and interests of the participants. What is already clear, however, is that a too strong institutionalisation, possibly linked to a grading in the sense of "IBME lecturer of grade XY," could be quite problematic, as this would create an assymetrisation that could stand in the way of the necessary or at least helpful psychological group processes. Lecturers might then behave in a similar way to many students, trying to undermine the fulfilment of requirements, etc.

A limit of the workshops carried out became clear in the respect that layers 1 and 2 are in a certain sense only presented virtually. However, whether further developments of tasks, their use, and so on, modify the didactic contract or lead to the intended mathematical learning actions had to be left open here. Clarifying this in exchange with the students represents, in a sense, the objectification step that cannot be achieved in this way, but which is immanently linked to IBME and is reflected in the necessary inter-connectedness of the three layers. A workshop detached from such clarification steps can therefore only be a step to initiate processes. In the IBME sense, it is an important step, but it would be best to make it an ongoing activity, also to enable further group processes and a deepening of the discussion and reflection on mathematical teaching.

Finally, we outline below three aspects of progress.

1. How is the progress related to issues in mathematics professional development? Professional development is a very broad field that also addresses so-called personality development. The workshops focused on professional aspects and the provision of tools (e.g., Table 7.1, Table of dimensions for IBME task, and Figure 7.3, Levels of inquiry and intuitive and formal thinking) with which academics can develop subject-related teaching. The tools have similarities and differences, depending on whether there is a stronger focus on the subject logic or on the process logic. All workshops showed how tools can be used and offered opportunities to do so. As far as we can see, the respective institutional teaching-learning contexts were always taken into account. In this way, local and, if necessary, small steps should be made possible. The expertise of the participants was used. In local contexts, the collaboration between the mathematics teachers and mathematics educators is a crucial point. This is not something that is specific for the university education community. Workshops provide opportunities for mathematicians and mathematics educators to discuss the problems of education at the university level, the pedagogy of teaching, the development of innovative teaching methods etc.

2. How is the relation of the workshops to inquiry-based practice and inquiry communities? All workshops were organised in such a way that for a short period of time CoIs have been created, in which conversations and discussions could take place on equal terms. The focus was more on the innermost layer, the development of tasks design, hence, on the use of the tools and possible products in the form of further developed tasks. In general, the establishment of a kind of CoI might be seen as a prerequisite for fruitful discussions. It is also noteworthy that another level of inquiry community occurred (at least in Norway and Spain) as the workshops are aimed at contributing to the development of CoI at the national level, promoting the collaboration between different universities.

3. How are the consequences for developments in teaching and students' mathematical learning? The development in teaching and mathematical learning was connected to the main goals of the workshop: the participants presented their tasks, debated on how to transform them to inquiry-based tasks and discussed what the inquiry-based task in the specific teaching situation can be. The concrete tasks made possible to experience that a certain systematic approach to tasks and their design can be helpful. The workshops were organised in such a way that the participants could connect with their professional competence, their teaching objectives, etc. One challenge that became evident for those who run these workshops is their own professional competence. The imagination must be stimulated in the relevant professional context for the participants.

7.5. Conclusions

With the experiences of the different workshops (Germany, Norway, and Spain) we have tried to highlight key elements for an effective teaching of the inquiry approach at university level. These materials are proposed from three local communities that take into account institutional and theoretical frameworks to support the professional development of lectures through teaching, research and participation in learning communities or communities of inquiry. The current understanding of university professional development for lecturer has some institutional constraints or tensions. With the examples proposed here, a design-based community that integrates research and participation in a learning community can better facility lecturer learning. From PLATINUM project's study and experience, we note several requirements for developing a design-based community. First, develop a strategy for linking research and practice perspectives in the programme. In the three-layer model we suggest a layered structure, but the big challenge is to operationalise design tools in the interaction of these layers. Second, to engage lecturers in designing instructional tasks and to detect their mathematical and pedagogical challenges. In order to facilitate lecturers' task designing, the workshops have proposed some starting points. Third, develop of strategies to allow teachers to incorporate theoretical ideas into their instructional task design. The adoption of the three above considerations could be taken into account for the design of a workshop applicable in various contexts.

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CHAPTER 8

Mathematical Modelling and Inquiry-Based Mathematics Education

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8.1. Introduction

Mathematical modelling (MM) is a powerful tool used by scientists and engineers to solve important problems for humankind. MM opens many possibilities for inquiry and has been included in the PLATINUM project as one of the Intellectual Outputs, IO5 (see Chapter 5 for the complete list). We consider MM an important part of the teaching and learning process. We believe that helping to develop students' modelling competencies we equip them with a valuable understanding of practical and theoretical concepts, prepare them for a life-long learning, and form them as critical citizens.

This chapter is organised as follows. In Section 8.2, we share our views on why we teach MM, noting that our teaching practices are adapted differently to suit local educational contexts (types of students, programs of study, institutional traditions, constraints, etc.). We proceed with the discussion of what are the most important to us characteristics of Inquiry-Based Mathematics Education (IBME) as a teaching approach. To explain authors' understanding of how the MM relates to IBME, the concept of 'active knowledge' is introduced in Section 8.3. The key idea of this concept is that in response to the use of IBME in the classroom students' engagement with MM activates previously acquired knowledge and facilitates its efficient use. In Section 8.4, partners present examples of the use of MM within IBME practices and comment on how students activate their mathematical knowledge. Each example shows multifaceted connections between MM and IBME. We reflect on the lessons learned from our contributions to the Intellectual Output IO5 in Section 8.5.

8.2. Mathematical Modelling and Inquiry-Based Mathematics Education in Our Teaching

Theoretical foundations of Inquiry-Based Mathematics Education (IBME) presented in Chapter 2 emphasise the importance of improving the balance between procedural and conceptual learning of mathematics through an inquiry approach that offers students opportunities for deeper engagement with the subject. On the one hand, inquiry-based tasks motivate and encourage students to get involved with the subject more actively by posing questions and trying to answer them, and by exploring processes and concepts. On the other hand, mathematical modelling (MM) tasks motivate students' engagement into activities that contribute to the development of their creativity and exploratory skills that are characteristic of professional mathematicians, with the aim of developing students' mathematical literacy and prepare them for professional life. Therefore, the main ideas of the IBME are well suited for the use in mathematical modelling and can be employed to motivate students to learn actively in and outside the classroom through individual and collaborative problem solving and project work. This is often achieved when students work with authentic tasks and are provided with a timely strategic support.

8.2.1. Why Do We Use Mathematical Modelling in Our Teaching? In their work on mathematical modelling and applications, Blum and Niss (1991) defined a mathematical model as a triple (S, M, R) consisting of some real problem situation S, some collection M of mathematical entities and some relation R by which objects and relations of S are related to objects and relations of M. Then, MM is the entire process leading from the real problem situation to a mathematical model. Whilst this definition seems easy to understand, in practice MM activity is very complex and there is a certain disagreement in the mathematics education community as to what exactly counts as a model/modelling, what the aims of MM are, and how it can be taught best (cf., Kaiser & Sriraman, 2006; Hernandez-Martinez et al., 2021).

It is therefore not surprising that the authors of this chapter have similarities and differences in their views of MM. In order to elucidate where these similarities and differences lie, we look at the perspectives on MM that Kaiser and Sriraman (2006) elaborated, and the questionnaire that Treffert-Thomas et al. (2017) developed based on those perspectives plus an additional one called "Enjoyment perspective." The five categories connected to goals of teaching modelling are:

(1) Realistic (or applied) perspective, which describes the aims of MM as pragmatic or utilitarian, that is, to solve practical problems in the way that applied mathematicians would do in their professional practice;
(2) Epistemological (or theoretical) perspective, which sees the aims of MM as theory-oriented, that is, to develop theory without paying too much attention to the realistic aspects of a problem;
(3) Socio-critical (or emancipatory) perspective, which characterises MM as aiming to develop critical understandings of the world and the role that mathematics plays in making important societal decisions;
(4) Contextual perspective, which characterises MM as a tool for psychological development, that is, MM activity should elicit the invention, extension, and refinement of mathematical (psychological) constructs;
(5) Educational perspective, which sees the aims of MM as pedagogical, that is, MM should foster the understanding of mathematical concepts and structure the learning processes.

Treffert-Thomas et al. (2017) complemented this categorisation with a sixth one (see also, Rogovchenko et al., 2020):

(6) Enjoyment (or affective) perspective, in which the aim of MM is the intrinsic satisfaction derived from engaging in MM activity.

The authors completed the questionnaire on MM and IBME, where part D is based on the items used by Treffert-Thomas et al. (2017),[1] and we next discuss the results from those categories where the majority of the partners agreed or disagreed. We all strongly agreed or agreed that MM should aim to develop skills in solving authentic

[1]`https://bit.ly/32h3cy1`

problems and that MM with real data leads to significant insights, both characteristics related to a realistic perspective. MM team members also strongly agreed or agreed that the aims of MM are to develop general problem-solving skills and develop students' critical thinking skills, which would correspond to the educational and socio-critical perspectives, respectively. Finally, MM team members strongly agreed or agreed that MM should be based on the theoretical understanding of the phenomenon to be modelled, which corresponds to an epistemological perspective.

These similarities are reflected in how each of the authors views MM and how s/he operationalises her/his views in their teaching as a local community of inquiry. For example, partners at Brno University of Technology (BUT) believe that MM activities should be based on or motivated by applications of mathematics to real life problems (realistic perspective), but this does not exclude pure mathematics activities (epistemological perspective). Partners at Borys Grinchenko Kyiv University (BGKU) believe that the main purpose of using MM in the educational process is for students to acquire the knowledge of new mathematical facts, master new mathematical methods useful for studying different phenomena and processes, improve conceptual understanding of mathematics and advanced mathematical thinking, gain some experience in applying mathematical knowledge, develop collaboration and communication skills, deepen motivation for lifelong learning, and so on. That is, MM for partners at BGKU is a tool for forming students' mathematical competence, which mainly corresponds to an educational perspective. Partners at the University of Agder (UiA) believe that MM should allow students to reach a certain mathematical maturity, to have knowledge of different areas of mathematics, to develop critical thinking and the ability to collaborate and communicate efficiently, with all these components embedded in traditional mathematics courses. They also believe that MM should stimulate joy and excitement of MM in students, and bring about creativity and inspiration, particularly in the advanced educational levels. These views correspond to the realistic, educational and enjoyment perspectives.

We should be aware that these views are mediated by a myriad of factors that affect practice. For example, partners at UiA need to deal with the fact that there are no dedicated courses where MM is taught, so they include this important aspect of mathematics in mathematics courses they teach. Partners at BGKU are responsible for teaching of students in mathematics and ICT undertaking a larger variety of courses, including those aimed at real-life applications, and hence MM plays a more prominent role in their pedagogical strategy. Partners at BUT are new to the use of MM in teaching but have plenty of experience in using MM in professional settings, where theory is valued. All these contexts and circumstances shape what we value in MM and what role we ascribe to MM in teaching (Hernandez-Martinez et al., 2021). Therefore, while we agree on several basic features of MM and the ways it should be taught, these features might look quite different in each of the partners' practices.

Finally, we discuss questionnaire items in part D where most of the partners disagreed. Partners were split in their opinion that every student should learn modelling, if the aim of creating a model is to obtain a solution or if solving word problems constitutes MM. We hypothesise that these disagreements stem from the realities and situations in which each partner operates. For example, for partners at BUT, word problems would not constitute MM because they are inauthentic while for partners at BGKU these types of problems might provide important tools to achieve their teaching goals. For partners at UiA and BUT, the process would be more important than the solution to a problem because it is through the process of MM that students become interested, engaged or even creative. On the other hand, for partners at BGKU, correct and complete solutions to problems carry a great deal of value because for it indicates if students have achieved the learning that the teachers desire them to do. Hence, we see disagreements as part and parcel of the different circumstances in which partners find themselves but not as a barrier for discussing MM.

The partners agree that a simplified four-step cycle (Understanding task—Establishing model—Using mathematics—Explaining result) described by Blum and Borromeo Ferri (2009) represents a convenient format for the work with modelling tasks with students. We also agree that successful students' work on modelling tasks requires certain mathematical maturity, knowledge in different areas of mathematics, critical thinking and ability to collaborate and communicate efficiently; MM at a higher professional level brings also creativity and inspiration which turn it into an "art of MM" rather than a process. One important goal in teaching MM is to share with students the excitement and joy of mathematical modelling (Rogovchenko et al., 2020) which can be experienced in a mathematics classroom even with an entry or intermediate level modelling projects and problems.

8.2.2. How Do We Use Inquiry in Our Teaching? The ideas of inquiry-based mathematics Education (IBME) can be traced back at least to the work of the American philosopher and educator John Dewey (1859-1952) who published two cornerstone monographs "How we think: A restatement of reflective thinking to the educative process" (1933) and "Logic: The theory of inquiry" (1938). In the latter book, Dewey defined inquiry as "the controlled or directed transformation of an indeterminate situation into one that is as determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole" (p. 108). Nowadays, student inquiry is characterised as "an educational activity in which students are placed in the position of scientists gathering knowledge about the world. Students direct their own investigative activity, completing all the stages of scientific investigation such as formulating hypotheses, designing experiments to test them, collecting information, and drawing conclusions" (Keselman, 2003, p. 898). This definition emphasises active participation and learner's responsibility for constructing this knowledge (de Jong & van Joolingen, 1998). Modern views on inquiry-based education develop further Dewey's educational philosophy promoting learning through reflective inquiry which combines inductive and deductive methods and emphasises pragmatic efficiency of knowledge and connections to real-life situations and professional practice (Artigue & Blomhøj, 2013). In line with Dewey's educational philosophy, inquiry in the mathematics classroom often starts with the discussion of realistically looking situations which may naturally lead to modelling.

> The starting points for mathematical inquiry are the multiple live issues that students possess; mathematics becomes the set of tools from which they can choose to help carry out their inquiries. In this type of mathematics class, the teacher becomes a skilled guide who can help shape student inquiries, aiding in the construction of mathematical models and the selection of appropriate mathematical tools of inquiry and in supervising the evaluation of such activities. (Stemhagen & Smith, 2008, p. 34)

Pedaste et al. (2015) conducted a systematic literature review identifying the core phases of IBME and their involvement in learning distinguishing five distinct general inquiry phases, some also split into subphases. The list of phases includes *Orientation*, understood as the process of stimulating curiosity and addressing a learning challenge through a problem; *Conceptualisation*, the process of stating hypotheses or stating theory-based questions, which splits into subphases of *Questioning* and *Hypothesis Generation*; *Investigation*, the process of exploring, experimenting, planning, and collecting and analysing data with the two subphases of (i) *Exploration, Experimentation* *and Data Interpretation* and (ii) *Conclusion and Discussion*, understood, respectively, as the process of drawing conclusions from the data analysis and answering the hypothesis or research questions and the process of presenting findings, communicating with others and engaging in reflection, with the subphases *Communication* and *Reflection*. Inquiry activities are organised in cycles, which combine different phases. This fits especially well the process of learning mathematical modelling because recent research links modelling competency with the ability to successfully perform all steps in a modelling cycle (Blomhøj & Højgaard Jensen, 2003; Blum & Borromeo Ferri, 2009; Blum & Leiß, 2007; Blum & Niss, 1991). Different models of a modelling cycle are used in mathematics education ranging from a seven-step model for research and teaching (Blum & Leiß, 2007) to a simpler four step schema (Understanding task—Establishing model—Using mathematics—Explaining result) deemed to be more appropriate for students' work (Blum & Borromeo Ferri, 2009).

We explored partners' views on IBME by asking them to respond to parts A-C of the four-part questionnaire "Relevance of inquiry for mathematical modelling."[2]

In part A, the respondents were asked to indicate their preferences to 34 key inquiry activities listed by Pedaste et al. (2015) with the understanding that not all activities may be equally well suited for the inquiry in a mathematics classroom; this is clearly reflected in the answers. It turns out that partners did not indicate particular interest in the three activities related to the first phase, Orientation. Partners viewed the following five activities as very relevant or relevant for inquiry-based learning;[3] phase names are written in italics in the parentheses; the items are ordered from the highest ranked on the top of our list to the lowest ranked at the bottom):

(5) Determining what needs to be known, Define problem, Identifying the problem, Identification of question or questions (*Conceptualisation*);

(11) Investigate, Observe, Observation, Collect my evidence, Conduct observation, Explore, Exploration, Initial observation (*Investigation*);

(24) Construction, Reasoning with models, Problem solving and developing a course/ experiment (*Conclusion*);

(27) Evaluating success, Evaluate, Evaluation, Evaluate action, Evaluate inquiry, Comparing new knowledge to prior knowledge, Test the explanations (Falling between *Conclusion* and *Discussion*);

(28) Discuss, Debate, Share and discuss my inquiry, Discussing with others, Communicating new understandings, Elaborate, Communicating results, Argument, Discussion and presentation of new content, Communication, Learner communicates and justifies explanation, Present inquiry (*Discussion*).

On the list of inquiry activities where the partners' views on relevance diverged the most, we find three items: (14) Sign system exploration and (21) Transmediation (both from the phase Investigation) and (23) Celebration (from the phase Conclusion).

Part B regards Essential Ingredients in inquiry-based mathematics education (Artigue & Blomhøj, 2013). Summarising the results, we observe that the partners valued most highly two items: Pose questions and Inquire—the 5 E's: Engage, Explore, Explain, Extend, Evaluate—both grouped into What Students Do, and that the partners expressed differing views on the two items from the group Classroom culture: Shared sense of purpose/justification, and Shared ownership. The views on the ingredients of IBME collected under the umbrellas of Valued outcomes, Teacher guidance and Type of questions were much less pronounced. This suggests that all

[2]The questionnaire and the summary of the answers of six team members can be found at the following link: `https://bit.ly/32h3cy1`

[3]Activity numbers are from Pedaste et al. (2015).

partners indicate clear interest in using inquiry as the learning strategy and value multifaceted experiences that inquiry offers students. Not surprisingly, classroom culture in the Czech Republic, Norway and Ukraine differ significantly which is reflected in the answers.

Part C of our questionnaire is based on the list of Components of Inquiry Process (Artigue & Blomhøj, 2013), indicate three favourites: New experience/question, Plan and conduct investigation, and Interpret data. Much less agreement between the partners was observed with regard to the remaining components: Possible explanation, Existing idea, Alternative ideas, Bigger idea, Prediction, and Conclusion. This again points towards partners' prioritisation of the organisation of students learning through inquiry emphasising questioning, analysis, and validation of results, all three components critical for the proper structuring of the modelling activities. The final part D of the questionnaire was based on the paper by Treffert-Thomas et al. (2017); our answers to this part were already discussed in Section 8.2.1.

The sources of mathematical inquiry in IBME emerge not only from mathematical objects themselves, but also from daily life problems, industrial and technical problems, processes, and phenomena in nature, and even from art and human artefacts.

> In relation to IBME, the concept of modelling offers a systematic way of understanding and working with the relationship between mathematics and problem situations or phenomena in other disciplines and in extra-mathematical contexts in general. From a learning perspective, modelling can thus be a bridge between the mathematical concepts and ideas and real-life experiences. Through modelling activities, the learner can make sense of the concepts as well as gain new insights into the problem situations modelled. (Artigue & Blomhøj, 2013, p. 805)

Inquiry cycles and mathematical modelling cycles discussed in the research literature exhibit striking similarity; therefore, the work with mathematical models leads to "valuable understanding of inquiry as a more general process with different particular realisations in different disciplines and contexts" (Artigue & Blomhøj, 2013, p. 805).

8.3. Active Knowledge: Connecting IBME and MM

Introducing the notion of mathematical modelling competence, Blomhøj and Højgaard Jensen (2003) highlighted the following steps in the modelling process:

(a) Formulation of the task and identifying the characteristics of perceived reality.
(b) Selection of the relevant objects and relations, use of idealisation.
(c) Translation to mathematics.
(d) Use of mathematical methods.
(e) Interpretation of results.
(f) Evaluation and validation of the model by comparison with data.

We believe that these steps require the following abilities from the students:

(a) Curiosity, motivation, exploring, engagement
(b) Exploring, engagement
(c) Engagement
(d) Engagement
(e) Evaluation
(f) Evaluation

Drawing on these ideas, we introduce the concept of active knowledge to explain how the modelling process is related to inquiry-based learning and how this relationship enhances the active knowledge formation. We want to distinguish active knowledge from passive knowledge in a following way: active knowledge is used on the regular basis; passive knowledge is what we recognise when we encounter it but do not use often. This can be compared with the use of languages: active and passive vocabulary. Transformation from passive to active knowledge in MM can be seen as an activation of the knowledge previously acquired by students during the modelling process in response to the use of IBL in the classroom. On the other hand, active learning describes the process of gaining knowledge based on learner's activity and agency; students acquire an important role of co-creators of new knowledge. We see the process as follows: students use active learning to obtain knowledge; activate it during MM (learn how to use it); and use active knowledge for theoretical and professional tasks on the regular basis. The connection between the ways of gaining knowledge and different areas of applying it within the active knowledge framework is shown in Figure 8.1; we use the concept of active knowledge to explain the passage from "how to gain knowledge" to "when to use it." Students use active knowledge both during the study process (solving realistic tasks with help of modelling, using modelling to develop mathematical concepts) and as professional tool (creation of mathematical theory, innovation activities, solving professional tasks).

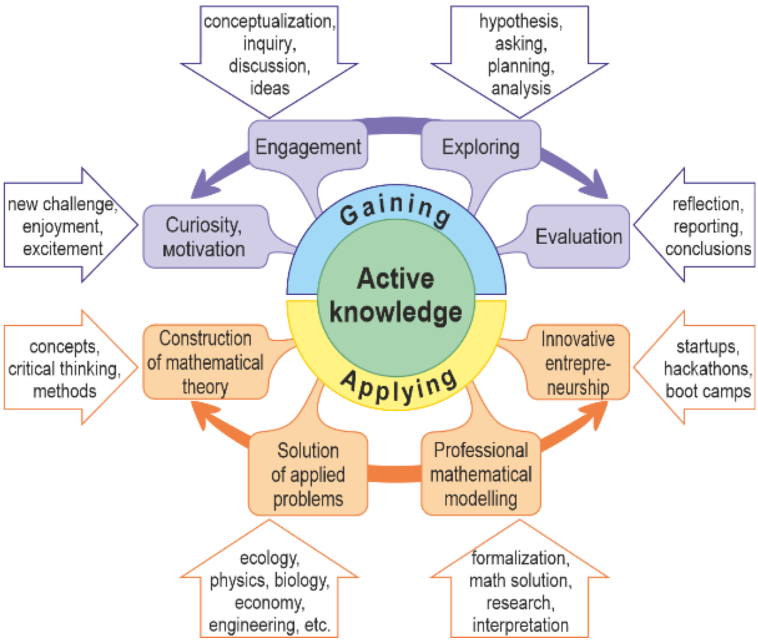

FIGURE 8.1. Main components of *active knowledge*.

We believe that teachers can promote inquiry within mathematical modelling as the form of active knowledge; this brings students closer to the atmosphere of mathematical discovery which is usually associated with the work of professional mathematicians. Students are encouraged to reflect about new material in an explorative manner, by asking and answering questions which help to understand mathematical concepts, the reasoning in the proofs, logical chains of arguments in the solutions of problems. When students understand that passiveness in the class does not help to learn mathematics efficiently and accept the challenge of being challenged, the routine work in the classroom turns into an exciting adventure into the Universe of mathematics. Ambiguity and confusion experienced on times by students should not discourage them as they lead to small but important discoveries and victories; uncertainty is the feeling often experienced by research mathematicians and mathematical modellers creating and applying new mathematics in their work. As Byers (2007, p. 78) pointed out, "ambiguity can be the doorway to understanding, the doorway to creativity." By designing inquiry tasks, especially modelling problems, and including them in our teaching, we provide opportunities for students' learning and engagement in critical reflection. On the other hand, mathematics teachers also engage in critical reflection during the design, testing, and refining tasks; critical reflection is even more requested when teaching-learning interactions occur in a context of genuine inquiry. Furthermore, existing students' knowledge can be activated when teachers design tasks that link students' learning to an authentic inquiry, as it is done in academic and industrial research. Finally, teachers also engage in critical reflection when they adjust their thinking by discussing their practices with other teachers and researchers, as was the case in PLATINUM. The three cycles of inquiry related to the active knowledge are illustrated in Figure 8.2.

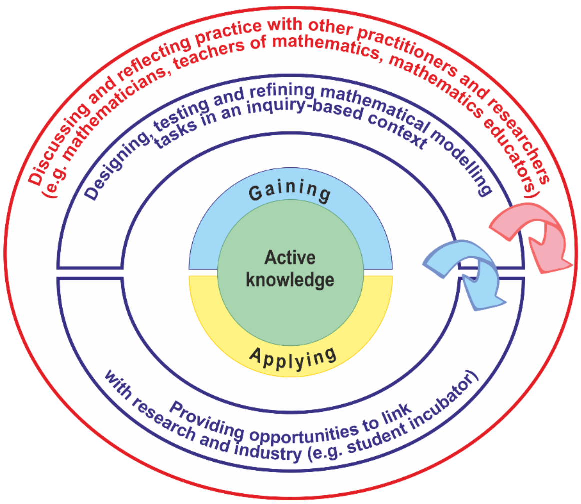

FIGURE 8.2. Active knowledge and mathematical modelling within the three-layer model of inquiry.

The idea of active knowledge is reflected in the examples from the three partner universities discussed in Section 8.4. First, we observe how Ukrainian mathematics undergraduates create models for practical problems which lead to a rigorous formal definition of a definite integral by using their previous experience of calculating areas of elementary geometric shapes and the knowledge of the additive property of area. Then we follow the process of the knowledge activation of Czech undergraduate students who relate calculations of the volume and surface area for a sphere and a cylinder in the final years of the high school, definition of the derivative of a function, its basic properties and optimisation applications from a calculus course, and real-life experience with the shapes of different beverage cans to design their own can that is optimal from both mathematical and practical standpoints. Finally, Norwegian master's students activate the knowledge of physics laws, basic calculus, differential equations, and computational skills to verify the applicability of the existence and uniqueness theorem for a differential equation constructed to model a leaking bucket. In all three examples, active knowledge featured multifaceted connections between inquiry with mathematical modelling.

8.4. Examples From Three PLATINUM Partners

In this section, we describe partners' experience with the use of mathematical modelling in their teaching and explore how the inquiry was organized by the lecturers and perceived by students.

8.4.1. Borys Grinchenko Kyiv University. An important characteristic of mathematical competence and, at the same time, a necessary condition for the effective application of mathematics for solving applied problems is the proper mastery of mathematical concepts. Despite a large number of theoretical and empirical studies on modelling and related mathematical activities, not many examples promote the use of modelling as a teaching philosophy aimed, in the first place, at the formation of students' conceptual understanding. We agree with Gravemeijer (1999), who argued that formal mathematics should be created by students themselves, and believe that ideas of emergent modelling that encourage and stimulate the process of discovery (construction) of mathematical theory by students themselves are useful. This is especially true for advanced mathematics because many key mathematical notions and structures, like the concept of a limit of a function and related notions of derivative, integral or the method of mathematical induction can be presented initially to students as imperfect models introducing intuitive, non-rigorous ideas about mathematical concepts and structures, or even as metacognitive models for the process of thinking about them. Departing from authentic real or real-like situations originating usually in an extra-mathematical domain, students construct the initial, naive understanding of a new concept. The teacher's task is to offer such stimulating problems. Then, through the abstraction from the subject content of the specific problem, a mathematical model of a rigorous mathematical concept is created on the basis of the preliminary intuitive non-rigorous concept. During the shift from the real world back to abstraction, a rigorous, formal concept gains a new quality, it carries with it an imprint of reality becoming an efficient tool for its study, a "building block" for mathematical modelling. Therefore, mathematical models and mathematical concepts develop simultaneously, mutually stimulating each other's development. Thus, we can view the formation of the rigorous mathematical concepts and their consequent application as a cyclic triad presented in Figure 8.3 where our views on MM and concept formation align with the French tradition of Chevallard (1999), Brousseau (2002), and other authors (García & Ruiz, 2006; Dorier, 2006) who consider all mathematical activity including problem solving in pure mathematics as modelling.

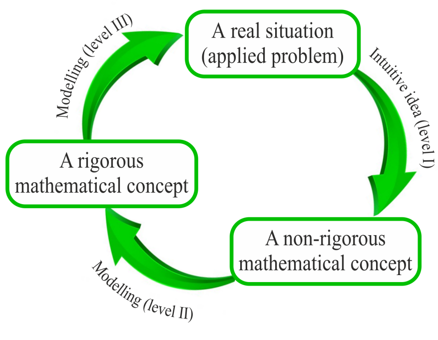

FIGURE 8.3. The triad of formation and application of a mathematical concept.

The cyclic triad presented in Figure 8.3 is implemented at BGKU throughout the entire study period. Depending on the year of study, discipline, and specialisation, teaching accents are shifted in accordance with the changes in learning objectives and the course content. The IBME approach can be used effectively at every level of this triad because its goal is not limited only to finding the right answer (which is often not available in modelling tasks), but to find a good approach to the solution. Modelling requires much more than a mere retrieval and use of the previously gained knowledge, it requires a deep understanding of concepts, facts, processes, methods, and an ability to apply them under new conditions working continuously on the border between the known and the unknown.

During the first years of study, students at BGKU go through fundamental mathematical disciplines where the modelling is efficiently used to assist in the formation of rigorous mathematical concepts and methods. Conceptual understanding of mathematics lays down good foundations for the study of advanced mathematical subjects, including those oriented at the applications and solution of practical problems coming from various branches of science, business, and engineering. At this stage, for educational purposes students are mainly trained on “toy” problems designed for the direct application of mathematical concepts, facts, and methods. The subject and complexity of applied problems, both real and pseudo-real, eventually increase following the developments in the study curriculum and relevant mathematics disciplines at the bachelor level. The list of applied and interdisciplinary disciplines includes such subjects as operations research and econometrics where the problems generally do not require the use of the full mathematical modelling cycle or interaction with specialists.

In accordance with the methodological model presented in Figure 8.3, teaching of “real” mathematical modelling for students majoring at BGKU in mathematics starts only at master’s level within the programme “Mathematical Modelling.” The purpose of the programme is to provide students with the solid training in mathematics emphasising the-state-of-the-art theories and methods that have wide applications in different fields of science and professional practice, including the basic methods of mathematical modelling. The study curriculum includes the in-depth study of the following disciplines: fundamentals of mathematical modelling, systems analysis, forecasting, applied functional analysis, dynamic systems, applied mathematical and computer modelling. Students must complete an undergraduate internship and write a master’s thesis in which they develop a mathematical model for a particular field (economics, finance, computer science, physics).

One of the possibilities to apply theoretical knowledge in mathematics and mathematical modelling is a university business incubator (UBI) created to provide practice-oriented applied learning and to increase students’ motivation for studying mathematics. The model of UBI uses existing successful practices in Poland, Israel, Norway, and Estonia; it reflects both the market needs and peculiarities of higher education. The UBI at BGKU provides a creative platform for the development of students’ innovative projects in various fields of science including the development of mathematical models for the solution of practical problems in business and industry; its structural organisation is presented in Figure 8.4.

We present now an example illustrating the first level of the triad, namely the use of mathematical modelling for the formation of the concept of the Riemann integral in the Mathematical Analysis class for the first-year mathematics undergraduates at BGKU taught by Dr. Mariia Astafieva. The purpose of the lecture was to form the concept of a definite integral, make students understand what classes of applied problems can be solved by using definite integral and practice modelling skills. The lecture was conducted using an IBME approach, in particular, structured and guided inquiry. Interactive teaching methods used in the classroom included brainstorming, small

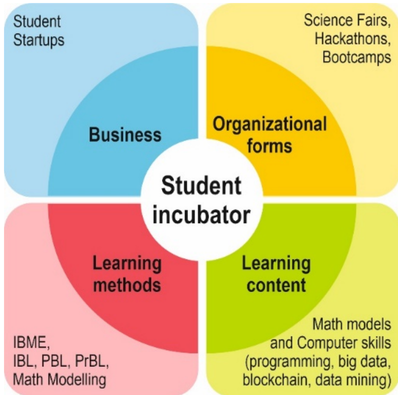

FIGURE 8.4. Organisation of the University Business Incubator at Borys Grinchenko Kyiv University.

group discussions of the problems followed by the presentation of the outcomes to the class, reflection, and the whole class discussion.

To involve students into cognitive and explorative activities, four applied tasks were given where students had to find (1) the area of a curvilinear trapezoid, (2) the mass of an inhomogeneous rod; (3) the distance travelled with a variable speed; and (4) the volume of the output at variable productivity. To stimulate the student activity, the work was initially organised in small groups (two groups of four students), each group received identical figures cut out of paper (Figure 8.5), scissors and rulers.

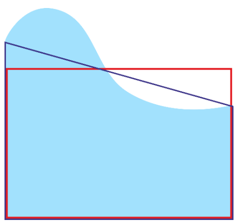

FIGURE 8.5. Replacing the figure: with a trapezoid or rectangle.

Both groups had to suggest how to find the area of the given figure and determine it in just one minute. After a one-minute discussion, the group's spokespersons presented the solution idea and suggested an approximate value for the area. Group N° 1 noticed that the figure looked like a rectangular trapezoid. Therefore, their proposal was to replace it with an ordinary trapezoid (i.e., to replace the curved "side" with a line segment, see Figure 8.5) and use the formula for the area of the trapezoid. Approximate value of the area: $(11 + 7) \div 2 \times 12 = 108\ \text{cm}^2$.

The idea of Group N° 2 was to cut off the "hump" with scissors and fill the "hole" with a piece that was cut off. That is, to replace the original figure with the rectangle (see Figure 8.5). This group had as many as four different answers ($108\ \text{cm}^2$, $102\ \text{cm}^2$, $99.6\ \text{cm}^2$, and $105.6\ \text{cm}^2$), because each student built "his/her own" rectangle.

To facilitate students' formation of the concept of a definite integral, the teacher encouraged them to think about the pros and cons of the proposed methods. In a teacher-led discussion, students also suggested how to improve the procedure for determining the approximate value of an area so that the calculation error is minimised. In particular, they easily concluded that the methods proposed by both teams were not very successful. Their only advantage is simplicity. The biggest drawback is that the obtained approximations are very rough. The productive idea of improving the procedure in order to achieve better accuracy of the result came up with more difficulty. However, the teacher was not in a hurry to readily suggest the idea using instead a sequence of questions that eventually prompted the required idea to cut the figure into vertical strips, which, like the whole figure, will be curvilinear trapezoids, and then to add the areas of all the strips found by the approximation method of any of the teams.

At the teacher's suggestion to analyse the solution, the student S1 stated that we did not solve the problem because our procedure gave an approximate but not the exact value of the area, required in the problem. The problem identified by the student is important in the context of constructing a definite integral. Thus, the student correctly recognised the mathematical essence of the problem.

The identified problem encouraged a new search for correct answer illustrated in the following excerpts (L=lecturer, S=student(s)).

Excerpt 1

L: Assume we cut the figure into one-centimetre-wide strips, calculated the approximate values of the area of each of them, added them up and got a value which (with a certain error!) approximates the area of the whole figure. Is it possible to reduce this error?
S: (chorus) Yes. It is necessary to cut the figure into narrower strips.
L: How wide? Half a centimetre? One millimetre?
S: (chorus) As narrow as possible.
S1,S2: The narrower the strips, the more accurately will the area be calculated.
L: Right. So, can we now draw a conclusion about the exact value of the area?
S1: This will be the limiting value! We already did that when we were looking for the value of the instantaneous velocity.

Describing in mathematical terms the sequence of actions required for the solution, that is, creating a mathematical model of the definite integral, students answered lecturer's questions by making reasonable assumptions and determining the required parameters, as illustrated in the following excerpt.

Excerpt 2

L: Is it important to divide the figure into the strips of equal width?
S2,S3: No, it is not. It is important that the strips are narrow. But when they are of the same width, it is easier to calculate the area.
L: Shall we choose trapezoidal strips, as suggested by group N^o 1, or rectangular, as suggested by group N^o 2?
S1,S4,S6: It also doesn't matter because the strips are very narrow. But it is better to choose rectangular form because the area of the rectangle is easier to calculate.
L: So, we will assume each narrow strip to be rectangular. And what is its height?
S5: Well, let's take approximately about half of the measurement between the points on a "hump" and in the "hole."
S1: We can actually measure this height anywhere.
L: I also believe that there is no need to "aim" at any specific point between the "hump" and the "hole". Since the width of the strips goes to zero, any perpendicular to the base of the strip can be considered the height of the rectangle. Do you agree?
(...)
L: Finally, when we write down the expression for the area of a "stepped" figure made of n rectangles, it is necessary to pass to the limit. What limit do we need?
S2,S7: When the number of strips n goes to infinity.
L: Does everyone agree?

Pause. Nobody replies.
L: Why do we want to increase the number of stripes indefinitely? What is the purpose?
S3,S7,S8: To make the strips narrower.
L: Okay. And by increasing their number will we achieve this goal?
S1: No! Look (*demonstrates, cutting a curved trapezoid into two strips of approximately the same width, and then continuing to cut in half only one of the two parts*), the number of strips increases, but one strip remains wide.
S7: Okay. But if you cut into strips of the same width, then with an unlimited increase in the number of strips, their width will approach zero. So maybe it's better to divide into strips of the same width?
L: No need. I think there is a way out of this situation. We will require that if the width of the widest strip approaches zero, so do the widths of all remaining strips.

The lecturer projected Figure 8.6 on the screen, introduced necessary mathematical notation ($\Delta x_k = x_k - x_{k-1}, \xi_k \in [x_{k-1}, x_k], k = 1, 2, \ldots, n$) and asked student $S8$ to recall all solution steps writing them down on the blackboard up to the final result:

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k) \Delta x_k, \quad \text{where } \lambda = \max_k \{\Delta x_k\}. \tag{8.1}$$

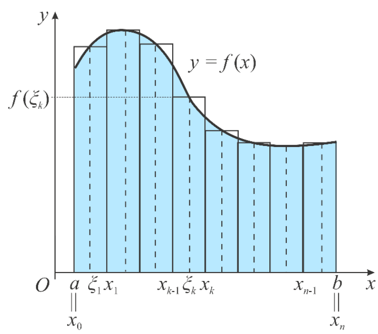

FIGURE 8.6. Replacement of a curvilinear trapezoid with a stepped figure.

After that students worked all together finding the mass of a heterogeneous rod and then, in small groups, they found the distance travelled by a particle at variable speed and production volume presenting group solutions to the class. Comparing solutions to all four problems, the students concluded that the mathematical model for these problems is based on the use of the limit as in Formula 8.1, where the function f defines, respectively, the equation of a curve that limits the curvilinear trapezoid, linear rod density, variable speed, and variable productivity; the same sequence of actions leads to this mathematical model. It is worth mentioning one important observation made by students: in all problems the required quantity A that they calculated (area, mass, distance, volume of manufactured products), possesses the additive property:

$$A(\phi_1 + \cdots + \phi_n) = A(\phi_1) + \cdots + A(\phi_n).$$

Students actively assisted the lecturer who introduced on the board the concept of a definite integral of a function $f(x)$ from a to b and explained the relevant notation. Then they revisited all four problems, wrote solutions as definite integrals, and discussed the context in which the definite integrals were used: geometric (area), physical (mass), mechanical (distance travelled), economic (production volume). In the end of the class, students were asked for the feedback about the lecture. The lecture and students' feedback were discussed by the community of inquiry at BGKU. In particular, the team members focused their attention on lecturer's strategies and actions taken to achieve the educational goals set for the lecture and reflected whether these goals were achieved, and if not, then why (see Table 8.1).

Goals set by the lecturer	*Strategy/Action to achieve the goals*
Creating positive motivation	Solution of applied problems in various fields
Involvement of students in mathematics exploration	Use of IBME approach, in particular, structured inquiry and guided inquiry. Students independently hypothesised, discussed ideas, conducted their own explorations, and drew conclusions. The lecturer encouraged students to be critical of these ideas, to focus on solving the problem. In fact, in students' work, all stages of the 5E-model of inquiry were implemented (Bybee et al., 2006). Interactive teaching methods: brainstorming was effectively used to mobilize students to find productive ideas, discussion of the problem in small groups, followed by presentation of the results, group discussions.
Intuitive understanding of the concept of a definite integral through the sequence of steps needed for its construction	Inductive approach and use of informal considerations. The lecturer did not suggest the action plan to students, they found it themselves. The lecturer set aside time for students working in small groups to discuss ideas and find solutions.
Conceptual understanding	The lecturer did not rush and allowed enough time for discussions (setting the time limit of only one minute in the beginning was an element of the game that created excitement and the atmosphere of competition). She followed closely discussions in the groups joining them unobtrusively when needed becoming a peer participant (see, for example, Excerpt 2). The lecturer supported students during the discussions by providing constructive feedback. She did not answer the questions directly encouraging students to look for answers on their own. The lecturer patiently led students from real situations to abstraction and construction of a formal, rigorous definition.
Mastering mathematical language	Using symbolic representation of the problems
Mathematical competence	Using the existing knowledge in a new situation (solving four different applied tasks)
Formation of active, independent, creative thinking	IBME supported the curiosity of students and their desire to be independent in the search for new knowledge; it stimulated students to ask questions and look for answers, which ultimately contributed to the improvement of students' knowledge, their ability to apply it in new situations, and an overall development of students' advanced mathematical thinking.

TABLE 8.1. The discussion of the lecture by the members of the BGKU community of inquiry.

The feedback from students signalled that the goals set by the lecturer were achieved. It should be noted that students worked actively and with visible pleasure throughout the lecture. This is what they were saying about the class:

> We studied the definite integral at school. And, perhaps, the teacher told something similar there. Maybe I forgot. But I only remember from school how to calculate, for example, $\int_1^3 x^2\,dx$ and in today's lecture, I could not even think of a definite integral as a limit. (...) I also do not know where the formula for the calculation of the integral came from (the one called Newton-Leibnitz formula). Most likely, the teacher just wrote it down on the board and showed how to use it, but I remember it. But now I am sure that in the following lectures we will find this out too. The teacher never tells us to just remember, but tries to make us understand, and we often deduce this or that formula ourselves, notice some fact, pattern. And I really like this kind of training.

> From school I only remember how the area of a curvilinear trapezoid is found with a definite integral. But only during today's lecture I understood why this is so and how do we come to this. (...) Now I understand that in the same way as we were looking for the area of a curved trapezoid, the mass of an inhomogeneous rod, the volume of production, the path travelled at a variable speed, we can find other quantities. For example, the length of a curve. It is necessary to break it into parts and replace each part with a line segment. And then add the lengths of all the segments and look for a limit. And it will be some kind of integral.

> It's hard for me to remember the definitions of concepts that are studied in mathematical analysis, such as limits. But the lecturer always introduces us to such concepts on concrete examples. If it weren't the specific examples, I would have never remembered those "epsilon-deltas." The same applies to the abstract and complex definition of a definite integral. But after I cut the figure into narrower and narrower strips and looked for the approximate value of the area, and then we attempted to find an exact one, we obviously needed to pass to the limit when the width of the strips goes to zero and then I understood how to construct a definite integral and what does it mean. Now I think I can formulate the definition correctly and remember it.

The lecturer also commented on the chosen strategy and the effectiveness of lecture:

> During the lecture a new concept was introduced. The most important task was to form its conceptual understanding because it creates the basis for active knowledge, lays the foundation for the conscious construction of its generalisations (multiple, curvilinear and surface integrals). The definite integral, like many other concepts of mathematical analysis, is not a model of certain objects but rather a model for the ways to study objects and understand the ideas that underlie their construction. That's why I chose the IBME strategy: I tried to organise the classes so that students acted like professional mathematicians building their understanding of mathematical concepts (in this case, the concept of a definite integral). In particular, they asked questions, hypothesised, discussed and substantiated their own thinking, evaluated the thinking of other students in a constructive and supportive environment. The creation of an adequate mathematical model organically combines all the key areas and forms of mathematical activity, launches all the necessary psychological mechanisms in their interaction. The four "real" situations proposed for modelling were presented as inquiry-tasks. Clearly, in a week or so none of mentioned tasks would have any inquiry potential for these students. The tasks would become simple routines for training the skills in application and calculation of the integral.
>
> As for the efficiency of the lecture itself, it is difficult to analyse it right away as well as the overall achievement of the goals. In particular, it takes time to assess students' conceptual understanding of the notion of a definite integral, how well have they formed this concept. Clearly, it takes much longer than just one lecture. Since the conceptual knowledge becomes an instrument for the subsequent cognitive and practical work, it should not only be acquired by own efforts and correspond to person's natural curiosity,

it must be well organised, reliably and optimally placed in the long-term memory, and be ready for further use. For this purpose, the initial perception is not sufficient, one needs a long practice of the knowledge application in different contexts.

There was an episode in the lecture when due to the nature of inquiry and questions posed it was possible to reveal a gap in students' understanding of the concept that they studied earlier prompting that this concept did not develop into a component of active knowledge. To find the area of a curvilinear trapezoid, Group N° 2 proposed to replace it with a rectangle (Figure 8.5). This prompted me to ask students impromptu an inquiry question: "Is there a rectangle whose area is exactly (and not approximately!) equal to the area of a curvilinear trapezoid?" It was suggested as a homework. And guess what turned out? The next day, 4 students (out of 8) said that they think (intuitively feel) that such a rectangle exists, but do not know how to prove it. Three students answered the question in negative because "it is impossible to know exactly what the height of the rectangle should be." And only one student gave the affirmative answer and substantiated it by "the property of a continuous function (in our case, it was the area) to take on all intermediate values (Bolzano-Cauchy Theorem)."

So, most of the students didn't notice that the area was a continuous function, or couldn't apply the property of a continuous function in different unexpected to them context. The reason, obviously, is that the continuity of the function was studied a long time ago, in the beginning of the first semester, and the students did not activate the relevant knowledge due to its insufficiently frequent use or this knowledge was not properly stored in the long-term memory and didn't become active.

This example confirms once again the following:

- An indicator for the correct formation of active knowledge is the ability to apply it for solving problems of practical and exploratory nature.
- The use of the IBME and its application in problems from various disciplines, including mathematical modelling, are expedient for the formation of conceptual understanding of mathematics and active knowledge.

8.4.2. Brno University of Technology. For partners at Brno University of Technology (BUT), MM means activities motivated by applications of mathematics to real life problems. We believe that MM should contain at least the formulation of a concrete problem, development of an abstract model, and mathematical work with the model (not only simulations, that is part of an engineer's job!). As mentioned in Section 8.2.1, we believe that the 'mathematical work with the model' part can also include pure mathematics activities (e.g., 'playing' with model components and studying consequences) that are not motivated by applications. Being mathematicians, we practise such 'playing with the model per se' that we see as a 'pure math' component of MM. Therefore, we do not see MM as a 'proper subset' of applied mathematics.

We acknowledge that teaching MM at a technical university like BUT, where the faculty in the Mathematics Department mostly offers service courses, represents certain challenge. Two major reasons for this are: (1) elevated number of topics 'packed' into a few semesters of mathematics leaves little space for MM activities; (2) modelling and work with the models are key topics in other specialised engineering subjects. We believe that including more mathematical activities (e.g., analysis) into engineering activities would provide students with a better insight into the modelled problem and consequently contribute to a more efficient learning.

Taking a closer look at the concepts of IBME and MM, we realise that these approaches are similar in many aspects and use similar means to achieve different goals. It is not easy for us to unravel and analyse where an individual aspect belongs and what it contributes to, IBME or MM. However, we feel that teaching MM without inquiry is just a dry transfer of knowledge regardless of the needs of students and learning MM without inquiry is unproductive and unnecessarily difficult. Taking into account the three-layer model of inquiry and MM described in Section 8.3 (see also Chapter 2) and the challenges we see in teaching MM introduced in the preceding paragraph, we realise that educational practices at BUT can, with some restrictions, accommodate inquiry in the first two layers (inquiry into students' learning mathematics and inquiry into teaching mathematics).

Motivated by

- the positive experience of our colleagues at BGKU with MM as an educational approach,
- our own experience with IBME within the PLATINUM project,
- the revealed proximity of the two approaches, and
- the possibility of achieving better long-term learning outcomes,

we decided to incorporate a MM activity into the dense teaching schedule. Here we exemplify our experience with a task that we believe to fall within the intersection of IBME and MM.

The task has been tested in three lessons with groups of 6–12 first-year engineering students, mostly male. Due to the COVID-19 outbreak, the teaching had to be realised in a virtual environment, a combination of an MS Teams online meeting and a shared virtual space substituting the whiteboard. The students were supposed to work in groups in the virtual environment. The formulation of the task was brief: *Given the volume of 0.5 litre of a liquid, minimise the material needed to make a can that would contain the liquid.*

This task has been given to the students in a Calculus I course after they learned the necessary prerequisites: derivatives, applications of the derivative to analyse a graph of a function and to search for local and global extrema. The students were familiar with the examples of applications leading to finding local extrema.

During the lessons, we were able to identify the following elements of the four-step MM cycle:

1. Understanding task. In this introductory phase, an informal formulation of the task has been made more precise. The students reflected about possible shapes of the can and investigated real samples of half-litre cans.

2. Establishing the model. All groups succeeded in turning the task formulation into a mathematical description. Students searched for expressions for the surface area and the volume of a cylinder and a sphere and expressed them as functions dependent on one unknown variable. They recognised that the function defining the surface area must be minimised.

3. Using mathematics. In each group, there were students who suggested to solve the optimisation task using derivatives. All students agreed to that and participated in the solution process. The fact that the minimum value of the function which defines the surface area is achieved at the stationary point has only been commented on, but not verified mathematically.

4. Explaining the model. In the final phase, the students compared results for cylindrical and spherical cans and observed that, as expected, the spherical shape has a smaller surface area. However, they also reasoned that making a spherical can would be technically more complicated and thus more expensive and, last but not least, a spherical can would be difficult to drink from. The relation between the optimal radius and height has been discussed, as well as possible modifications of the task.

We were also able to identify the following phases of the inquiry process and activities:

1. Orientation. In the beginning, students discussed possible can shapes and investigated real samples.

2. Conceptualisation. The students had to figure out that they need to know how to calculate the volume of a solid and its surface area. They identified the problem as an optimisation problem, and that the derivative might be a convenient tool.
3. Investigation. Students searched on the internet for formulas for the volume and the surface area of a cylinder and a sphere, and they measured the dimensions of real cans for comparison. They predicted that the sphere would have minimal surface area. They investigated functions that define the surface area for extrema and claimed that the extremum that has been found is the minimum.
4. Conclusion. The students identified that the sphere would have the minimal surface. However, they used reasoning to conclude that cylindrical shape is more practical for a real-life application.
5. Discussion. At the end of the lesson, students were asked for reflection and feedback on the lesson. They also discussed possible generalisations of the results they obtained to volumes other than 0.5 litre and extensions of the problem to more realistic conditions, e.g., when some free space inside the can is needed to open it without spilling the content.

All three groups came to a solution within the given time. However, in all lessons the student teams made some mistakes. The teacher did not warn the students about that and let them find out themselves. This approach has been valued positively by the students in the feedback. One of the students commented:

> Well, I take it positively. I liked the moment when we calculated a wrong derivative and you (the teacher) let us continue and we made further calculations with that, until we got to the point when it was clear that something is wrong. For example, when I work on a test and make such arithmetic mistake, I get to the point where we were not taught what to do because what we are doing is wrong. Nobody taught us how to do things wrongly, so it is difficult to get back and start again. Here (in the lesson) we could get back and that was good.

Some teacher's reflections after the activity:

- prior to the first lesson, there was some anxiety about the teamwork in a virtual environment;
- the choice of an appropriate shared virtual space (whiteboard) was a challenge, but the work in MICROSOFT ONENOTE was satisfactory;
- all three lessons went well and brought a new experience both to students and the teacher;
- the main disadvantage of the activity was the excessive time demand/consumption caused by the virtual environment;
- it was good to acknowledge that students use the knowledge from other subjects that may have a positive effect on the development of ideas and reasoning;
- it would be interesting to try the activity in a physical classroom setting and compare the outcomes.

In conclusion, note that we perceive the inclusion of the problem of mathematical modelling with elements supporting interest in the standard course of mathematics as fruitful and meaningful. According to our experience, the students were made to think out of the boxes, which helped them not only in understanding the topic but also in improving their problem-solving skills.

8.4.3. University of Agder. At the University of Agder (UiA), there are no dedicated courses in which mathematical modelling (MM) is taught; but the study curriculum assumes that mathematics and engineering students gain some knowledge of MM and methods of applied mathematics. Under these circumstances, efforts are made to include modelling tasks in traditional mathematics courses in the form of small group projects or individual assessments. Calculus, Linear Algebra and Differential Equations courses at UiA are particularly suited for this purpose. For illustration, we discuss an example of a modelling task introduced in a Differential Equations course for the fourth-year engineering students who neither had taken dedicated modelling courses nor had previous modelling experience, but all had some basic knowledge of Calculus and Linear Algebra. The complete analysis of this activity and students' work from the commognitive perspective (Sfard, 2008) has been reported in Rogovchenko (2021), where further details may be found.

The assessment involving multiple modelling tasks was suggested to students in line with the process of modification of the course traditionally taught to master's students in Mechatronics at UiA. The course upgrade was motivated by the lecturer's intention to connect better the knowledge students gained in physics and mathematics courses by introducing several MM assignments rooted in engineering or physical applications in the form of graded course projects. The lecturer set several pedagogical goals: to enrich students' mathematical narratives about the nature of differential equations, promote students' advanced mathematical thinking and the use of mathematical language, contribute to the development of general modelling routines, explain how known mathematical ideas and procedures can be combined and employed to generate new ones, and motivate an explorative approach to MM as a particular problem-solving strategy. In addition, organisation of students' work in small groups introduces important elements of collaborative learning to the classroom and enhances students' social skills.

Forty students in the course (38 males and 2 females, all in their twenties) worked in small groups of two to three students on different sets of modelling problems for one week, discussed their solutions to problems and produced individual written reports. The selection of MM tasks was primarily linked to the subject area of engineering studies, mechatronics, and the complexity level of the problems ranged from closed to open-end problems. Students were asked to employ mathematical methods for finding solutions and use mathematical software of their choice, Maple or MATLAB, to support their work. In addition, the analysis of the validity of the mathematical model regarding its correspondence to the real-world experience or data was required. Students audio-recorded group discussions themselves in the absence of the lecturer and provided recordings at her request for research purposes. Afterwards, group solutions to various problems were presented by each group in a whole-class session. Students' individual written reports were graded as a part of the course work; the mark counted towards the final grade.

A MM task presented below has been designed for the topic Existence and Uniqueness Theorems (EUTs) for initial-value problems for differential equations. Contrary to traditional teaching practices requiring merely formal verification of the conditions of the theorems, students engaged this time with EUTs through modelling problems. Another interesting experience with the use of nonstandard problems for testing students conceptual understanding of EUTs is described in the conference paper by Treffert-Thomas et al. (2018).

Problem. Consider a cylindrical bucket of constant cross-sectional area A with a hole of cross-sectional area a at the bottom of the bucket. The small hole is plugged, and the bucket is filled to height h_0. A clock is started as the plug is removed and the water begins to leak out of the hole. Construct a DE model to determine the height $h(t)$ (m) with respect to time t (s). Take $g = 10$ m/s^2. Choose your values for A and a so that the ratio $A/a = \sqrt{5}$.

(a) Explain all your steps while setting the model.
(b) Take $t_0 = 0$, $h_0 = 4$, set the IVP, explain its physical meaning.
(c) Solve the IVP and observe that the solution is defined for all t but after some time it is no longer a realistic description of the height. What physical event occurs at this moment?
(d) Build a realistic continuous solution to this problem and show that the solution is valid for all times t. Is this solution continuously differentiable?
(e) Do these results contradict the Existence and Uniqueness Theorem? Explain your reasoning in detail.
(f) Plot the solutions found in subsquestons (c) and (d), and analyse the graphs.

Students suggested several physical descriptions of the problem, some of which are illustrated below in Excerpt 3, and discussed corresponding mathematical models. Many students used diagrams to illustrate the translation from verbal description to mathematical formulation and combined in the process of solution several familiar routines. They set up and solved an IVP; this required several steps including the integration of a differential equation, identification of the general solution, and application of initial conditions for finding a particular solution. The analysis of the mathematical discourse presented in students' written solutions suggests that they have developed the ability "to express things in the language of mathematics" (Schoenfeld, 1992, p. 337). In Excerpt 3 from the discussion in Group 1 one can clearly witness the "repetitiveness, and thus patterns which is the source of communicational effectiveness" (Sfard, 2008, p. 195). It was important for this group to agree on the common solution method and to "indorse" the narrative, but not all groups came to an agreement in the end; in such cases, students presented their individual, different versions of solutions.

Excerpt 3

S11: I used the Bernoulli equation to set the differential equation.
S12: I did something similar, but I started from the Conservation of Energy Law to find the velocity out...
S13: I also used the energy law, and worked with some constants and found a nice equation...
S11: Yes, we can use different values for constants, but I chose to have the simplest...

Students in Group 2 used physics laws to derive the following initial value problem:

$$\frac{dh}{dt} = -2\sqrt{h}, \quad h(0) = 4.$$

Formal integration yields the exact solution $h(t) = (2-t)^2$ to the IVP which is obviously valid for all times but due to the problem setting should be considered only on the interval $[0, 2]$ until the bucket empties completely. From the instant $t = 2$ and further on, the bucket is empty and the second 'piece' of the solution to the problem on the half-axis $[2, \infty)$, $h(t) = 0$, can be obtained only by the reasoning in context. Conditions of the Existence and Uniqueness Theorem are not satisfied when the height $h = 0$; this occurs at the instant $t = 2$ and yields multiplicity of solutions to the given initial value problem. During the discussion about 'realistic' solutions, students also pointed out the existence of a formal mathematical solution for 'negative times' obtained by reversing time and argued about the meaning of solution in physical terms ("the water level will approach infinity"). Students' explorations were facilitated by the use of a computer algebra system (CAS); the relevant discussion illustrated in the following Excerpt 4 demonstrates the explorative character of the mathematical discourse.

Excerpt 4

S21: The solution we got is a parabola. (...) After $t = 2$ the solution is no longer realistic. What physical event occurs at this moment? What occurs is that the tank becomes empty and then some sort of refilling starts to happen at the tank, which would not obviously happen with the real tank... It would be very practical for my car (laughter), but unfortunately this is not the case.

S22: We agree that the solution is not realistic after this point, like the tank starts to magically fill in again (laughter).

S21: The way I started to solve this is to make the solution a piecewise function and say that it follows the original solution up to the point when the tank is empty and the second part of the piecewise function is zero for all values of t larger or equal than 2.

S23: Yes, we can use different values for constants, but I chose to have the simplest... I also tried to fit the exponential function, like saying it is linearly independent, but it did not fit very well so I ended up splitting the function.

The attempt of student $S23$ to fit an exponential function can be interpreted as an explorative routine in the use of CAS. Other students in this group solved the differential equation and plotted graphs manually; surprisingly, students did not solve the differential equation analytically with CAS although they already knew how to do this from other assignments. In Figure 8.7(a), the student plotted the formal solution to the problem and then plotted a piecewise defined function corresponding to the 'realistic' solution arguing: "As our solution is a parabola, the reasonable thing to suspect is that after the level has decreased to its bottom value, it will start increasing again. As we plot the graph, we can see that at $t = 2$ the container is empty, and mathematically it starts filling again. So, the solution stops being a realistic description of the height after the container becomes empty." For the 'realistic' solution in Figure 8.7(b), the student defined a function $h(t)$ by two expressions, $(2-t)^2$ for $0 \le t \le 2$ and 0 for $t \ge 2$.

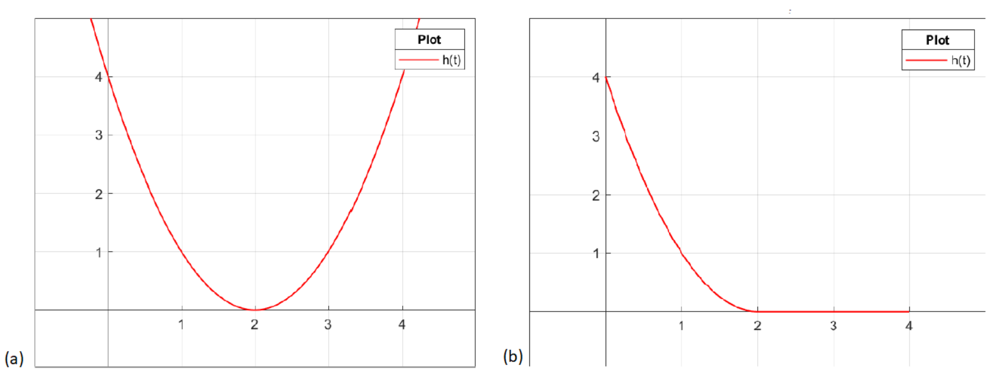

FIGURE 8.7. (a) Student's graph of a formal solution; (b) Student's graph of the 'realistic' solution.

Not surprisingly, in the development of a theoretical mathematical discourse, engineering students felt less confident with the use of rigorous mathematical concepts of continuity and differentiability and often relied on geometric arguments in explanations as illustrated, for instance, in Figure 8.8.

From out first part of the piece-wise function:

$$\frac{dh(t)}{dt} = -2 * (2 - t), \qquad t \in [0, 2\rangle$$

$$\lim_{t \to 2} \frac{dh(t)}{dt} = 0.$$

This gives a line that starts at (0, -4) and ends in (2, 0). Differentiating our second definition gives us 0 for all t and is also continuous. The differentiated functions meet at the point (2, 0). We can therefore say that the function is continuously differentiable and of class C^1. Which is enough for our problem.

FIGURE 8.8. Student's reasoning regarding continuity and differentiability of the solution.

The use of mathematical language in this fragment can be described as immature and the explanation provided by the students sounds rather intuitive, but computer simulation helped to develop the mathematical discourse and supported it. In fact, similar sets of graphs are present in the student's report twice: in the explanation of the solution as shown in Figure 8.8 and in the answer to subquestion (f) of the problem where the explanations to the graphs were explicitly requested.

The analysis of written reports shows that students relied on different representations (realisations) of the modelling task: mathematical description with the help of a diagram (visual), mathematisation using an appropriate differential equation (symbolic), graph plotting (visual), solution of the differential equation with the help of the CAS (symbolic). Students' written reports document striking differences in their ability to use CASs and demonstrate that technology was mainly used as a computational and verification tool and, to some extent, as a helpful visualization tool, but, similarly to the findings of Doerr and Zangor (2000), CASs did not become a transformational tool, nor a data collection and analysis tool.

8.5. Conclusions

Theoretical and empirical research indicates that students' success with MM tasks requires "a well-developed repertoire of cognitive and metacognitive strategies as well as a rich store of mathematical concepts, facts, procedures, and experiences; vicarious general encyclopedic knowledge of the world and word meanings; and truly experiential knowledge from personal experiences outside school or in more practical school subjects" (Stillman, 2015, p. 796). Much of this description is included in some form into the concept of active knowledge introduced in this chapter to relate MM to IBME. MM is not easy to teach and one of the main difficulties is the dependence of learning on the specific context; this requires that MM has to be learnt specifically (Blum & Borromeo Ferri, 2009). None of the three examples presented by PLATINUM partners were related to teaching MM per se but MM was embedded into different contexts. The first example of discovery of the definition of the definite integral with a modelling approach is close to the educational perspective of Realistic Mathematics Education (RME) whose fundamental principle is guided reinvention (Freudenthal, 1991). As pointed by Artigue and Blomhøj (2013), "in RME, modelling, and especially mathematisation, plays an essential role as a vehicle for the conceptual knowledge aimed at with no clear distinction being made between mathematisation of extra-mathematical situations and mathematisation within mathematics" (p. 805). This describes the approach used by the colleagues at BGKU to promote conceptual understanding of the definite integral through modelling tasks. On the other hand, the PLATINUM partners at BUT and UiA fostered students' understanding of applications of mathematics in Calculus and Differential Equations courses through modelling tasks which in both cases required active knowledge to relate mathematics to not very complex but realistic daily situations. All three examples confirmed recent research findings that "working with modelling in mathematics and in other subjects can thereby lead to valuable understanding of inquiry as a more general process with different particular realisations in different disciplines and contexts" (Artigue & Blomhøj, 2013, p. 805).

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CHAPTER 9

Evaluation of Inquiry-Based Mathematics Education

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9.1. Introduction

The PLATINUM project is a joint effort to develop an approach for teaching and learning mathematics at university level that will improve the balance between procedural and conceptual learning of mathematics and build a community of inquiry that will disseminate this approach across European universities. It promotes inquiry-based practices while encouraging collaboration across regional, European and international institutions. This chapter focuses on one of the main intellectual outputs on the project: "Guidelines and recommendations for quality assessment in Inquiry Based Learning environment" (IO6). One of the main goals of the project is not only to provide lecturers with the tools to implement inquiry-based practice, but to offer practical guidelines that enable them to independently monitor their progress in mastering Inquiry-Based Mathematics Education (IBME) methods and students' engagement with more efficient learning approaches (see Guideline Document available on the PLATINUM website). The preparation of this guide took into account the design, testing, and appropriate instruments that enable an in-depth insight into teaching innovations at local level. Based on the experience of the PLATINUM consortium, case studies will be reported and analysed through a cross-case analysis methodology. We will examine different evaluation and measurement tools which have been used within IBME environments in the Czech Republic, the Netherlands, Spain, and the UK. To support the multi-faceted nature of inquiry-based learning (see Chapter 2) we will critically assess available evaluation tools and criteria and adopt those giving a deeper insight into IBME. We assume that local aims and institutional conditions for IBME activities can vary significantly. Therefore advice on the experience of four national teams will be shared. Investigation on the contexts where the research and evaluation tools were developed will allow us to facilitate the transfer of knowledge to other colleagues from other institutions interested in building and fostering the progress and implementation of IBME in higher education. Our intention in this chapter is not to cover all possible evaluation tools exhaustively. Instead, we would like this chapter to serve as an inspiration for other communities to adjust what they find valuable in the evaluation methods described here to their setting.

Since IBME takes place at several interrelated layers as shown in Figure 9.1 (see also Chapter 2), the selection of possible approaches in this cross-case study can offer insight in the different interrelations of IBME processes from different perspectives. In contrast to traditional evaluation, within the inquiry-based approach the focus is

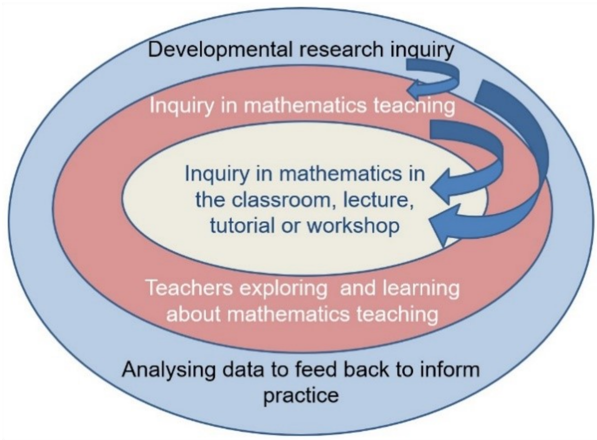

FIGURE 9.1. Interrelated layers in the three-layer model of inquiry.

placed on evaluating the whole learning process covering all layers of the three-layer model of inquiry. Such an approach has both an iterative and a longitudinal nature.

9.2. Research Methodology

A qualitative methodology based on cross-case analysis is used as the methodological research perspective (Borman et al., 2006; Khan & VanWynsberghe, 2008) in order to explore similarities and differences between cases in the PLATINUM consortium.

We purposefully select four cases from different countries (Czech Republic, the Netherlands, Spain, and the United Kingdom) in order to contrast features of the evaluation tools, to implement the methodology of inquiry, and to inform others wishing to adopt it. A number of criteria were taken into account for this choice:

- Different cultural and social contexts.
- The IBME inquiry model consisting of three layers. Inquiry in:
  - (a) engaging with mathematics in inquiry-based teaching-learning situations with students;
  - (b) exploring teaching processes, the didactic and pedagogies involved in student inquiry, and their use in teaching-learning situations to achieve the desired student outcomes;
  - (c) the entire developmental process in which participants reflect on practices in the other two layers, and gather, analyse, and feedback data to inform practice and develop knowledge in practice.
- Some concepts related to IBME such as (1) evaluation of conceptual learning and teaching of mathematics, (2) monitoring students' engagement in IBME, (3) reflection in communities of inquiry on own teaching practice, and (4) professional development of university mathematics lecturers.

This chapter has two dimensions, a theoretical dimension and a practical dimension of design and implementation of evaluation and reflection tools in university teaching practice. The process of preparing this cross-case study has developed in three phases: (1) setting up detailed guidelines for the assessment and evaluation of IBME environments, (2) collecting instruments for evaluation of IBME currently used in the PLATINUM partner universities, and (3) cross-case study analysis of the collected cases.

Multiple cases are taken to establish the range of generality and conditions of applicability of the IBME at university level. The comparative case studies are particularly useful to understand and explain how IBME has been used and which categories have been taken as most relevant. In what follows we will present the case studies in turn. In each of the case studies we will emphasise its characteristics and what contribution is made to further developing the three-layer model of inquiry shown in Figure 9.1.

9.3. Presentation of the Cases

9.3.1. Case in the Czech Republic. At Masaryk University (see also Chapter 13) inquiry-based teaching and learning practices do not have a long tradition. These practices have been implemented in the Mathematics and Statistics I courses in the Faculty of Economics and Administration, and Mathematical Analysis in the Faculty of Education within the PLATINUM Project for the first time.

Our teaching modules are not completely inquiry-based. Inquiry-based activities have a form of small units (either separate tasks or a sequence of linked tasks) incorporated into a traditional curricula. For this reason, the tools used at Masaryk University for courses evaluation cannot be solely applied to IBME as both the traditional procedural approach and the inquiry-based conceptual approach are complementary parts of teaching and learning. Thus, it is not possible to evaluate separately competences achieved solely via inquiry-based tasks. Therefore we decided to evaluate the IBME teaching units individually. The evaluation from the students' perspective has primarily two aims:

- Did students achieve the intended knowledge?
- Were the students active and did they participate actively in the learning process?

The evaluation from the lecturers' perspective mirrors the students' evaluation and follows a similar pattern in asking:

- Were the tasks designed so that they encourage students' thinking and lead to the desired learning objectives?
- Were the tasks designed so that students were engaged and motivated to work on the tasks?

To pursue these aims, we decided to use questionnaires and adopt an experimental design of treatment-control, to observe lectures and seminars, and to organise lecturers' discussion meetings.

Questionnaires. Immediately after the selected seminar or lecture with IBME units, students received a link to a questionnaire with questions related to the benefits of the IBME task from their perspective. The questionnaires combined both questions focusing on learning objectives and questions associated with students' engagement. They had also a space for students' free comments, which proved to be highly beneficial for further development of IBME tasks.

Treatment-Control experiment. The lecturer of two comparable parallel seminar groups on Algebra taught one seminar group with traditional procedural teaching and the other with inquiry-based tasks. The inquiry-based tasks were contained in worksheets that encourage collaboration in small groups. Due to the COVID-19 restrictions and the need to allow students to work in small groups, the ZOOM platform was used as it offers a breakout rooms option and allows an observer to visit and observe students in these virtual rooms. Two weeks later, students from both seminar groups were given the same assessment of the knowledge and skills acquired.

End-of-semester project. In the course Mathematics 2, students completed an inquiry-based end-of-semester team project. The assignment included problems applying linear programming in economics, finance, and management. Teams of three or four students were asked to build a mathematical model of the problem, choose and use appropriate software to solve the problem, interpret the results, and answer additional questions using sensitivity analysis, shadow prices, etc. Groups elaborated the solution independently and met the lecturer every week during office hours to ask for advice on their solution. Finally, students presented the solutions during a lecture slot. After the presentations, the students were sent the feedback questionnaire. Their responses were meant to help lecturers evaluate the activity.

Observations. A qualitative dimension to the evaluation was also added. An observer attended the lectures and seminars where inquiry-based tasks were used and took field notes. Before the COVID-19 related distance learning period, observers were present in class taking field notes to describe the structure of the lessons and the timing of the tasks. Further, the filed notes were complemented by the observers' comments on the students' behaviour, engagement and their inquiry development. During the distance learning period due to the COVID-19 pandemic, the observers were present in the online lessons via the distance learning platform. These observations are used at two levels of inquiry: they help to evaluate (1) students' engagement in the inquiry process and (2) the inquiry-based units from the lecturers' perspective in association to the learning objectives.

One of the observers had a dual role: she was both observer of a seminar group of one of her colleagues and she was teaching the same content in a parallel seminar group. Thus she could utilise her evaluation of inquiry-based activities in her teaching and to share this experience with other colleagues.

Discussion meetings. At the beginning of the PLATINUM project, a Community of Inquiry (CoI) was established, as reported in Chapter 13. One of the purposes of the CoI was to hold meetings to evaluate the IBME units and to discuss further development of IBME units in future. As our community is new to the IBME approach, the evaluation of our experience with IBME units will be valuable for other colleagues at our university. At the meetings, we evaluate many aspects of IBME tasks implemented in our teaching. These include the compliance with the learning objectives and coherence with the traditional curriculum as well as technical aspects such as the timing or the reactions of students who are not used to discussion in the mathematics classroom and do not feel comfortable when risking being wrong when volunteering contributions to the solution of a task.

9.3.2. Case in the Netherlands. At the University of Amsterdam (UvA) the inquiry-based mathematics education (IBME) was implemented in the mathematics courses in the Bachelor Psychobiology and in the Bachelor Biomedical Sciences. A strong community of inquiry has developed around these two courses. The courses had a blended learning design using the digital tools RSTUDIO, SOWISO,[1] and the learning management system CANVAS. During the COVID-19 pandemic the course was online and it used also MS TEAMS. Next to the interactive lectures and tutorial sessions that included short small-group sessions, individual online asynchronous learning activities took place in SOWISO in combination with RSTUDIO. In several parts of the course students worked on realistic problems and discussed solutions following an inquiry based learning approach (see Chapter 12, UvA case study). During the COVID-19

[1]SOWISO is a cloud-based environment specifically designed for learning, practising and assessing in STEM courses, see (Heck, 2017).

pandemic the IBME activities took place in MS TEAMS channels in groups of 4 to 5 students. The teaching staff team collaborated in the format of a Community of Inquiry via a private MS TEAMS channel. In all study years the exam was remote and it was taken in SOWISO.

The evaluation of inquiry-based mathematics education (IBME) activities took place on all three levels of the three-layer model shown in Figure 9.1 (cf., Chapter 2): student level, lecturer level, and the developmental research inquiry level. Different instruments were used in this evaluation process. Some of the instruments were developed as part of the PLATINUM project, some were already in use before or they were adapted. The evaluation study was done in two academic years: 2018-2019 and 2020-2021. Henceforth we focus on the evaluation process that was applied to the inquiry-based mathematics module for first-year students in Biomedical Sciences, which is presented in more detail in Chapter 12.

Instruments for students. Two types of instruments were used to evaluate IBME from the student perspective: (a) an online questionnaire and (b) semi-structured small-group interviews. The evaluation instruments for the IBME tasks were integrated into the instruments that were also used for the evaluation of other aspects of the course. The semi-structured interviews were done only in the third year that the Biomedical Sciences course ran.

The online questionnaire used for the evaluation of students' perception of IBME was applied as a pre- and post-test. The questionnaire was administered online in SOWISO. In the pre-test, at the beginning of the course, biographical data and information about the student background were collected. In the post-test the students were also asked to reflect on learning mathematics. The questions in the test were also on mathematics anxiety, test anxiety, and motivation and engagement. Students also took a diagnostic mathematics test. Standard questions sets from standard instruments and translated into Dutch were used to measure mathematics anxiety (Hopko et al., 2003), test anxiety (Spielberger, 1980), and motivation and engagement (Martin, 2007). In the post-test questionnaire, the following three 5-point Likert scale questions were included that were specifically oriented towards the experience with IBME tasks:

- Q1: In some mathematics tasks you had to find out/discover things by yourself. Such an approach of "inquiry-based learning" appeals to me.
- Q2: A small inquiry task to be carried out in pairs for example as a bonus task, seems to me a useful extension of the course.
- Q3: There were sometimes short tasks embedded in the lectures (for example, inventing a method for numerical differentiation and practising with line element fields and direction fields). I learnt much from these tasks.

The semi-structured interviews were conducted after the IBME tasks and took place in the last two weeks of the course before the examination. For this purpose students were invited in small groups of 10. Participation in the interviews was not compulsory but very much encouraged. To get more responses and to lower the generation gap the senior and the junior lecturer were not involved in this stage of the evaluation. The interviewers were the teaching assistants who had attended the course as biomedical students one or two years before and had been involved in the design of the course as members of the CoI. These teaching assistants invited the students for the participation in the interviews. Each semi-structured interview was taken by a team of two teaching assistants; one was asking questions and the other was taking notes. The interviews took place online in MS TEAMS. The questions in the semi-structured interview were clustered in four groups: (1) similarities and differences with the secondary school mathematics content and working style, (2) support in working with RSTUDIO, (3) learning materials in SOWISO, and (4) orientation on inquiry-based learning. In the part about inquiry-based learning the students were asked three open-ended questions:

Q1: To what extent does the teaching material encourage thinking about mathematics and its applications in Biomedical Sciences? Did it change your ideas about doing mathematics at all?

Q2: "Having to figure something out for yourself or together with other students" gives a picture of mathematics that does not have to follow a prescribed route or provides no ready-made answers to questions. Mathematics is then seen as a tool to better understand processes or situations and not as a standard procedure to arrive at a correct answer (think, for example, of different regression methods or different techniques for numerical differentiation from which an underpinned selection must be made). Two questions: (a) How new is this to you and how do you feel about it? (b) Do you feel encouraged to do such assignments in the course?

Q3: Would you like to perform more or fewer open-ended assignments and why?

Instruments for lecturers. The procedure for this evaluation level was slightly different for the two academic years in this study. In the study year 2018-2019, the principal lecturers had established an IBME community of inquiry (CoI) together with two junior lecturers. During the course the lecturers met once a week for one hour after the last session of the week with students (face-to-face). Reflective discussions according to the IBME framework and structured oral evaluations were used as instruments during these weekly meetings. The meetings were recorded and minutes were taken. The three lecturers also wrote narratives as their personal reflections. In the study year 2020-2021, the principal lecturer established the IBME CoI together with one junior lecturer and three teaching assistants. The three teaching assistants had followed this module in previous years. The meetings started already four weeks before the start of the course to discuss the course materials and assignments biweekly online in MS TEAMS, and weekly when the course had started. The teaching assistants reflected on their own learning experience as students and the team members collaborated on the development/adaptation of the (new) materials using their reflections.

The questions for the semi-structured interviews were developed by the teaching assistants in collaboration with the lecturers. The pairs of teaching assistants who did the interviews wrote a report of each interview and shared it in the IBME CoI. The results of the students' questionnaire (pre- and post-test) and the analysis of the reports of the semi-structured interviews are presented and discussed in the UvA case study in Chapter 12 of this book.

Instruments for developmental research inquiry. One PLATINUM project team member joined to the lecturers' Community of Inquiry of the Biomedical Sciences course. She attended the meetings of this CoI as an observer of the process on the level of the developmental research inquiry. She observed also a lecture given by the senior lecturer in this CoI in which a short IBME task was used, and a group IBME session where students worked on a longer IBME task based on biomedical research data and the programming language R. The instrument on this level was making observation notes and writing narratives based on observation notes.

9.3.3. Case in Spain. At the Complutense University of Madrid (UCM) inquiry-based mathematics education (IBME) is implemented in the mathematics courses in the Bachelor Mathematics, Bachelor Mathematics and Engineering, Bachelor Mathematics and Statistics, Bachelor Computer Engineering, and in Programmes of Professional Development for mathematics lecturers. The evaluation of inquiry-based activities took place on three levels: (1) the student level, (2) the lecturer level, and (3) the level of lecturers' professional development. Different instruments were used in this process. This section focuses on the lecturers' professional development. The case presented is about the professional development of novice lecturers, in particular within the training unit about teaching Rolle's Theorem: "Intuition on Rolle's Theorem and its extensions." The materials and evaluation instruments are original and specifically developed for the PLATINUM project.

Professional development and teaching context. The Faculty of Mathematics at UCM develops courses for university teaching qualification of novice lecturers. These courses aim to provide university lecturers and research assistants with educational tools that enable them to better design, implement, and analyse teaching and learning processes. Three organising principles guide the design of these resources:

- To enable lecturers and research assistants to make informed decisions on what they teach and how they teach it.
- To train novice lecturers and research assistants who are becoming lecturers, in the growth of their conscious awareness: self-awareness as lecturer, awareness of discipline, awareness in guiding others by teaching them to learn, and by learning to learn.
- To develop lecturers' professional identity through a continuous reflection on their professional role and their specific vocation.

All novice lecturers participating in the PLATINUM professional development course on inquiry-based mathematics education had to design mathematical tasks or units following the inquiry approach to be implemented in the classroom. These tasks were presented at team meetings and discussed together before they were implemented. The lecturers were also observed during the teaching of a mathematics unit (this means between one and three lectures/sessions) and the sessions were video recorded. The PLATINUM community of inquiry gave feedback, watched the video of the lecture, and reflected on the teaching, the behaviour of students during the lecture and the anticipation of the lecturer on the activity of the students and evaluated these.

In the proposed conception, research and development are mutually involved. Professional development is viewed from a reflexive position concerning practice. The aim is for the novice lecturers to join in the PLATINUM project over the practice, questioning and analysing it, and even transform it according the approach of inquiry-based mathematics education.

It is important to remark that there exist two types of context: (1) the professional development course or formative situation, involving the trainer (professor in mathematics education) and trainees (novice lecturers), and (2) the teaching situation in which the lecturers work with undergraduate students. These situations produce different levels of activity and practice for professional development and the lecturers-in-formation (see Figure 9.2).

Different instruments are used in this evaluative reflection process: (1) the lecture plan and the proposal for the inquiry based tasks, the planned teaching, learning and (formative) assessment activities in the time frame of the session; (2) the video recording of the lecture/session; (3) the peer feedback, the observation report of the observer of the session (in the case presented here, the lesson was observed by two members of PLATINUM, one from Spain and one from England); (4) the students' evaluation

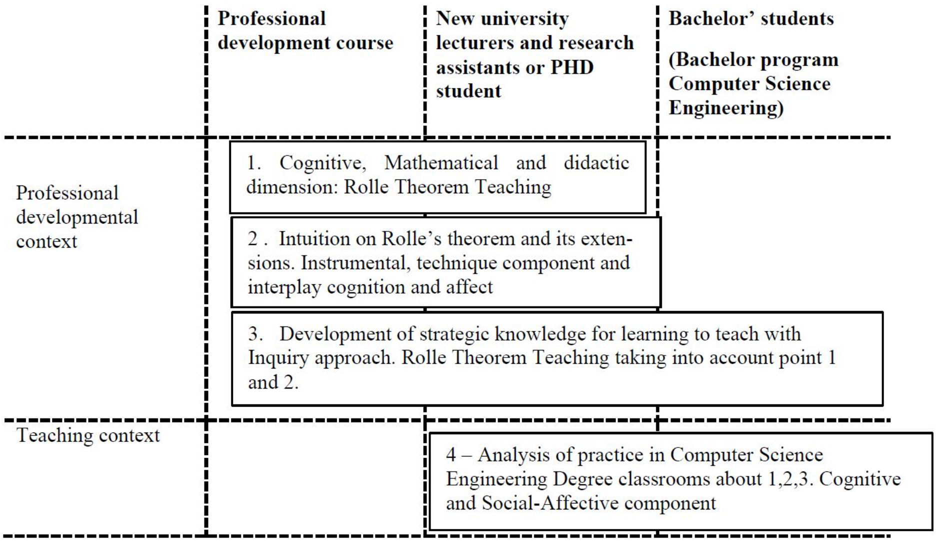

FIGURE 9.2. Professional development and teaching context at UCM.

(questionnaire); (5) the novice lecturer's evaluation (semi-open questionnaire) and interview; and (6) the reflective report of the novice lecturer based on self-observation, peer-observation, and the students' evaluation.

Here we consider two evaluation instruments: (1) a student evaluation questionnaire, and (2) a questionnaire for new lecturers' evaluation.

Instruments of evaluation for professional development — Teaching Rolle's Theorem. Prior to the presentation of the evaluation instruments, we present the learning tasks on the lesson "Rolle's Theorem teaching: Intuition on Rolle's Theorem and its extensions."

Overview plan of inquiry-based tasks. Calculus is a first year subject of the bachelor program in computer science engineering. It is an introductory course that starts with the definition of real numbers and the construction of sequences and series, and it covers differentiation and integration of functions of one variable with applications. Weeks 9 to 11 of the course focus on functions and their derivatives. The main concepts regarding this subject had been already learned in high school, but since the backgrounds of the students vary greatly a revision and reconstruction of some these ideas is deemed necessary. Rolle's theorem and its extensions (the Mean Value Theorem) is one of the big theorems in Calculus because it establishes a connection between continuity and differentiability. Though it is a very simple and intuitive theorem, it requires the understanding of limits, continuity, and differentiability. The goal in the mathematics unit is to clarify different concepts regarding functions and to redefine them with intuition. The inquiry-based tasks are articulated in:

(1) A revision of the known concepts: who is who?
(2) Intuition on Rolle's Theorem and its extensions.
(3) Understanding the concept of derivative. For more information see (Luque, 2019).

Some instruments. Two evaluation tools are described below.

(1) *The student questionnaire.* The questionnaire was administered to a group of 34 students enrolled in the Bachelor's Computer Science Engineering after receiving

theoretical and practical background on the concept of functions and derivability.[2] The questionnaire was divided into three parts:

(i) Functions and derivatives concepts through the inquiry process.
(ii) Rolle's Theorem: mental image and intuitive understanding. Students were asked to explain in their own words Rolle's theorem and to identify in different plots of graphs whether it is possible to apply the theorem.
(iii) Grading the understanding and interplay between concepts continuity and differentiability: metacognitive and affective factors can inhibit the correct utilization of students' knowledge.

(2) *The novice lecturer questionnaire.* At UCM the PLATINUM evaluation tools have been elaborated within the Design-Based Research Collective (2003). Two hypothesised dimensions that constitute mathematical knowledge for teaching were the focus: mathematics content knowledge and mathematics pedagogical content knowledge. In identifying the elements of these dimensions, the design of inquiry-based tasks and the development of these tasks in the classroom (Figure 9.2) are considered for the evaluation instrument for novice lecturers.

The questionnaire is intended to ensure that the novice lecturers' practice is based on self-observation and the students' evaluation, and that novice lecturers reflect on their teaching practice and the strategic knowledge that they have developed for teaching. The semi-open questionnaire is structured into three parts:

(i) Regarding inquiry-based tasks design: it was inquired at three moments, viz., before, during, and after the implementation time.
(ii) Regarding mathematics conceptual topics: intuition on Rolle's Theorem and its extensions and interplay between concepts. The teaching plan aimed to balance all three components of mathematical representation (graphical, numerical, and algebraic) to enable the students to view ideas from different standpoints and develop their intuition and a holistic perspective of each concept. It allows to review the results of the students in the sense of (i) the learner's ability to state the theorem and apply it to reasoning tasks, (ii) the influence of concept images in his or her reasoning about the theorem, (iii) the learner's ability to perceive the relationship between Rolle's Theorem and other related mathematical concepts and mathematical attitude, and (iv) metacognitive and affective factors possibly inhibiting the correct utilisation of knowledge that the students should use to solve a problem. We note that this block of questions in the questionnaire cannot be seen in isolation; it is in close relation to the questions posed to students in their questionnaire.
(iii) Regarding the mediation of the PLATINUM Community of Inquiry. This group has had a significant influence on the lecturer's professional development, offering teaching intuition. In this section of the questionnaire there is a reflection on the tacit knowledge dimension acquired. Some aspects considered are the following:
    - goal setting in motivating and guiding oneself to attain the desired end goal;
    - systematic problem solving by using resources and orientations;
    - personalising situations by appropriating one's strengths and weaknesses.

    They are asked to develop a narrative of their experience.

9.3.4. Case in the United Kingdom. The type of evaluation described in this case differs from the previous cases. It involved reflecting on the work of small, informal groups of mathematics and mathematics education lecturers coming together

[2]See the PLATINUM website: `https://platinum.uia.no` and (Gómez-Chacón & Luque, 2019).

to talk about their teaching practice and the connected educational research. These meetings were not part of a training organised by the institution, but they did originate from the desire to discuss mathematics-specific teaching and learning issues. As described in Chapter 15, Loughborough University (LU) in the UK has a long tradition of formal and informal collaboration between mathematicians and mathematics educators in Communities of Inquiry (CoIs). These CoIs can take the form of small groups of colleagues reflecting about teaching and learning mathematics, or are supported via funded projects that aim at involving students in the creation and testing of inquiry tasks. In this chapter, the evaluation process of one informal CoI involving mathematics lecturers and mathematics educators colleagues is summarised.

Some ideas about evaluation. What we discuss in this section is not and cannot be a rigorous evaluation of the impact of taking part in a CoIs on the participants' teaching practices. What are described here are some ideas that can help colleagues trace some of the outcomes of small, often informal, reflection both on the practice and on educational research for those who have taken part in the CoI. The main inspiration for these principles is the work that Pawson and Tilley (1997) report on realistic evaluation. The guiding principle of realistic evaluation can be summarised by the following quote:

> Whereas the question which was asked in traditional experimentation was, "Does this work?" or "What works?", the question asked by us in realistic evaluation is "What works for whom in what circumstances?" (Tilley, 2000, p. 4)

Of course, to follow the principle of realistic evaluation for large interventions (not only educational) is complex and requires a well-structured team of researchers versed both in quantitative and qualitative research. However, from this work we can find three guiding principles that can be useful also for more informal qualitative evaluations of smaller activities. These principles are:

- Focus on the context where the evaluation was introduced. What are the contextual characteristics that may lead one activity to be successful in one implementation and not in another?
- Small evaluations can ask bold questions regarding in our case the effectiveness of informal CoIs. Pawson and Tilley (2001) argue in one of their writings that even very small informal interventions, if guided by theory, can contribute to the refinement of that same theory.
- It is important to focus not only on the outcome of the intervention (did participation to a CoI of colleagues discussing teaching and learning change the practices of those who took part in it?) but also on the mechanism that lead to such intervention. The description of such mechanism will help others ascertain whether that intervention has the potential to be successful in their own context.

In what follows we discuss how these principles have guided the evaluation of one of the case studies that took place a Loughborough University.

The teaching group: a small informal CoI. As described in Chapter 15 this was the work of a small group of mathematicians and mathematics educators (all teaching mathematics or statistics at university in the same institution) who met four or five times per year for three years to discuss topics related to the teaching and learning of mathematics at university level. The details and general aims of these meetings are described in Chapter 15. Here we want to focus first of all on what was evaluated, and on—paraphrasing Tilley (2000)—what worked for whom in what circumstances. The questions we asked were:

- What activities were conducive to effective reflection on practice?

- What activities had a visible impact on practice?
- What activities were not deemed to be beneficial?
- What were the aspects that facilitated participation to the sessions?
- What were the aspects that prevented participation to the sessions?
- What were the contextual factors that facilitated (or prevented) participation to the sessions?

Given the nature of the questions and of the activity that is investigated we collected two distinct types of data: the documentations discussed throughout the existence of the group meetings and a series of semi-structured interviews with stakeholders. The documentations consisted in research papers that the participants suggested as reading, materials related to teaching that were brought to be shared (e.g., questions in exam papers, or suggestions for feedback to students), or simply questions for discussion by the group. Analysis of the documentation collected indicates that one of the main concerns of the group was summative assessment practices in mathematics. This reflects a general preoccupation in the institution where the CoI was based and in the UK more at large with issues related to assessment. During these sessions we would both discuss concrete examples of exam questions volunteered by one of the participants and research papers on the topic. The analysis of such documents is very important on this evaluation as it allows the analysis of the contextual factor that guided the interest of the CoI.

Regarding the semi-structured interviews, stakeholders are considered to be not only colleagues who took part in the meetings of the teaching group but also those with responsibility in the mathematics department connected to teaching. Therefore in the case of the Teaching Group, stakeholders were not only those who took part in the session of the CoI but also those with responsibility for teaching in the mathematics department, such as the head of the department and colleagues who had shown an interest in teaching and learning of mathematics (from our experience) but did not take part in the sessions of the CoI. Interview questions included in the interview schedule were:

- When you started at Loughborough University, were you new to teaching mathematics at university level? What kind of students/year groups have you taught or are you teaching now?
- What aspect of your teaching are you particularly pleased with, or alternatively, are thinking of changing?
- Consider a course that you have taught more than once. Did you make any changes from one year to the next? Why?
- Do you think you teach like your colleagues?

These questions aimed to investigate from the general to the particular and aimed to ascertain participants' perceptions of the benefits or drawbacks of having a CoI like the Teaching Group in the department. Through the analysis of the interviews it was possible to understand the trajectory of the CoI, which stopped meeting in July 2019. The analysis of the data also allowed us to understand the role of the 'value' that was put on such initiatives by the Institution and the fact that without even informal institutional support such activities cannot flourish.

9.4. Contribution of the Cross-Cases Study: Challenges and Issues

The choice of the cross-case study methods was made to highlight the differences that can occur in the implementation of Inquiry-Based Mathematics Education (IBME) in different contexts and at different layers of the theoretical model. Therefore there is a range of foci of evaluation and evaluation tools. The cases described in the chapter intend to be an example of foci and tools that others can follow when evaluation their own IBME in their own context. In this section we synthesise commonalities and differences in the evaluation using IBME in four contexts and to what extent they become challenges for future implementations.

A first observation is how each partner contributed to the three-layer model of inquiry and the notion of a Community of Inquiry (CoI). Since local aims and institutional conditions for IBME activities can vary significantly, advice on the experience of seven national teams will be shared. The cases presented here contribute to enriching the layers and the interplay between layers shown in Figure 9.1). For instance, the selection made by Loughborough University focused on Communities of Enquiry (CoIs) between mathematicians and mathematics educators and offered ideas for the evaluation process of small CoIs. All universities cover all three layers of the three-layer model of inquiry; however, each university emphasises its specific area. The University of Amsterdam takes into account instruments for students and lecturers implemented in the mathematics courses in the Bachelor Psychobiology and Bachelor Biomedical Sciences, and Masaryk University with instruments implemented on the courses Mathematics and Statistics I at the Faculty of Economics and Administration, and Mathematical Analysis at the Faculty of Education.

The case of the Complutense University of Madrid focuses on the professional development of mathematics lecturers and offers an insight into the different interrelations between layers. It also proposes tools to evaluate the entire developmental process in which novice lecturers react on practices in the other two layers (teaching in the classroom and receiving feedback data to inform their practice and develop knowledge in practice).

In the analysis of the differences in the cross-case analysis, we highlight two: (1) the characterisation of the inquiry community and (2) how the choice and integration of various tools for evaluation has taken place.

Communities of inquiry. To support the multi-faceted nature of inquiry-based learning showcased by the PLATINUM project and presented in the intellectual output about evaluation, a key element has been the type of community of inquiry. Some highlights are the previous trajectory in mathematics education or the member composition (mathematicians members-only, or a mixed community of mathematical educators and mathematicians). For instance, the Czech Republic team's expertise in statistics and statistics education, and in the nature of the IBME has enabled them to trial an experimental design and a more quantitative approach. The LU experience focuses on the realistic evaluation approach, and the UvA and UCM cases combine a natural approach with design-based research that includes different cycles of monitoring and evaluation.

Choice and integration of various tools for evaluation. The evaluation tools used in each case reflect the nature of the CoI and the activities evaluated. Each team critically assessed the available evaluation tools and criteria and adopted those to their context giving a deeper insight into the working of their IBME. This chapter aims at equipping the readers with similar tools to critically evaluate tools that allow them to adopt what is most suitable to the situation investigated.

9.5. Conclusions

We have presented four implementations of IBME in the teaching practice. In the experience of evaluation of very different CoIs we highlighted how each one contributed to developing further the three-layer model of inquiry that the PLATINUM

CoI adopted (Figure 9.1, see also Chapter 2). We have deliberately sought to compare cases that differ in their forms of evaluation implementation in order to find similar processes or outcomes in the IBME PLATINUM approach. We believe that this case-oriented approach emphasises diversity in the selection of cases. Its potential lies in its ability to extend lessons learned in individual cases to inform another case and discover similar processes in unexpected contexts. In examining the differences between the cases we have covered both learning about mathematics with students and learning about teaching and learning mathematics with lecturers. The common aspect that the cases presented have is the engagement in the CoI and the subsequent engagement in developing the work of the CoI in the light of what was learned through the process. This—as we have seen previously—is the third layer of the CoI and the one that needs developing in time.

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Part 3

Learning About Teaching: Case Studies

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CHAPTER 10

Introduction to the Case Studies in PLATINUM

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PLATINUM is a project in which the main focus is the development of teaching and learning mathematics through an inquiry-based approach in which our students, through their own inquiry in mathematics, engage more deeply and develop more conceptual mathematical understandings. The project spans seven countries with partners in eight universities. In each of these universities the partner team has engaged with ideas about inquiry-based learning and teaching in mathematics and, in so doing, members have developed their own practices.

In our submission to the EU Erasmus+ programme, we promised a book in which each partner would write a case study, from their inquiry community, in which they would discuss their learning and development through engagement with the six Intellectual Outputs of the project (see Chapters 2 and 5 and below).

This chapter provides an introduction to these case studies and a brief perspective on the main elements of each case study.

10.1. Inquiry-Based Mathematics Education—Basis for Our Case Studies

10.1.1. The Idea and Nature of a Case Study. We anticipated (and indeed expected) that each partner team would take a developmental focus related to their own history and academic culture and that these would therefore differ from one country, from one institution to another. As the project has developed, it has been interesting to see these focuses emerge in relation to our agreed three-layer model (Chapter 2) for exploring inquiry-based activity. The idea of a case study was intended to give each partner an incentive to reflect on their development in a very local and personal way that could provide readers with an insight into their developing experiences. With communication through meetings, workshops, dialogue and writing, a vision of these local experiences began to emerge and we started to perceive a local 'essence' in that experience. We were all encouraged to write 'narratives' that captured our own experience of participating in this project, and it was interesting to see how the narratives developed with the project. Early narratives were written in a rather formal manner in which the narrator could be seen as an outsider reporting on an observed event. Gradually we started to see individuals taking an insider role, reporting their own activity, decisions, issues and feelings. It is these more personal narratives which provide a deeper insight into the essence of participants' experience. In writing our case studies, we were encouraged to include extracts from these more personal narratives and you will see how this is done in different ways as you read.

10.1.2. Our Developmental Processes. When we began this project, every partner group and every individual within these groups had a vision of the project and a corresponding vision of inquiry-based mathematics education. Undoubtedly, these visions differed. Some were related to extensive experience of developmental activity in inquiry-based mathematics learning and teaching (often abbreviated to IBME—Inquiry-Based Mathematics Education). This may have included theoretical perspectives and a knowledge of related literature. Others were coming new to IBME, perhaps with some feelings of uncertainty, even insecurity. Overall, it is fair to say that we were all concerned about our students' learning of mathematics and the extent to which they engage with mathematical concepts. The levels of experience varied considerably, together with perspectives on what IBME might look like in practice, what the difficulties might be in achieving it, and whether it was really possible to realise our aims in the local environment and culture. Project organisation has, fundamentally, respected these differences and tried hard to work with them sensitively. While the Intellectual Outputs (IOs), as written in our proposal, have provided guidance for our developmental activity in each partner group, there has been space and encouragement to work locally according to our own visions of what could be involved. For example, IO1 provided a theoretical framework, IO2 an expectation to work together and form an inquiry community in our partner group; IO3 gave a lead on the design of inquiry-based mathematical tasks for use with our students. The extent to which we have focused on each of the IOs has been for each group to decide. Sharing our activity and perspectives through our project workshops has enabled us to grow in understanding, both as a project community and as partner communities, of what might be possible and how we would interpret the expectations of the IOs. The local teams, each developing as an inquiry community, have explored possibilities in their own ways and the result, we believe, offers a richness of experience and outcomes. The diversity of essence makes clear for readers the many ways in which IBME can be interpreted and experienced at university level.

10.1.3. Issues and tensions. For all of us in the project, the developmental process has had many elements, paths, directions and experiences: many satisfying, rewarding, illuminating; others more challenging, disturbing, worrying. When we try out new practices – new ways of presenting mathematics, new activities for our students, new ways of being a teacher, new ways of expecting students to learn—the outcomes may not be what we had envisaged or hoped for. While this is likely to be a great learning experience, it can also be depressing and demotivating. Issues can arise due to factors such as the available lecture theatre or tutorial room, students' responses to what we have asked them to do, technical limitations, educational infrastructure. They can also arise due to our own ways of presenting ourselves and interacting with students. In our awareness of these possibilities, we may unwittingly influence our students against the practices we would like to promote.

What is perhaps important is not our narrow judgmental evaluation of such outcomes, but our *inquiry* into the factors involved and how any of these might be changed to afford outcomes more in line with our goals and associated visions. This can be where *inquiry*, as well as being a factor in and for mathematics and its learning, can be at the centre of our developmental process. If, when we plan a lecture or seminar, we see our actions as a design stage for something we will try out and reflect on its outcomes, we may become aware of a range of factors that were not visible before, but which can be modified subsequently. The outcomes inform us, give us insight into what is possible or not, and why. In these respects, we work in the second layer of our model. This approach is sometimes called 'action research' (Elliott, 1991) or 'design research' (Design-Based Research Collective, 2003) depending on how it is carried out. When such an approach is informal, the developmental outcomes are informative and encourage us to reformulate and try again. When the approach is more formalised, it becomes a research approach, fulfilling the criteria for validation, verification and trustworthiness of interpretation and formalisation. In these respects, we are working in the third or outer layer of our model and producing outcomes that can be shared more widely to inform our professional and/or research community (see Chapter 2).

In these case studies we find examples of all the elements mentioned above. We hope that they inform and inspire you as reader to engage with IBME in your own environments, inquiring into your own practices and their development and making possible for your students to gain a more inquiry-based perspective of mathematical concepts.

10.2. Elements of Our Inquiry Activity in the Case Study Chapters Which Follow

10.2.1. References to Didactics and Pedagogies. Fundamental to all teaching, even if not stated or recognised overtly, are concepts and practices under the headings *didactics* and *pedagogy.* These terms are perhaps redolent of education courses in teacher education programmes. However, mathematics-teacher-education programmes exist primarily in pre-tertiary education and, to date, there are far fewer educational programmes directed towards teachers in higher education. As Winsløw et al. (2021) have pointed out, there is a growth of general educational programmes in universities, although much less that is subject-based (e.g., mathematics-based). General educational programmes tend to deal more with pedagogy than with didactics which is highly subject related.

Just briefly, didactics of mathematics deals, practically, with the ways in which teachers who *know* mathematics transform this *knowledge* into activities for learners. Such activities include listening to exposition or explanation from a teacher, making sense of examples provided by the teacher, working on mathematical problems (perhaps with their peers), engaging with mathematical tasks carefully designed by the teacher to focus attention on key elements of mathematics. Theoretically, didactics addresses the relationships between the engagement in mathematical activity and the learning of mathematics and is the province of 'didacticians' of mathematics in university education. In PLATINUM, we have focused rather more on the practical side of didactics than the theoretical side.

In contrast, pedagogy in mathematics learning and teaching focuses on the ways in which activity with students is organised. So, for example lectures to several hundred students are a form of pedagogy. Organising students into small groups to work on carefully designed tasks or problems is another. As with didactics, pedagogy has its own theoretical bases, often addressing learning and what it means to learn. For example, general theories include constructivism or behaviourism; more particularly related to mathematics are the theories around problem solving, or in our case inquiry-based learning of mathematics (IBME). As with didactics, pedagogy in PLATINUM has been much more practically focused.

In the chapters which follow, you will find considerable focus on didactics and pedagogy, even where these terms are not used explicitly. This is because we are fundamentally addressing what teaching and learning mathematics mean for us. Especially a pleasure in reading the case study chapters has been the ways in which different authors have explored the educational literature to inform their writing, or have used Information and Communications Technology (ICT: see below) to find new ways of exploring mathematical concepts. It is clear from reading these chapters what a valuable experience PLATINUM has proved to be in terms of our own learning in these areas.

10.2.2. The Three-Layer Model—a Basis for Each of the Cases. The PLATINUM project can be seen to draw on a range of theoretical perspectives informing inquiry-based learning and teaching and relating to the educational perspectives of those speaking or writing. Despite this theoretical diversity, at the centre of PLATINUM has been the *three-layer model* which has provided a basis for both theory and practice in our activity. In our proposal to Erasmus+, we promised to develop a (theoretical) framework which could guide our work in PLATINUM: the three-layer model provides this framework (Chapter 2).

Much of the published work about inquiry-based teaching and learning relates to pupils or students in classrooms and their inquiry into mathematical concepts, as well as the design of tasks for this purpose. This has of course also been central to PLATINUM. However, PLATINUM has gone further to see teachers' design of tasks as an inquiry process in which we develop our knowledge of task design through an iterative, cyclic, process (plan, act, reflect, feedback) in which we refine our plans at each cycle. The result might be a prototype task developing its potency at each stage or the experience gained by the designer in seeing the task in use, or indeed as a teacher putting it into use (see Chapter 12). In all these cases inquiry in each cycle leads to new knowledge and awareness for the teacher/designer. This is a (natural) professional development process which does not depend on formal training. The third layer of the model makes this whole process less intuitive and more explicit. We seek to justify each stage of the process, presenting evidence for our claims for learning and development, perhaps through the reflections of those involved or by analysing data from the inquiry activity.

10.2.3. Community of Inquiry (CoI). *Community of Inquiry* is a fundamental concept in our theory of inquiry in PLATINUM. It spans the entire three layers of the framework and crops up in all of the chapters below, in some cases very frequently. We have taken the idea of CoI from the literature (e.g., Cochran Smith & Lytle, 1999; Jaworski, 1998; Wells, 1999) and have built its use in PLATINUM on projects in Norway in which teachers at school level worked collaboratively with didacticians in a university to develop the mathematics learning of pupils (e.g., Goodchild, 2008; Jaworski, 2008). PLATINUM is, we believe, the first use of this theory in the teaching and learning of mathematics in university education. The many references to CoI in these chapters provide evidence that this theoretical construct (CoI) has been taken up in practice by these university teachers and assimilated into their thinking about and language of teaching development. A CoI can consist of 2 people or 20 or 200 people; its characteristics are that its members inquire into their learning and into their practice. So, we can have a CoI between students learning mathematics, between teachers designing a teaching unit, or between didacticians and teachers, together, analysing the learning outcomes of teaching. Possibly the best way to find out what CoI means is to read these chapters and build up a picture from their differing ways of describing this concept.

10.2.4. Working With Students Who Have Identified Needs. As teachers we are all aware of the great diversity of needs of our students, although not always confident that we know how best to recognise needs and provide support. In Chapter 4 we read:

> Taking this diversity into account, we prefer to use the social model of disability whereby difficulties are seen as a product of social circumstance, removing the onus from the individual and giving the responsibility for inclusive learning environments to educational

> institutions. “This is in contrast to the medical model of disability that concentrates on the impairment as the cause of the disability” (Drew, 2016, p. 30). (p. 50)

In PLATINUM, led by colleagues with expertise, we have been introduced to a variety of particular needs (e.g., physical needs caused by sight or hearing loss, neurodiversity including dyslexia, ADHD) with advice and discussion about how the social setting (e.g., of mathematics teaching) can be prepared or adjusted so that we do our utmost to *include* the students and *address* their needs. Chapter 4 offers a comprehensive perspective on what *addressing* might involve. *Inclusion* is more complex. As Chapter 4 asserts:

> The social model requires that educational institutions take on responsibility and break down barriers in order to ensure an inclusive learning environment. (p. 50)

For many colleagues in PLATINUM, the contents of Chapter 4 are new, informative and demanding on teachers, especially where the institutional position is not so clear.

Our focus has been twofold: (a) in the design of tasks and teaching units, ways in which the design takes into account a diversity of needs (see Chapters 6 and 12); and (b) regarding pedagogy, it has been up to each of us to use pedagogies that are as widely inclusive as possible. Examples include, the design of teaching such as that described in narratives in Chapters 11 and 13; using computer software to provide alternative insights in Chapter 15; dialogue between teachers and students in Chapter 18. While we do not claim to have developed extraordinary expertise in considering diverse needs, our awareness of particular needs has been enhanced and we have begun to consider inclusion in our inquiry more generally, particularly in the second layer of our model where we inquire into new teaching approaches alongside new design of tasks etc. As we read in Chapter 4:

> While students undertake inquiry-based instruction, teachers inquire how to implement some of the Universal Design ideas into their lectures and seminars. Such development is continuous and clearly needs the feedback not only from students but also from experts on inclusive education in order to evaluate the effectiveness of implemented recommendations and plan other modifications of the course. (p. 68)

We hope our expertise will grow through our inquiry and invite readers to use inquiry as a means of including students in mathematics more conceptually.

10.2.5. Use of Information and Communications Technology (ICT). In our proposal to Erasmus+, we indicated that one of our areas for inquiry would be the use of ICT in developing inquiry-based tasks for teaching and learning. We are aware that this is a major area of research in school-based development, but not yet so in university education (although, see Gueudet, 2017). In PLATINUM we have incorporated digital tools and methods into our design of tasks or our teaching approaches. For example, in Chapter 12, we see the use of a free software for generating graphs or geometrical situations for exploration, GeoGebra,[1] and the use of an open source programming environment for exploration of dynamical systems, Rstudio,[2] along with a cloud-based environment for students' on-line inquiry-based learning, SOWISO.[3] In Chapter 15, we see a package called Autograph[4] used with students to inquire into operations with complex numbers, providing a pictorial way of 'seeing' the complex concepts that are involved.

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[1] `www.geogebra.org`
[2] `www.rstudio.com`
[3] `www.sowiso.com`
[4] `https://completemaths.com/autograph`

During the COVID-19 pandemic, teachers were required to teach on-line according to government regulations interpreted by our universities. The use of video to present lectures or demonstrations became more common as did the use of communication platforms like ZOOM[5] and MICROSOFT TEAMS[6] to organise online meetings, to allow online collaboration of students, and to provide group chat functionality and screen sharing (e.g. an online whiteboard or a ONENOTE document). Flipping the classroom was a strategy some partners used for introducing inquiry teaching in an online setting.

An important consequence of these varied uses of ICT, many of which were intuitive or inquiry-based, is that we have become more aware of and more experienced in alternative approaches to stimulating our students' learning of mathematics. The common lecture is no longer the only way of teaching at our disposal. We see inquiry opportunities in all these digital approaches.

In consideration of the use of ICT in *school* mathematics, there has grown a wide literature on ICT potential and use, much of it theoretically based. Here we find theoretical concepts that could well apply to teaching mathematics at university level. PLATINUM has not so far engaged with such theory, but inquiry-based mathematics education at university level could valuably do so.

10.2.6. Inquiry-Based Tasks in All the Chapters. As suggested above, in these case study chapters you will find many examples of inquiry-based tasks of differing sorts, serving different purposes. Some are first attempts to think about a different style of question from the more traditional questions we are used to. Some are more sophisticated in their design, being created to fulfil different didactical purposes. The teaching units associated with these indicate something about the particular task, the pedagogy associated with the task and the overall didactic expectations related to students' learning. Chapter 6 has provided a comprehensive discussion of such tasks and teaching units, their design and use. In addition, each partner group has provided more detailed examples of tasks and teaching units which can be found on the PLATINUM website.

One way to begin to think about teaching through inquiry is to start with tasks which have inquiry-based characteristics, try them out with students and learn from students' responses. This can lead to the adapting of new elements of didactics and pedagogy more generally in teaching. An issue here of course is that students themselves are not used to such tasks and teaching and can be very resistant to it. They may see it as making unfair demands on them – requiring new forms of involvement without telling them precisely what is required, what is right and what is wrong. Many of the teachers in PLATINUM have faced such responses from students (see for example, Chapter 11). Finding ways through this didactic/pedagogic minefield can be seriously discouraging, making us lose confidence in what we want to achieve. One way to cope with this and come through it is to discuss it with colleagues in a CoI, share teaching approaches and the insights we learn from using them, and find ways of convincing students that the whole inquiry process is worthwhile. It is worth recognising that the PLATINUM insights and findings have developed over three years of working in CoIs and exploring the use of inquiry-based tasks with our students—this has not happened 'overnight'. Several of the cases below show evidence of such development.

10.2.7. Reference to the IOs. As explained in earlier chapters, the ERASMUS+ programme required us to declare Intellectual Outputs which would be developed by PLATINUM; we declared and developed six IOs as described in Chapters 2 and 5.

[5] `www.zoom.us`

[6] `www.microsoft.com/en-CA/microsoft-teams/group-chat-software`

The responsibility for leading activity within the IOs was spread across the partners. Each IO has at least one chapter dedicated to it in Parts 1 and 2 above. However, there are many overlaps.

IO1 and IO2 permeate all the cases, bringing a rich panorama of practice to the more theoretical elements of inquiry, the Three-Layer Model and Community of Inquiry. The tasks-based substance of IO3 also permeates widely, with the cases showing differing scenarios and types of task. These three IOs might therefore be seen to capture the inquiry basis of PLATINUM. However, the other three IOs are no less important. IO4, focusing on professional development for new lecturers, sets the scene for supporting colleagues in developing inquiry approaches to teaching and learning. New lecturers are not the only ones needing support with inquiry-based teaching. As suggested above, even many experienced lecturers need some kind of support when seeking to use inquiry for the first time. The approaches suggested in Chapter 7 can apply to all lecturers: those more experienced with inquiry-based teaching can engage with their colleagues in the ways suggested in Chapter 7 and it is likely that all will learn from this activity.

The other two IOs are somewhat different. IO5, focusing on *modelling*, extends the focus of IO3 to the design of tasks of a modelling nature. This focus is especially valuable when working with students from other disciplines such as engineering, science or economics. Drawing teachers from these disciplines into the dialogue can be valuable in developing a more comprehensive approach to teaching these students rather than isolating mathematics from them, and hence encouraging students to underestimate its value to them.

IO6, focusing on evaluation, offers a perspective on gathering evidence from our teaching activity regarding the extent to which students are indeed learning what we have set out to teach. Presenting an example of an evaluation instrument, the IO addresses the approaches that partners have used to evaluate teaching and learning (see Chapter 9 and Chapter 18). As you read the case studies below you will gain some insight into how PLATINUM has addressed such issues. However, in this area there is much more to be done and you might consider how this relates to your own teaching and its outcomes.

10.3. Introduction to Each of the Case Studies

In this section, there will be a short introduction to the main aspects of each case study, drawing out the diversity and commonality of the experiences/essences discussed there. The purpose of these short abstracts is help you to decide where to start and which cases might be most interesting to your own thoughts on inquiry-based teaching.

Where we refer, below, to the *teaching* discussed, we use the term "teacher" to describe the person working with the students, avoiding terms like professor, assistant professor, lecturer or assistant lecturer unless the term is particularly meaningful to what is described. In most cases, we have found that the experiences reflected cut across these boundaries.

10.3.1. Teaching Students to Think Mathematically Through Inquiry: The Norwegian experience. In this chapter, we find an account of two teachers' very different experiences of teaching in which they introduce aspects of inquiry. The two teachers, both mathematicians, together formed a *Community of Inquiry*, with support from colleagues with more didactical experience. An aim for their teaching was to motivate students to take more responsibility for their own learning.

With reference firstly to a course on *Ordinary Differential Equations* (ODEs) for *engineering students* and secondly to a *Multivariable Calculus* course for *first-year bachelor's students*, we find examples of the mathematical tasks used and references to ways in which students have responded. Recorded dialogue from small-group activity with students provided insights to students' responses to their inquiry activity. The teachers each reflect on the influence the design of their course had on students, the challenges from institutional factors affecting outcomes and their own associated learning. We see here, mathematicians finding it useful to address the mathematics education literature within the PLATINUM frame of Inquiry Community; they demonstrate important elements for students' learning through inquiry and the challenges faced when students and inquiry do not seem to mix well.

10.3.2. Design and Implementation of an Inquiry-Based Mathematics Module for First-Year Students in Biomedical Sciences. In discussion of a first-year mathematics module for biomedical students, the three authors 'set the scene' of their *Community of Inquiry* (CoI), the roles of its members, and their aim to use inquiry-based teaching with students. They show how the structure and content of the module evolved through meetings of the CoI, each recorded for later analysis, and their use of digital inquiry (involving SOWISO, GeoGebra and R/RStudio). The inquiry-based nature can be seen through examples of designed tasks (in Calculus/ODEs) and emphasis on students' critical thinking going beyond traditional courses. The research literature in IBME and feedback from students raise issues in the CoI for teachers' reflection on their teaching, on the progress of the module, their use of ICT and their students' experience of a new style of teaching and learning. The redesign of the module for a second year demonstrated their opportunity for putting this learning into practice in changed circumstances.

10.3.3. The First Experience with IBME at Masaryk University, Brno. Mathematicians and teachers work together across disciplinary boundaries in a large and diverse team to introduce inquiry-based activity in courses in statistics and mathematics. Having no previous education for teaching, and no knowledge of inquiry-based activity, they bring readers into the collaborations formed within their Community of Inquiry and the ways in which practices developed. Reflections of members of the CoI on the elements of their inquiry into teaching, the tasks they designed and used and responses of students provide a rich tapestry of learning for both students and teachers. We gain insight into teaching of statistics and mathematical analysis in which traditional practices are modified or replaced in inquiry ways, informed by the research literature. Observation of teaching by others in the CoI encouraged sharing of experiences and issues, and promoted learning for all. The reality of juxtaposing traditional and inquiry-based practices raises many issues for the team and for their future development of mathematics teaching in the national and institutional contexts.

10.3.4. In Critical Alignment With IBME. This chapter focuses on a group of mathematics education researchers teaching a mathematics education course for student teachers in Germany. The researchers focus on the teaching and the challenges they face in addressing tensions between the theories they teach in the didactics of mathematics and the traditional practices, beliefs and customs in the educational system and the students they teach. The tensions are reflected at two levels: First, as symptoms of inescapable institutional-societal or current teaching-learning conditions, and second, as the contents of theories they teach, which constitute both obstacles and conceptual opportunities to learning. Starting from a mathematical problem involving questions about the nature of graphs with or without inflection points, they reveal a range of issues, both mathematical and sociocultural, that affect their relationship with their students and affect their own development as teachers and researchers. They weave the PLATINUM concepts of Community of Inquiry and Critical Alignment with theories from German educators and social scientists to present counterpoints between theory and practice in teacher education.

10.3.5. Two Decades of Inquiry-Based Developmental Activity in University Mathematics. Focusing on inquiry-based learning and teaching in projects in which they have engaged, a group of mathematics education researchers presents and reflects on examples from their own practices in research and teaching. Seeing themselves as a Community of Inquiry, they consider their inquiry-based activity with other colleagues and with students, reflecting on experiences from which they have learned as practitioners. These include the inquiry of a mathematician into making examination questions more inquiry-based; of a teacher-researcher designing mathematical tasks for her students involving computer-based inquiry; and of a teacher reflecting on the issues raised when working with groups of engineering students in traditional or inquiry ways. These examples recognise that inquiry-based teaching is not a simple matter, but its challenges create insights for learning about teaching.

10.3.6. Teaching Inquiry-Oriented Mathematics: Establishing Support for Novice Lecturers. Inquiry activity here draws on a long experience in university education of innovative teaching/learning activity in mathematics and mathematics education, involving mathematical problem-solving and forms of IBME. The chapter focuses particularly on professional development for new lecturers designed by a team of four experienced teachers with fields of research in mathematics or in mathematics education. The design process for the materials to be implemented is divided into four phases: Discover, Define, Design, and Develop. The chapter shows how these phases are applied, the activities in each phase, the mathematical tasks implemented by the new lecturers with students, and reflections of both lecturer and students on their activity and learning. An example of matrix factorisation illustrates the implementation of this process in some detail, showing its outcomes for lecturers and students in terms of critical attitudes to mathematics, to learning mathematics and to mathematical meaning and processes. Finally, the authors recognise how the whole process of design and implementation contributed to their own learning as teachers and researchers.

10.3.7. Development of a Community of Inquiry Based on Reflective Teaching. In one university, as this chapter reveals, there was no immediate group of people available for forming a Community of Inquiry (CoI) and alternative ways had to be explored, including inviting colleagues from neighbouring universities. A key expectation of the CoI was to provide observation of teaching to allow feedback and discussion of the observed teaching and learning. Colleagues observed each other's teaching and shared their practice. It allowed inquiry into diverse classroom settings, their affordances and constraints for teachers and students to be considered. Mathematical tasks were designed and shared. The Covid pandemic, with on-line learning and teaching, provided new environments to be explored. Overall, a relatively stable CoI was achieved that allowed reflective teaching to be shared and inquiry-based activity to be established in this community. Challenges, achievements, and experiences of the CoI are discussed, with narrative reflections from its various members.

10.3.8. Experience in implementing IBME at the Borys Grinchenko Kyiv University. In this chapter, we learn that an educational community was formed at the university, including colleagues from several universities, to address low motivation of students in choosing mathematics programmes and to share understandings of IBME. This Community of Inquiry (CoI) addressed issues related to conceptual versus procedural learning approaches drawing on a range of literature. PLATINUM members led the CoI in suggesting inquiry-based approaches to teaching and learning and one member led a course in Mathematical Analysis to enable the community to observe and address processes and issues. An open questioning approach was taken by this teacher with encouragement for students to address the questions and to ask their own questions. Extracts from the teaching and examples of students' responses suggested that teaching had motivated students and engaged their interest, thus also motivating the CoI to engage further with IBME approaches.

10.4. Concluding Thoughts

The PLATINUM partners invite you to read our chapters described briefly above. We believe that each one offers insights into engaging with inquiry-based teaching and learning. Although this book can offer only brief examples of inquiry-based tasks, inquiry-based ways of working with students, or inquiry-based theoretical analysis, you will find further examples on the PLATINUM website `https://platinum.uia.no`. It may be that you have experiences that could add to the richness provided in this book and on the website. And of course you can contact corresponding authors of chapters directly.

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CHAPTER 11

Teaching Students to Think Mathematically Through Inquiry: The Norwegian Experience

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We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself, to consider matters as a historian does, to take part in the knowledge getting. Knowing is a process, not a product.
Jerome Bruner (1915-2016), American cognitive psychologist

11.1. Mathematics Education at the University of Agder

The University of Agder (UiA)[1] is a public university located in the southern part of Norway on two campuses, one in a larger city of Kristiansand where the university administration and most faculties are situated and another in a smaller town of Grimstad, about 45 kilometres distant from the main campus. UiA is one of the youngest universities in Norway, yet its history dates back to 1839 when the Teacher Training School was established at Holt rectory. Being one of the major driving forces for the regional development, UiA is also internationally oriented; it contributes to many international projects in education and research as a leading organisation (as in PLATINUM) or as a partner. The university is the home to about 13,000 students and 890 academic staff. It is organised in six faculties: Faculty of Engineering and Science, Faculty of Fine Arts, Faculty of Health and Sport Sciences, Faculty of Humanities and Education, Faculty of Social Sciences, School of Business and Law and has a Teacher Education Unit.

The University of Agder is acknowledged as one of the national leaders in mathematics education, mathematics teacher education, mathematics teachers continuing professional development, and mathematics education research. It has Norway's longest running master programme and the largest PhD programme in mathematics education. In the recent evaluation of education research commissioned by the Research Council of Norway,[2] The Mathematics Education Research Group at Agder (MERGA)[3] at UiA was rated as outstanding; it was granted a priority research centre status by the University of Agder in 2018. University of Agder hosts the Centre for Research Innovation and Coordination of Mathematics Teaching (MatRIC),[4] the only National Centre for Excellence in Education specialised in teaching mathematics. MatRIC is funded in 2014–2023 by NOKUT (the Norwegian Agency for Quality Assurance in Education), an independent expert body under the Ministry of Education and Research;[5] it also receives financial aid from the university.

[1] www.uia.no/en
[2] www.forskningsradet.no/en/
[3] https://bit.ly/2Y8Cntx
[4] www.matric.no
[5] www.nokut.no/en/

Mathematics is taught at UiA mainly within the Faculty of Engineering and Science as a service subject with the largest cohorts being engineering students on the campus of Grimstad, economics students and teacher candidates in Kristiansand. A handful of dedicated and hardworking mathematicians teaches a modestly sized group of bachelor students in mathematics on the campus of Kristiansand. As part of the Pure and Applied Mathematical Analysis Research group (PAMAR),[6] mathematicians also conduct research in fluid mechanics, functional analysis, ergodic theory, ordinary, partial, and stochastic differential equations, variational methods, mathematical modelling, statistics. In the report *Research in Mathematics at Norwegian Universities*[7] commissioned by the Research Council of Norway, the research in the period 2006–2010 was evaluated. With regard to the University of Agder, the homogeneity of a small mathematics group and scarce available resources were pointed out. This certainly affects the possibilities of course offer, which is not as wide as desired; for instance, there are no dedicated courses on mathematical modelling at UiA. On the other hand, due to relatively low student enrolment in several programmes, it is not economically feasible to tailor, for instance, Calculus or Linear Algebra courses to particular needs of different study programmes. For instance, Calculus courses are offered in the bachelor's programme in Mathematics, the 5-year master's Advanced Teacher Education programme in Mathematics, and the 1-year university preparatory programme; students in these three programs have different backgrounds and educational needs.

The Department of Mathematical Sciences at UiA has a long tradition of mathematics teacher education and teacher education with many students pursuing a master's or a PhD degree. Many staff in the faculty have teaching and research interests in mathematics education; they are supported in different forms by the University, Faculty, Department, MERGA, and MatRIC. A few years ago the department started a master's programme in mathematics with very small groups of 2-4 students recruited in the previous three years. A PhD programme in applied mathematics is now offered by the department but it currently has only one student, working in functional analysis; he defended his PhD thesis recently. A number of bachelor's mathematics courses for engineering students in the departments of engineering sciences and ICT on campus Grimstad are taught by a small mathematics unit composed of instructors with different backgrounds including mathematics, geophysics, astrophysics, engineering, etc. Some courses are taught to large cohorts of engineering students with different specialisation and some are tailored to special needs of specific study programs. For instance, Mathematics 1 is offered to students in five bachelor's programmes: Civil and Structural Engineering, Computer Engineering, Electronics and Electrical Engineering, Renewable Energy, and Mechatronics, whereas Discrete Mathematics is taught in the 1-year Programme in ICT, bachelor's programme in Computer Engineering, and a 5-year master's programme in Artificial Intelligence.

Traditionally, there has been very little collaboration between mathematicians and mathematics lecturers in Grimstad and Kristiansand who were separated not only by 45 km of distance between the campuses but also by their affiliation with different departments and study programs, even though within the same Faculty. The situation started improving after the Centre for Excellence, MatRIC, was established at UiA with the focus on mathematics teaching and learning for specialisations other than mathematics.

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[6] `https://bit.ly/39YkuTJ`

[7] `https://bit.ly/3sTXNsv`

11.2. The MatRIC-PLATINUM Community at the University of Agder

As explained in Section 11.1, the University of Agder has a very good mathematics education environment; this contributed positively to the development of the local community of inquiry (CoI) and a larger PLATINUM community in general. Many activities organised within the PLATINUM project are especially relevant to MatRIC since the Centre focuses on mathematics teaching and learning within the university study programmes in non-mathematics disciplines such as engineering, natural sciences, economics, and teacher education. The main activities of the Centre are related to its five networks for Digital Assessment, Modelling (led by the second author), Teacher education, Simulation & Visualisation, and Video. Therefore, MatRIC supports relevant educational projects that enable sharing and development of effective use of video, digital, web-based, and emerging technologies in teaching, learning, and assessing mathematics. The Centre is very much interested in the use of most recent research discoveries in psychology and education in teaching, learning, and assessing mathematics and works to identify, understand, and evaluate effective innovation in practice.

During the first years since its establishment in the end of 2013, MatRIC arranged many interesting events including a Video Colloquium, a Mathematical Simulation and Visualisation Symposium, and a Computer Aided Assessment Colloquium. The second author organised two Mathematical Modelling Colloquia in 2015 and 2016 with invited speakers from Denmark, Germany, Mexico, the Netherlands, Norway, Portugal, Sweden, UK, and USA. These events brought together mathematics educators, scientists, engineers, computer scientists and economists in cross-disciplinary teams to produce workplace simulations and realistic tasks for mathematical modelling. Several PLATINUM team members met at these events to discuss the role of mathematical modelling in university education; these first contacts led to the alignment of research interests with the subsequent establishment of new collaborations. Not surprisingly, mathematical modelling became one of the important directions in the development of the PLATINUM project. Another important initiative taken by MatRIC was the organisation of the Mathematics Teaching Induction Course, first in collaboration with the Norwegian University of Science and Technology (NTNU) in 2015–2016 and later on in collaboration with the German Centre for Higher Mathematics Education; the most recent one was arranged in 2019–2020.[8] The experience of the first author with the organisation of the very first induction course for newly appointed and less experienced university lecturers in mathematics was very useful for the organisation of the related professional development activities in Intellectual Output 4 of the PLATINUM project (see Chapter 7).

The PLATINUM project was supported by MatRIC from the very beginning due to its relevance to the main goals of the Centre whose strategic policy envisions that effective mathematics teaching and learning result in motivated students gaining fundamental subject knowledge and understanding the important role played by mathematics in modern society. Several PLATINUM project partners met at educational events organised by MatRIC; many stimulating discussions regarding possible applications for external funding for research or educational projects were initiated there. MatRIC funded a number of partner meetings where the draft of the main ideas of the PLATINUM project were conceived and parts of the application for the EU funding through the Erasmus+ programme was prepared; this is described in more detail in Chapter 5 of this book. During the project, MatRIC and PLATINUM collaborated

[8] `www.matric.no/articles/130`

to provide the best educational experience to students, training them to understand better fundamental mathematical ideas and to be capable of applying these ideas for solving problems encountered in daily life and at the workplace. MatRIC's vision "Students enjoying transformed and improved learning experiences of mathematics in higher education" perfectly aligns with the goals set for the PLATINUM consortium and for the local team at UiA whose ambition is to teach students so that they enjoy mathematics and appreciate its relevance as a powerful tool for effective problem solving. Although MatRIC spans a much larger area of interests, when it comes to teaching mathematics at the university level, it is quite difficult to separate the core interests of MatRIC and PLATINUM communities due to intricate visible and invisible links between the two; therefore, we quite often refer to PLATINUM CoI at Agder as a "MatRIC-PLATINUM team."

Daily work of the PLATINUM community of inquiry at UiA has been influenced by the changes in modern views on mathematics teaching which contrast but also complement the traditional professor-centred approach. Promoting inquiry-based methodology in our teaching, we motivate students to take more responsibility for their own learning and engage actively in constructing their understanding of mathematical subjects by combining individual studies, small group work with peers, and whole class discussions. Our explorations of new ways of teaching were encouraged by the recent empirical research which reports an about 6% improvement in examination scores in active learning classes whereas students in traditional mathematics classes were 1.5 times more likely to fail the exams (Freeman et al., 2014). Remarkably, both results were consistent not only across STEM disciplines but also across different class sizes (smaller classes with fewer than 50 students perform even better).

11.3. Promoting Conceptual Understanding in a Differential Equations Course for Engineers

A lack of conceptual understanding in mathematics and a wish to skip theory in favour of framed colourful formulas in the textbook and step-by-step recommendations do-it-this-way are often characteristic in teaching mathematics to engineering students. Ditcher (2001) pointed out that quite a few engineering students take an instrumental approach to their studies with a "motivation to pass exams in order to obtain a degree (and hence a job), rather than being driven by an interest in learning" (p. 25). However, many professional engineers highly value advanced mathematical thinking. For instance, Devlin (2001) stressed that "the main benefit they [software engineers] got from the mathematics they learned in academia was the experience of rigorous reasoning with purely abstract objects and structures. Moreover, mathematics was the only subject that gave them that experience" (p. 22).

Therefore, teaching future engineers is always a challenging task that requires a compromise between theory and rigour on the one hand and procedures and applications on the other hand. For many years, university courses in Ordinary Differential Equations (ODEs) have been an important part of engineering education (Francis, 1972). The research indicates that an inquiry-oriented approach to teaching ODEs contributes significantly to students' knowledge retention (Kwon et al., 2005; Rasmussen & Kwon, 2007). Nevertheless, students' experience in difficulties distinguishing between the meanings assigned to different types of solutions (general, particular, stationary, etc.) which becomes a challenge for students' learning ODEs (cf., Arslan, 2010; Raychaudhuri, 2007, 2013).

The research suggests that "if more time were spent in classrooms with students engaged in working on cognitively demanding non-routine tasks, as opposed to exercises in which a known procedure is practised, students' opportunities for thinking and learning would likely be enhanced" (Simon & Tzur, 2004, p. 92). In this first case study, we discuss how a deeper analysis of non-standard problems on the Existence and Uniqueness Theorems (EUTs) helps students to make sense of differential equations and relate the concepts of particular and general solutions. This teaching experiment was inspired by an interesting paper by Klymchuk (2015) on the use of 'provocative' mathematics problems and by the work on students' conceptual understanding of key issues in differential equations by Raychaudhuri (2007, 2013).

Earlier research has shown that students usually form a habit of applying formulas or rules without checking conditions required for the application of procedures and theoretical results, tacitly assuming that they are satisfied. Furthermore, assessment questions are often formulated so that these conditions are automatically met, and, in most cases, students are not asked to verify them. However, "ignoring conditions and constraints might lead to significant and costly errors" (Klymchuk, 2015, p. 63). On the other hand, turning the exploration of theoretical results into inquiry can be very useful for deepening students' conceptual understanding:

> How often do we ask students to prove something only to realise that they do not yet understand the statement, let alone believe it is true? Whether you are teaching students how to develop formal proof techniques, teaching a course where proof is a routine part of the homework, or just expecting students to justify assertions informally, an inquiry-friendly option is to ask students to try examples and begin to make conjectures before writing proofs. Working through examples ensures that students understand the key definitions they will need in the proof. (Dorée, 2017, p. 181)

Although proof writing was not the goal in the course, turning standard testing of easily verifiable assumptions into challenging inquiry questions about EUTs that promote advanced mathematical thinking sounded very attractive to the authors.

Challenging the status quo, the first author, a mathematics lecturer, designed the set of six non-standard problems on EUTs aimed at enhancing the conceptual understanding of a group of 23 fourth year students in mechatronics enrolled in an Ordinary Differential Equations (ODEs) course. The lecturer's intention was to provide her engineering students with unusual situations "for which students had no algorithm, well-rehearsed procedure or previously demonstrated process to follow" (Breen et al., 2013, p. 2318). Contrary to traditional practices in mathematics courses for engineering students, problems were formulated in such a way as to engage students more deeply with important details of theoretical results focusing on the development of conceptual understanding rather than procedural skills. The lecturer wanted to explore how non-standard questions can be used to challenge students, develop their analytical skills, and contribute to conceptual understanding of important notions and ideas in an ODE course for engineering students. Furthermore, introducing the small group work in the project, the lecturer wanted to understand to what extent have individual work and group discussions contributed to students' conceptual understanding of EUTs and influenced their individual solutions submitted for assessment. The authors started to select tasks by looking up relevant material in the textbook. But this did not suffice, and they browsed related research literature for more inspirational ideas. Last but not least, the authors contacted Dr. Treffert-Thomas from Loughborough University requesting some methodological advice on the organisation of the teaching experiment. The combined efforts of two mathematicians and a mathematics educator led to the design of the final set of six problems. Having in mind both improved students learning and subsequent educational research, this small community of inquiry adopted a formative approach to research known in the literature as design-based research (Swan, 2020) where the set of the tasks has been designed, developed, and refined through several consecutive cycles of observation, analysis, and redesign, including the use of the feedback from students.

Students started by working on the problems individually, first during the tutorial time and then at home producing their own solutions to the problems (see sample problems in Figure 11.1). All problems required conceptual understanding of the EUTs and their correct application in situations that were different from those traditionally requested by most texts, where it was necessary to directly verify the assumptions and conclude whether a theorem could be applied or not. For instance, for solving the problems shown in Figure 11.1, students had to apply the theorem that states "if coefficients of a linear DE are continuous on a given interval, there exists a unique solution of the initial value problem on this interval." Students learned earlier in the course how to verify that a given function is a particular solution to a given ODE but Problems 1(a) and 2(a) (see Figure 11.1) both require to check for the general solution. This is a rather unusual problem for engineering students, not found in most standard textbooks for engineering and science students. In fact, it is not hard to verify that the given function is a solution to the given ODE (and students were able to do this) but to show that it is the general solution, one has to explain the role of the arbitrary constant (we refer to the 'first method' later on). Alternatively, one can derive the general solution using an integrating factor or variation of constants; this establishes the formula for the general solution (the 'second method').

Sample problem 1

a) Verify that $y(x) = \dfrac{2}{x} + \dfrac{C_1}{x^2}$ is the general solution of a differential equation

$$x^2\, y' + 2xy = 0$$

b) Show that both initial equations $y(1) = 1$ and $y(-1) = -3$ result in an identical particular solutions. Does this fact violate the Existence and Uniqueness Theorem? Explain your answer.

Sample problem 2

a) Verify that $y(x) = C_1 + C_2\, x^2$ is the general solution of a differential equation

$$x\, y'' - y' = 0$$

b) Explain why there exists no particular solution of the above equation satisfying initial conditions $y(0) = 0$; $y'(0) = 1$.

c) Suggest different initial conditions for this differential equation so that there will exist exactly one particular solution of a new initial value problem. Motivate your choice.

FIGURE 11.1. UiA examples of nonstandard ODE tasks.

The formulation of Problem 1(b) is also unusual for engineering students. The 'trap' was set for those who might erroneously believe that the integral curve associated with the solution $y = 2/x - 1/x^2$ passes through the two different points given as initial conditions (ICs). However, since both coefficients $p(x) = 2/x$ and $q(x) = 1/x^2$ are not defined at $x = 0$ and are continuous either on $(-\infty, 0)$ or on $(0, +\infty)$, but not on any interval including zero, two different solutions defined by the same expression exist on two disjoint intervals, each containing one of the initial points. In Problem 2(b) it was necessary to verify that both ICs cannot be satisfied because the slope of the solution to the given ODE passing through the origin cannot be equal to 1 at $x = 0$ whereas for solving Problem 2(c) one had to notice that the ICs were given at the point $x = 0$ where the coefficients of the DE have a discontinuity. Therefore, even though a solution may still exist and be unique, this cannot be deduced from the EUT since its conditions are not satisfied. It is possible to resolve this issue by modifying the ICs, namely, either by changing the initial point from $x = 0$ to any other value and use the EUT, or by modifying the ICs at zero and showing by direct inspection that the solution exists (the latter also requires the proof that the solution is unique).

After working on solutions individually, students met in small groups to discuss their individual solutions and agree on a common set of solutions to the assignment to be presented to the class. After the presentation of solutions to the entire class (each group presented their solution to one of the six problems), students were given an opportunity to work at home on the assignment finalising their individual solutions which were then submitted to the lecturer who graded the assignment and provided the feedback to the students. An important feature of this teaching experiment was the lecturer deliberately not interfering in the students' small group discussions which were organised outside the course hours; she also did not contribute to the classroom discussion when group solutions were presented, encouraging students to engage critically in the peer discussion.

The analyses of three sets of students' individual written solutions (solutions produced during the tutorial session, at home and final solutions submitted for grading) and recorded discussions in five small groups along with the audio recordings of students' final presentation of solutions and the lecturer's reflections on the activity provide a useful insight into the process of students' learning. For example, the lecturer noticed, quite unexpectedly, that students experienced certain difficulties with the correct mathematical meaning of particular and general solutions. This problem has been also reported in several research papers on students' conceptual understanding of ODEs (Arslan, 2010; Raychaudhuri, 2007, 2013). However, the students in our teaching experiment worked out collectively what the "violation of conditions of EUTs" means. They developed new understandings in this context that the lecturer did not foresee while designing the coursework. Furthermore, on some occasions, discussions within the group led students to adopt familiar routines at the expense of other ideas that could have been more appropriate and could have led to conceptual understandings. We provide two excerpts from the transcripts of self-recorded small group discussions to illustrate the success and difficulties experienced by the students. In what follows, the students are identified by two digits, so, for instance, S23 means the third student in the second small group.

Excerpt 1

S12: Since we got the solution, I just took the derivative of that and put it into the original equation, to see that two equals two, and that was the case, that was my verification.

S12: Mine as well.

S12: Mine too.

S12: So, I was the only one who actually did any work, [laughter] so I actually integrated the whole thing, and ended up with the right expression, so [...] your way of doing it is a lot easier.

S12: A bit more efficient at least.

S12: And I had a problem with the term in front of C_1, which should be minus, according to the task, I only got it positive because of the integration.

In this episode, four students in Group 1 discuss the solution to Problem 1(a). We notice that explaining the solution to the task, student S12 describes the verification procedure known for particular solutions and somehow 'melts together' the concepts of the general solution and particular solution using a much more general notion 'solution' and not paying attention to the important loss of meaning.

Excerpt 2

S21: How can we verify that this is the general solution?
S22: Obviously, differentiate the solution, put it into the differential equation and see if it is correct as usual.
S23: You can also say that it is a derivative, you can use the product rule to bring it together, to integrate.
S25: I did the same as you did, using the integrating factor, multiplying and then I just solved the equation because it is solvable.
S23: You did not use u times v derivative and you get it v derivative times u plus u derivative times v?
S25: Yes, I used the method for it, where you define $\mu(t)$ as the integrating factor and then multiply in, the same as we did in the first lesson.
S24: I also solved the equation by the integrating factor but I think it is easier just to differentiate it once and put it into the original equation and see if it is a correct solution.
S25: But there could be more solutions, they are not general solutions.

In the second episode, five students in Group 2 also discuss their solutions to Problem 1(a). S22 suggests the procedure to verify that a given function is a solution to a differential equation but does not explain why it is a general solution. Similarly to what was observed for Group 1, S22 also does not distinguish between the two different types of solutions. S23 concentrates his attention on particular details of the solution procedure. S24 tends to agree with S21 and the obvious lack of attention to the detail at this stage potentially leads to an incomplete solution. Reacting to this unfortunate situation, S25 tries to bring attention to other possibilities but the group mates do not recognise the importance of this suggestion and proceeded further to the discussion of the next task.

Summarising the discussion of Problem 1(a) in two groups, we observe that after the encounter with a multifaceted definition of solution in the university ODEs course (general and particular solutions, solutions to initial value and boundary value problems), different from students' previous experience in other courses, students changed their mathematical discourse and embraced new meanings of the familiar term 'solution.' Surprisingly, for many students the work with the EUTs was less confusing than the work with the fundamental for ODEs question regarding the difference between general and particular solutions. After the lecturer analysed students' written individual solutions, transcripts of small group work and presentations of solutions in the class, it turned out that students in the course can be divided into three main types with respect to the development of their skills and conceptual understanding: pseudo-learners, potential learners, and learners. This classification has been suggested by Raychaudhuri (2013).

> ***The learner*:** This student is in possession of a coherent cognitive structure, and tries to maintain and rebuild it on a continual basis. A student such as this acknowledges a conflict, and attempts to reorganize his or her cognitive structure while keeping all the previous connections intact. He or she may or may not be successful in this attempt, but it is his or her approach that indicates the individual's status as a learner.

***The potential learner*:** This student is in possession of a more or less coherent cognitive structure, but does not try to maintain or rebuild it on a continual basis. The student acknowledges a conflict, but does not want to go to great length to remedy it. Faced with a conflict the student often deals with it by letting go of one or more previous connections. In other words, they suppress the conflict by *patching* it with a temporary quick-fix solution

***The pseudo-learner*:** The pseudo-learner: This student stockpiles items of knowledge one after another in an almost *linear* structure where connections are primarily local (often via processes studied in a localized context). He or she will not recognize conflict (without a connected structure, questions of conflict do not arise) and will compartmentalize the conflicting pieces if they are pointed out. Either way, the conflict will cause no perturbation to their cognitive structure. (p. 1241)

We explain this rather general classification in the following table providing more specific details relevant for our example on the understanding of EUTs. The interested reader would very likely find relevant applications of this classifications to own students.

Student type	*Challenge (evidence: homework)*	*Skills development (evidence: group work & presentation)*	*Understanding (evidence: final homework)*
Student A, pseudo-learner	Did not understand the logic of EUTs.	Performed several procedural steps correctly without developing conceptual understanding.	Did not understand the difference between necessary and sufficient conditions; did not understand the essence of EUTs; submitted many incorrect solutions.
Student B, potential learner	Understood the main ideas of EUTs.	Provided mostly correct solutions without elaborating the details and without reference to theoretical results.	The final homework has been very little influenced by the discussions and presentations and contained some incomplete or inaccurate solutions.
Student C, learner	Understood the logic of EUTs but missed some important details.	Refined solutions supporting them with references to appropriate theoretical material.	Used the results of the discussions for improving individual solutions significantly.

TABLE 11.1. Classification of students on the basis of written work and oral contributions.

Looking for students' feedback on this teaching experiment, the lecturer distributed two questionnaires, in the beginning and in the end of activity. Prior to the experiment, students rated themselves as quite competent in mathematics (3.3 out of 5 on the Likert scale, 5 being the highest score, here indicating 'very competent'); they also believed they possessed mathematical knowledge sufficient for their needs as engineering students (3.8 out of 5). Reflecting about the activity, students found the tasks in the assessment interesting (4.1 out of 5 on the Likert scale), enjoyable (4.0 out of 5), and very challenging (4.4 out of 5). Most students recalled that it was nice to have discussions, both in small groups and in the class, and to be able to see and discuss alternative solutions suggested by the peers (12 out of 19). It seems that inquiry in small groups through discussions was one of the most enjoyable and appreciated components of the activity, as acknowledged in students' answers quoted below.

> Nice to have a discussion and hear other people's opinions and thoughts.
>
> The discussion was surprisingly interesting because you learn a lot when you have to explain your reasoning.
>
> I learned a lot by solving it for myself and then got alternative inputs and different ways of solving/evaluating.
>
> I found the discussion part interesting, and it was nice to see that the majority of tasks was solved in a similar way.
>
> To make individual solutions to a common problem may ultimately give a better solution in the end than to work as a group from the start.

For a more detailed analysis of the use of non-standard problems in an ODE course for engineering students, we refer the interested reader to the papers of Treffert-Thomas et al. (2018) and Rogovchenko et al. (2020).

11.4. Innovation Versus Students' Inertia and Institutional Constraints

The second episode describes a not-so-successful teaching experiment with the first-year bachelor's students in a standard Multivariable Calculus course. The course is offered to students in the Bachelor's Programme in Mathematics, Advanced Teacher Education level 8-13, the 5-year Master's Programme in Mathematics Education, and the 1-year Bridging Programme in Mathematics. The student population was quite diverse, although for most students it was their very first year at the university, there were also a few more mature students; several students had received (at least partly) school education abroad. This experiment has been conceived by the authors in collaboration with Professor Simon Goodchild, a mathematics educator, specifically with the PLATINUM project in mind. Therefore, upon our request, permission to teach the course in English was granted by the Head of the Department. During the preparation to teaching in this course, the second author carefully explored available teaching resources, searched for textbooks, both in print and online, as well as for relevant lecture notes featuring the combinations of keywords *inquiry*, *active*, and *calculus*. Unfortunately, only a few online resources were available, the most appropriate being "Active Calculus – Multivariable" prepared by Steve Schlicker and his colleagues at Grand Valley State University.[9]

The discouraging results of the literature search clearly indicated that setting a Multivariable Calculus course within an inquiry-based teaching framework would not be an easy task neither for the lecturer nor for the students. The three-fold team was meeting regularly (once or twice a week) before the course start and also during the teaching to discuss the learning goals, teaching materials organisation of lecturing, tutorials, and exams, as well as the problems for the use in the class. The mathematics educator attended most lectures; he was observing the teaching and taking notes; he also had several conversations with students regarding the course; lectures were recorded to allow for subsequent analysis. His written comments were discussed with the project team after the classes and possible adjustments to teaching were suggested to the course lecturer.

[9] `http://bit.ly/3bUFk9k`

In the very beginning of the course, the lecturer described to the class the goals and the organisation of the teaching and learning process. He emphasised that the course is demanding and clearly accentuated students' attention to their role as *knowledge explorers and gainers* and his role as a *team member* assisting students' learning rather than a lecturer. The lecturer thoroughly explained the peculiarities of the current course organisation. All important details regarding the course and exam organisation were discussed by the three-fold course team and carefully described in the course description posted on Canvas, the learning management system currently used at UiA.

> Learning outcomes are set at the beginning of each week. They state the knowledge and skills that the students should acquire every week and are important for students' progress through the course. [...] What is new and special about the course this semester: to facilitate students' conceptual understanding of the material and to contribute to its better retention, a form of active learning known as inquiry-based learning will be employed. This means that in addition to traditional lecturing, students will be also more actively engaged in learning during the lectures through discussions in small groups, questioning and exploration. Elements of inquiry-based learning will be also incorporated in some problems included in four non-compulsory problem sets (the total of twenty problems). Sixteen out of twenty problems will be quite similar to those in the main textbook but will be selected from the sources different from it and thus no answers or solutions to the problems will be known; four of them (one for each set) will be selected for the final written exam. Four problems out of twenty will have a distinct flavour of inquiry; one of these will be selected for the final written exam. Answers or solutions to the problems in these four sets will not be provided but students who seriously engage in their solution will receive a comprehensive feedback. The course team composed of a lecturer, an experienced mathematics education professor and an experienced mathematics professor will regularly monitor and timely adjust, if necessary, the course teaching and learning strategy and selection of teaching and learning materials.

In the first lecture, students were introduced to the SOCRATIVE app for mobile phones[10] and informed about its use during the lectures for getting fast feedback on students' progress in the course. To test the app, students were asked to answer two questionnaires, each with three questions, distributed during the break and right after the first class (see Table 11.2).

The total of 43 answers to questions 1-3 and 38 answers to questions 4-6 were received by the SOCRATIVE app; students' choices are reflected in Figure 11.2. The survey results were very encouraging and clearly indicated students' preparedness to work hard and engage. In fact, 90% of the students expected the course to be more difficult or much more difficult than other courses; 78% expected to spend at least 16 hours per week on this course; 90% claimed that attending lectures was necessary and very necessary; 86% thought that attending seminars was necessary or very necessary; 92% assumed that working on non-obligatory tasks was necessary or very necessary; and 68% expected the course to be at least moderately interesting.

Students in the course seemed to agree with the need to work harder and be engaged in so-called active learning defined by Bonwell and Eison (1991) as "anything that involves students in doing things and thinking about the things they are doing" (p. 2). Emphasising the importance of active engagement of students in learning, the lecturer also warned about specific obstacles associated with the use of active learning methodology. These would, in particular, include (1) the difficulty to adequately cover the course content; (2) limited class time available; (3) possible increase in the

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[10] `www.socrative.com/`

amount of preparation time; (4) the difficulty of using active learning in large classes; and (5) a lack of materials, equipment, or resources (Bonwell & Eison, 1991). In fact, the lecturer of the course and the two professors supporting him experienced all these factors, acknowledging that the organisation of active learning in a medium-size class represents a serious challenge.

Nevertheless, the team worked enthusiastically in the hope that the positive students' feedback to the survey will be also supported by their increased effort in learning the material in the course. To stimulate students' engagement with the material, the lecturer was suggesting quizzes with 1–3 problems for "discussion with a peer sitting

Question	*Possible answer*
Q1. When you compare this course with other courses you take; do you expect this course to be:	A Much more difficult B More difficult C About the same level of difficulty D Easier E Much easier
Q2. To be successful in this course, an average student is expected to work on course tasks outside of classes for 16–20 hours each week. How many hours do you expect to spend, studying this course outside classes, to be successful?	A. More than 25 hours each week B. About 20 hours each week C. About 16 hours each week D. About 12 hours each week E. Less than 7 hours each week
Q3. In your opinion, how necessary is it to attend the lectures to ensure success?	A. Very necessary B. Necessary C. No strong feeling D. Not necessary E. A poor use of my time
Q4. In your opinion, how necessary is it to attend the seminars to ensure success?	A. Very necessary B. Necessary C. No strong feeling D. Not necessary E. A poor use of my time
Q5. In your opinion, how necessary is it to work on all the tasks and problems, which are not obligatory, to ensure success?	A. Very necessary B. Necessary C. No strong feeling D. Not necessary E. A poor use of my time
Q6. How interesting do you expect the course to be?	A. Very interesting B. Moderately interesting C. No feeling either way D. Rather uninteresting E. Very uninteresting

TABLE 11.2. Questions and possible answers in two surveys.

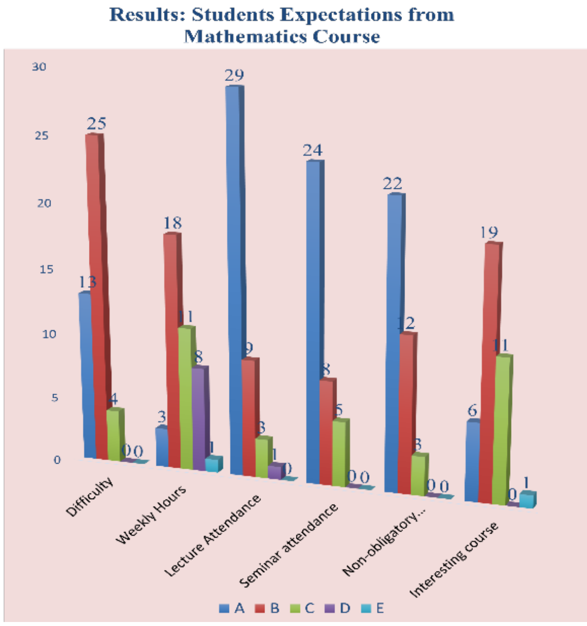

FIGURE 11.2. Students' answers to the six questions in Table 11.2.

next to you" two-three times during the lecture. The tasks required conceptual understanding of the material and very little or no computation. Students had to submit individual answers after 5–7 minutes of discussion with a classmate. The progress with the answering the tasks was projected on the screen and correct answers were marked with green bars. The student names were not visible to the class, only to the lecturer, who usually praised at the end students who answered questions correctly. The lecturer also commented shortly on the answers providing a short argument leading to the correct answer. Examples of the tasks are provided in Figures 11.3 and 11.4

(a) $$\int_0^{\pi/4} \sqrt{1-\sec^4 x}\,\mathrm{d}x$$

(b) $$\int_0^{\pi/4} \sqrt{1+\sec^4 x}\,\mathrm{d}x$$

(c) $$\int_0^{1} \sqrt{\frac{\pi}{4}+\sec^4 x}\,\mathrm{d}x$$

(d) $$\int_0^{\pi/4} \sqrt{1+\tan^2 x}\,\mathrm{d}x$$

(e) $$\int_0^{\pi/4} \sqrt{1+\sec^2 x\tan^2 x}\,\mathrm{d}x$$

FIGURE 11.3. Which integral gives the arc length of the curve $y = \tan(x)$ between $x = 0$ and $x = \pi/4$.

A Multivariable Calculus course at the University of Agder, like similar courses across the globe, is traditionally shifted towards computational aspects; this is often

$$\text{(A)}\quad \begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases} \quad 0 \le t \le 2\pi \qquad \text{(B)}\quad \begin{cases} x = 3\cos t \\ y = -2\sin t \end{cases} \quad 0 \le t \le 2\pi \qquad \text{(C)}\quad \begin{cases} x = 2\sin t \\ y = -3\cos t \end{cases} \quad 0 \le t \le 2\pi$$

$$\text{(D)}\quad \begin{cases} x = -2\cos t \\ y = 3\sin t \end{cases} \quad 0 \le t \le 2\pi \qquad \text{(E)}\quad \begin{cases} x = 3\sin t \\ y = 3\cos t \end{cases} \quad 0 \le t \le \pi \qquad \text{(F)}\quad \begin{cases} x = 3\sin 2t \\ y = 3\cos 2t \end{cases} \quad 0 \le t \le \pi$$

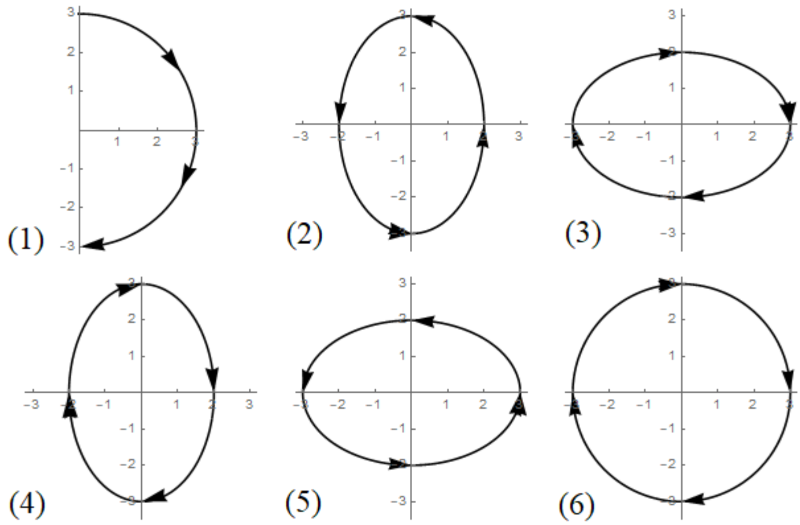

FIGURE 11.4. Which parametric equations A-F describe parametric curves plotted in Figures 1-6?

emphasised in most textbooks and in the teaching based on these texts. Not surprisingly, many students tend to memorise the formulas and algorithms without making an effort to understand them; problem solving in the class and at home frequently turns into predefined routines "repeat the steps after the lecturer" or "follow the procedure in the textbook's example." The empirical research indicates that even a simple reformulation of a traditional task as a question is useful for initiating students' inquiry and stimulating their learning.

> One step in teaching students to ask questions is to rephrase routine textbook exercises as questions that can be worked on in groups during class. This remarkably simple-to-implement shift can transform routine procedural exercises into questions that spark students' interest, deepen their conceptual understanding, encourage students to connect multiple perspectives, and inspire students to ask their own question (Dorée, 2017, p. 180).

The second author designed both tasks with the purpose of attracting students' attention to the key details important for the conceptual understanding of the material, taking inspiration from a limited selection of inquiry-oriented tasks available on the web. In the first, a slightly easier problem, students were asked to compare several possible answers which were intentionally designed to be alike; the choice of the correct one, the answer (b), requires the analysis of the main components in the equation for computing the arc length of a curve defined in Cartesian coordinates; to this end one needs to recall the general formula for the arc length along with the derivative of the tangent function. Although the assignment is not particularly difficult, 13 students out of 18 who registered for the class on SOCRATIVE submitted the answers and only 6 (about 46%) turned out to be correct.

The second task is much more challenging and requires students to associate six equations of parametric curves with their graphs. The students in their first year of bachelor's programmes did not see many similar examples, if any at all. In this task students have to pay attention to the intervals where the parameter t is defined in order to correctly identify the initial point and the direction of the motion along the parametric curve. If one does not check the details carefully, it is quite easy to confuse similarly looking graphs 1 and 6, 2 and 4, and 3 and 5. Since all parametric equations are also akin, this adds even more confusion to the task. Not surprisingly, only 4 out of 34 students (less than 12%) correctly paired all six equations with their graphs, six students made one mistake whereas quite a few students either did not attempt the solution at all, or did so only for the first few pairs.

Despite the lecturer's enthusiasm and willingness to engage students actively in learning mathematics supported by the generous advice from his two colleagues, both with extensive teaching and research experience in mathematics and mathematics education, the experiment, unfortunately, did not last long. Students' apparent understanding of the peculiarities of the course and the necessity to actively engage in learning did not help to change their reluctance to experience something new and challenging. Soon after the first few classes, a group of students complained to the study adviser and the department's head about the lecturer's too high expectations with respect to students' previous knowledge, their performance in the course, and a fear of receiving lower grades in Calculus II in comparison with top grades in Calculus I. Students also shared their concerns with the lecturer focusing, however, mostly on the language issue rather than on the lecturer's excessive demands regarding previous mathematics knowledge. Even though the lecturer reassured students that everything should settle down soon and they will receive all support needed to master the material, students were not convinced; the initiative of the mathematics educator to mediate the rising tension in a meeting arranged separately with students did not help. By the end of the second week of teaching, the head of the department—after several rounds of discussions with the lecturer, the mathematics educator involved in the experiment, the student adviser, and the study program leader—yielded to students' pressure and decided for the teaching to return to a traditional form, and we regretfully confirm that the experiment failed.

11.5. Lessons Learned

One of the distinctive features of both examples of teaching practice discussed in Sections 11.3 and 11.4 is that the authors were keenly interested not only in providing students with the learning opportunities to facilitate and promote conceptual understanding of mathematics but also in their own professional development as mathematics teachers as well as in contributing to mathematics education research. This is why, in both episodes described, the authors carefully looked up and analysed relevant research literature and asked active education researchers for methodological support. As fairly noticed by Jaworski (2006), "theory cannot show us what teaching should involve, but teachers and educators can search for clearer understandings of what teaching might involve; thus, we learn about teaching with the possibility to develop teaching" (p. 189). In both teaching experiments inquiry was used as a developmental tool and the authors worked with the mathematics educators in small communities of inquiry as described by Jaworski (2006) although with rather different arrangements. In the first case, the team was relying on the methodology of design research (Cobb, 2000) where cycles of design, testing, analysis, and redesign of the tasks over several academic terms were planned with the ultimate goal of creating knowledge for practitioners and mathematics education researchers. During the teaching experiment reported in Section 11.3, three team members met on a few occasions to discuss the design of the tasks and experiment settings and more frequently later on for the analysis of the learning activity and its redesign (the latter is not discussed in the chapter). In the second example (Section 11.4), the project team was prepared to work intensively during the entire academic term with regular meetings, extensive preparatory and follow up work, and a very active engagement of the mathematics education professor. This teaching experiment was designed primarily with the PLATINUM goals in mind and further plans for redesign and possible replication in partners' CoI. Both case studies described in this chapter fit the inquiry model in three layers (see Chapter 2). In the central layer, we have students engaging in inquiry in differential equations individually and with their peers, and in inquiry in calculus with their peers and the lecturer. In the middle layer, both authors engage in professional inquiry aimed at creating new learning opportunities for students. Finally, in the outer layer, the authors inquire with mathematics educators in wider communities of inquiry discussing implications of teaching experiments and creating new knowledge for professional use and professional development of university mathematics lecturers.

Did the outcomes of the two teaching experiments with different groups of students in different departments surprise us? The honest answer is: "not much," we knew well about possible gains and risks before we planned teaching experiments. The maturity of the group of engineering students in a graduate course and students' enhanced motivation contributed positively to the success of the first teaching experiment reported in Section 11.3; most students appreciated new learning opportunities created for them by the first author. On the other hand, in the second teaching experiment, after only five months at the university, many first-year students were not well enough prepared to unusual educational explorations; the fear of not being successful in the course with innovative elements turned out to be stronger than the wish to try new possibilities for learning differently through a more challenging and active engagement. Quite rapidly this fear developed into a panic for some students; they started seeking protection from innovation with the people responsible for the study program in the department which eventually led to the termination of the experiment.

In a very recent survey, Børte et al. (2020) recognised that "Higher Education institutions are, however, not always organised, structured, and led in ways that support and facilitate new approaches to teaching" (p. 11). They identified the existing barriers to active learning grouping them under three themes: (1) Leadership and organisation, (2) Teaching competence and training needs, and (3) Technology (ibid, p. 11). In our case, the most important factor which negatively affected the teaching experiment in Section 11.4 was related to the first theme: Although the team consisting of three professors carefully planned the experiment and the lecturer had sufficient experience with teaching Calculus courses using the same textbook for many years in Cyprus and Sweden, the department yielded to students' demands and requested to terminate the teaching experiment already in the end of the second week of teaching. Furthermore, analysing the prerequisites for student active learning to succeed as reported in the research literature, Børte et al. (2020, p. 11) identified the following three key components: (1) better alignment between research and teaching practices,

(2) a supporting infrastructure, and (3) staff professional development and learning designs. It seems that all three key components were in place in both teaching experiments, yet the first-year bachelor's students were much more reluctant to engage with active learning in Multivariable Calculus than the fourth year seniors in a Differential Equations course.

Summarising the lessons learned in the two cases discussed in this chapter, we confirm without hesitation that "the reform of instructional practice in higher education must begin with faculty members' efforts. An excellent first step is to select strategies promoting active learning that one can feel comfortable with" (Bonwell & Eison, 1991, p. vi). However, the very different outcomes in the two cases suggest that the wish, however strong, of the faculty to reform the classroom practice by introducing elements of inquiry-based learning is only a necessary, but not a sufficient condition. The most pronounced differences in the two teaching experiments are related to students‘ motivation for studying mathematics and interest in the subject, their academic maturity and readiness to innovation, and institutional support (or the lack of such).

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Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project
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CHAPTER 12

Design and Implementation of an Inquiry-Based Mathematics Module for First-Year Students in Biomedical Sciences

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12.1. Setting the Scene

How might you design and implement an inquiry-based basic mathematics module for first-year students in biomedical sciences? What mathematical content would you choose? What decisions would you have to make? What issues would you have to address? How would you gain insight into students' learning? In this chapter, these questions are addressed relating to a first semester module in the first year of the bachelor programme Biomedical Sciences at the University of Amsterdam, which started in the study year 2018–2019.

We, the authors, formed with teaching assistants a small community of inquiry, henceforth abbreviated as CoI (Chapter 2; Jaworski, 2008; Biza et al., 2014), for designing, teaching and evaluating the module by employment of a developmental and inquiry approach. This CoI had a common purpose in exploring the teaching and learning of basic concepts and methods relevant for a mathematical perspective on processes of change in a biomedical context, in promoting an inquiry-based mathematics education (IBME) approach, in trying to understand better teaching and learning processes in such an approach and in recognising issues that arise for lecturers and students. Members of the CoI had differing roles. André Heck was the module coordinator and principal lecturer. Marthe Schut and teaching assistants had responsibility for construction of tasks that they would use in tutorials. All CoI members involved in designing and running the module shared responsibility for design of the instructional innovation, for monitoring students' learning processes, and for continuous reflection on the teaching and learning in lectures and tutorials leading to modifications during this practice and listings of points of attention for next years of teaching. In other words, they engaged in research *in* practice, also called insider research (Goodchild et al., 2013). In terms of the three-layer model of inquiry outlined in Chapter 2 (cf., Jaworski, 2019), they did inquiry in mathematics and in teaching mathematics. The innovation included inquiry-based tasks, use of the digital environment SOWISO (Heck, 2017), and use of the programming language R (R Core Team, 2019) for exploring processes of change that can be described in the form of dynamical systems. Natasa Brouwer engaged in research *on* practice collecting and analysing classroom data (outsider research).

The CoI's aim was to develop a module for first-year students in biomedical sciences that would improve the mathematical component of students' biomedical scientific literacy and make their learning of mathematics through inquiry more enjoyable. This literacy has three aspects: (1) becoming familiar with and understanding basic mathematical concepts and methods (learning mathematics), (2) engaging in the kind of mathematics that biomedical scientists apply in their work (doing mathematics), and (3) gaining insight into the increasingly more important role of mathematics in biomedical research (learning about mathematics in context). Joy of mathematics learning was considered important because motivation for learning mathematics cannot be taken for granted in this population. The CoI's approach was design research (Bakker, 2018), which means that the design of instructional materials (e.g., computer tools and learning activities) was a crucial part of the research. The main focus of this case study is to reflect and report on

- the processes within and the products of the CoI,
- issues raised during its work,
- the students' processes and achieved outcomes in the module,
- the successes and failures in meeting the goals set by the CoI, and
- lessons learned.

To this end, data was collected from audio-recorded CoI meetings, design notes, observations in lectures and tutorials, student questionnaires and interviews, and module evaluations during the study years 2018–2019 and 2020–2021.[1]

This chapter is organised as follows. In Section 12.2 we give background information about the basic mathematics module, the student population, and the envisioned role of ICT. In Section 12.3 we report on the CoI and its developmental work in 2018-2019. This includes design of inquiry-tasks, student feedback, CoI's reflections, and their ideas for improvement for the next study year. In Section 12.4 we report on the CoI's revision of the module and experiences within the study year 2020–2021. We end in Section 12.5 with recommendations for lecturers based on lessons learned.

12.2. Background Information

In this section, we present a new perspective on biomedical sciences education and the role of mathematics herein, give information about the student population, and discuss the role of ICT in the developed module.

12.2.1. Mathematics for New Biology. In the past, basic mathematics was embedded in bachelor courses in biomedical sciences for brushing up mathematical skills of students with regards to mathematical functions, basic calculations that students need in laboratory work, and models of growth. In 2017, the curriculum was reformed in line with the advice of the Biosciences Committee of the Royal Netherlands Academy of Arts and Sciences (2011) about the importance of 'New Biology' in higher education. This is a cross-disciplinary science in which omics-based techniques of analysis ('omics' like genomics, transcriptomics, proteomics, or metabolomics) and system biology enable researchers to quantify biological processes in and around cells, organs, and organisms. The importance is also reflected in the 6th edition of the textbook 'Molecular Biology of the Cell' by Alberts et al. (2015), which is used throughout the bachelor programme. The authors of this book wrote in the preface:

> We now realize that to produce convincing explanations of cell behavior will require quantitative information about cells that is coupled to sophisticated mathematical/computational approaches—some not yet invented. As a consequence, an emerging goal for cell biologists is to shift their studies more toward quantitative description and mathematical deduction.

[1]The mathematics module ran three times during the PLATINUM project in differing circumstances. We report on the development and the revision phase.

This was also one of the main messages of the BIO2010 report (National Research Council, 2003), in which the following central abilities for future biomedical researchers were listed: knowledge of fundamental mathematical concepts and methods, quantitative analysis, and mathematical modelling.

The implementation of a New Biology curriculum explains the choice for a basic mathematics module in which first-year students get acquainted with systems biology and systems medicine. In mathematical terms this means an introduction to dynamical systems in the context of biomedical processes (see Segel & Edelstein-Keshet, 2013). This includes the study of mathematical models of growth, chemical kinetics, and quantitative pharmacology with the purpose that students see where and how mathematics is applied in biomedical sciences. It helps students understand that advanced mathematics methods and techniques are needed for quantitative modelling of processes of illness and health.

The basic mathematics module was taught for the first time in the study year 2018–2019 in 9 weeks with mathematics lessons spread over two courses (Basic Statistics and Basic Mathematics for Biomedical Sciences, part 1 & 2) from November up to April, with one 2-hours lecture and one 2-hours tutorial per week. In other course weeks, statistics was taught. This spreading of mathematics lessons across a period of four months was seen as an opportunity for making strong connections with other courses taught in the same period. However, it led to an instruction sequence with gaps, which made it difficult for students to keep oversight, and complicated the examination of two different subjects. So the basic mathematics module was redesigned for the study year 2020–2021 to a module with the same work load but now taught in a single semester block (in November and December), with one 2-hours lecture and two 2-hours tutorials per week, no parallel teaching of statistics, and with examination of only the mathematical concepts and methods.

12.2.2. The Student Population. The size of the student population attending the basic mathematics module is on average about 150 students. This means that it is rather difficult for lecturers to become well-informed about their students through direct contact in the short time that the module is run. For this reason, personal data of students such as mathematics background and study profile at secondary school level, mathematics anxiety, test anxiety, and motivation and engagement, were collected via questionnaires. In this subsection we report on data collected in the study year 2020–2021, but the outcomes in previous study years hardly differ.

Students' Mathematics Background and Study Profile. By students' mathematics background is meant the mathematics examination programme that students took at upper pre-university level, namely, Mathematics A or B. Mathematics A prepares for studies in social or economic sciences. Its core subjects are statistics and probability, and some calculus. Mathematics B covers the mathematics needed for exact sciences and technical studies, and mainly covers calculus. A substantial percentage (25%) of the first-year students had taken Mathematics A, which prepared them less for exact sciences. Their marks in the module exam were less good and only 50% of them passed the exam, compared to 90% of the students with Mathematics B.

Related to the mathematics background is the students' choice of study profile at upper pre-university level because it restricts the possible combinations of subjects in the examination programme. Mathematics B is obligatory in the 'Nature & Technology' (NT) profile, which is required for most studies in exact sciences and engineering. In the 'Nature & Health' (NH) profile, which prepares for studies in medicine and biology, pupils choose between Mathematics A and B. Slightly more than half of the participants of the module (55%) had an NH profile, and within this group of students 45% had chosen Mathematics A. Many students with an NH profile were less prepared for exact sciences: their marks were less good and only 69% of them passed the exam, compared to 97% of the students with an NT profile. It follows that mathematics background is the most influential factor for study success, even though the module was designed such that it would not disadvantage students with Mathematics A.

Students' Mathematics Anxiety and Test Anxiety. The level of mathematics anxiety amongst first-year students was measured via the Dutch translation of the Abbreviated Math Anxiety Scale (Hopko et al., 2003). The rescaled mean AMAS score (3.5) was low on a scale from 1 (no anxiety at all) to 10 (panic) and was a bit less than students' self-estimates (4.3). No statistically significant differences in mathematics anxiety regarding mathematics background, study profile and gender were found.

The level of test anxiety amongst first-year students was measured via the Dutch translation of the Test Anxiety Inventory (Spielberger, 1980). The mean TAI score, rescaled from 1 to 10, and the students' self-estimates were 4.6 and 5.5, respectively. A significant difference was found only in gender: female students reported a higher level of test anxiety than male students.

No correlations were found between the above emotional experiences and students' exam results. However, the lecturers noticed that students with a higher mathematics anxiety level asked more questions anonymously on the online forum of the module than other students.

Students' Motivation and Engagement. The 'Motivation and Engagement Wheel' framework (Martin, 2007) includes thoughts (motivation) and behaviours (engagement) that play a role in learning and consequently in course performance. Both are subdivided into adaptive and maladaptive forms. Adaptive thoughts consist of Self-Belief, Valuing of School, and Learning Focus, whereas adaptive behaviours consist of Planning, Task Management, and Perseverance. Maladaptive thoughts include (Test) Anxiety, Failure Avoidance, and Uncertain Control, whereas maladaptive behaviours include Self-Sabotage and Disengagement. These scales can be assessed via the 'Motivation and Engagement Scale – University/College' (MES-UC) instrument.

A significant difference was found only in gender for Planning, Anxiety, and Uncertain Control. Female students reported better planning of their work, assignments, and their study, but they were also more worried or felt more nervous (e.g., about work, assignments, exams) and were more uncertain (e.g., about how to do well or how to avoid doing poorly).

Only the group of maladaptive behaviours was significantly and negatively related to the exam mark. In the study year 2020–2021, with only online teaching during the COVID-19 pandemic, maladaptive thoughts could have been overshadowed by worries connected to the pandemic.

12.2.3. The Role of ICT Envisioned by the Module Designers. Kaput (1992) distinguished three modes of computer use in education, namely as an educational medium, as a set of tools, and as a toolmaker/mediumbuilder. The CoI that designed the basic mathematics module adopted the first two modes.

The educational medium was SOWISO (Heck, 2017), a cloud-based environment for learning, practising and assessing mathematics that allows randomised examples and exercises with automated feedback (Figure 12.1). The designers of the module created tailormade GeoGebra-based tools (Figure 12.2) and simulations (Figure 12.3) with the Easy Java/JavaScript Simulations (EjsS) toolkit[2] (Garcia et al., 2017),

[2]EjsS is an authoring tool for non-programmers to create interactive simulations in Java or Javascript, mainly for teaching or learning purposes. (see the website `www.um.es/fem/EjsWiki`)

which served as tools for promoting students' conceptual understanding of mathematics through inquiry

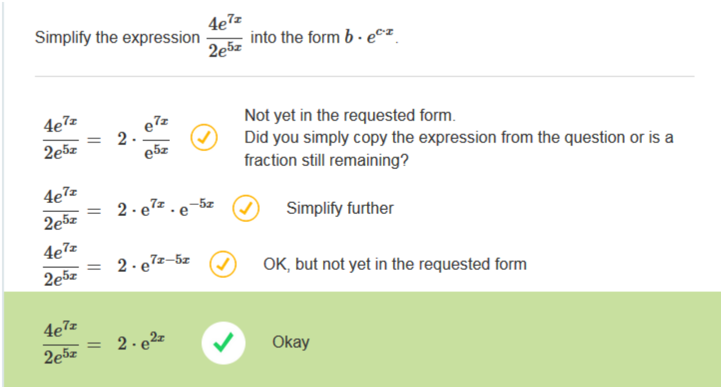


FIGURE 12.1. A simplification task in SOWISO with feedback.

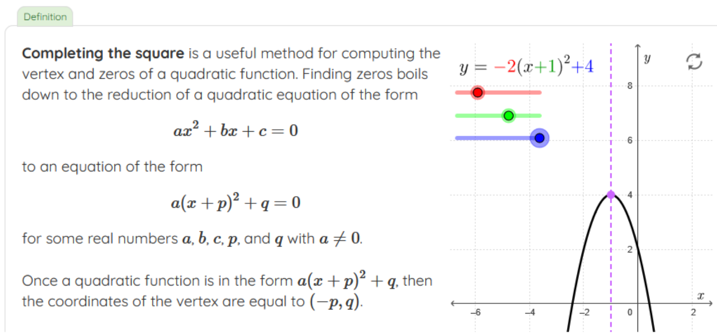


FIGURE 12.2. A GEOGEBRA tool embedded in a SOWISO theory page.

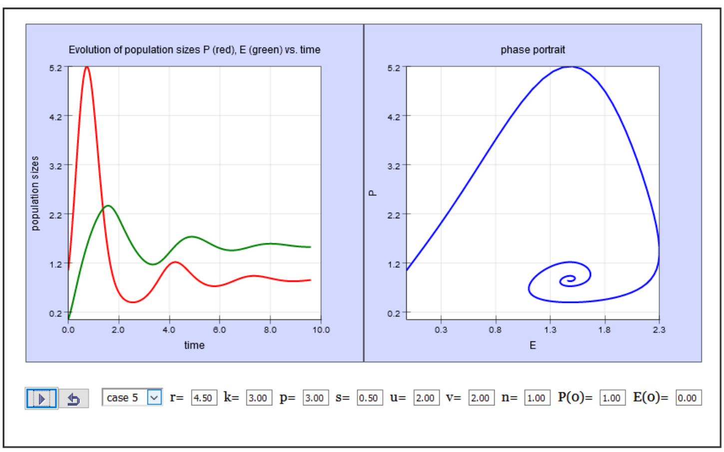


FIGURE 12.3. An EJSS-based user interface for exploring the immune response model of Mayer et al. (1995). The case of chronic coexistence of virus and antibodies is shown.

Figure 12.4 illustrates how the R programming language (R Core Team, 2019) and RSTUDIO (RSTUDIO Team, 2019) were used in the module as expressive tools that allow students to solve mathematical models numerically in the same way as biomedical scientists use this mathematical software.

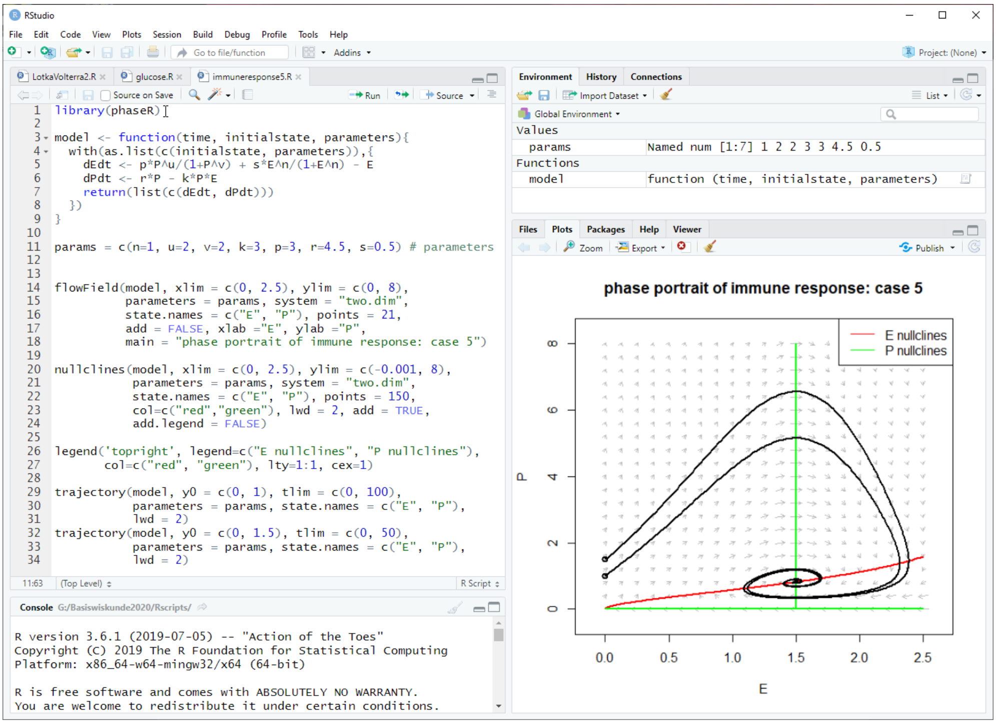


FIGURE 12.4. Worked-out solution in R and RSTUDIO of the case of chronic coexistence in the immune response model.

The module designers created microworlds for students to familiarise with fundamental concepts of ordinary differential equations (ODEs) and with ways to compute, visualise, and analyse solutions. These microworlds were implemented in GEOGEBRA, EJSS, and COACH[3] (Heck, 2012). Examples are microworlds for drawing a direction field of a system of ODEs and solutions in the phase plane (Figure 12.5), and diagrams for single neuron models (Heck, 2019).

The CoI applied principles of the original Universal Design framework Story (2010), adopted in PLATINUM to support students with identified needs (see Chapter 4), in the design of ICT-tools for student inquiry. We illustrate this in Figure 12.6 with a randomised GEOGEBRA-based exercise for drawing a lineal element at a random point for a randomly generated differential equation. The creator of this exercise applied the principle of 'simple and intuitive use' here. His thinking in doing so was that the students' learning curve of using a slope field tool would be reduced by a tool menu that contains only the necessary tools for drawing a lineal element, selecting and deleting objects. He applied the 'tolerance for error' principle in the sense that the user can sketch a reasonable approximation of the lineal element to get it marked

[3]COACH is an activity-based, open multimedia authoring environment that is designed for STEM education and offers students a versatile set of integrated tools for inquiry of natural phenomena, mathematics, science, and technology (see the website www.cma-science.nl)

as correct. The 'size and space for approach and use' principle is supported by the full-screen button at the lower-right corner of the GeoGebra-based tool that rescales the tool to fit the whole screen. In addition, visually impaired persons are supported in SOWISO by the display of mathematical formulas via MathJax, which works with any ARIA screen reader and can be brailled or transformed to speech output, and by zooming of formulas in ways that are adjustable to one's wishes.

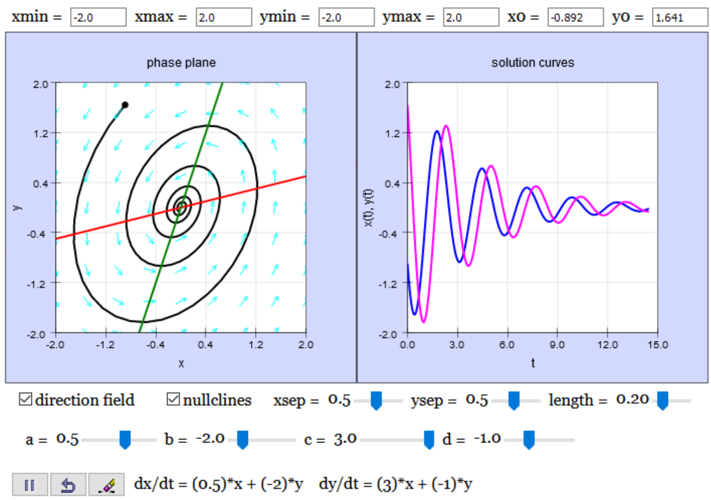


FIGURE 12.5. An EJSS-based dedicated tool to explore solutions of a linear system of two ODEs.

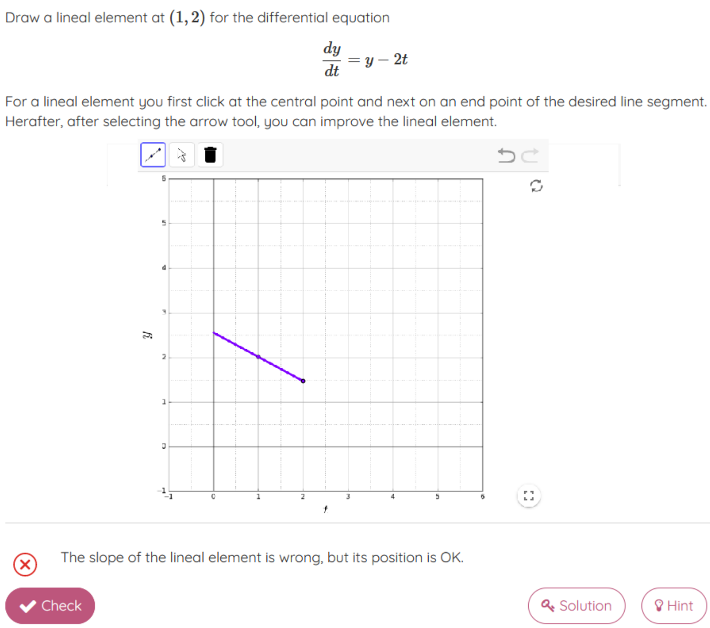


FIGURE 12.6. A screen shot of a randomised GeoGebra-based exercise in SOWISO with automated feedback.

12.3. Work of the CoI on the First Version of the Module

This section is a retrospective analysis of the design of the module taught in the study year 2018–2019 and experiences with the instructional materials in practice. The CoI consisted of the authors and one teaching assistant, who was a master student in physics employed for both instructional design and teaching in the tutorials. The analysis is based on voice recordings and notes of eight CoI meetings, classroom observations, and module evaluations by students. The ideas about IBME and what was learned from existing inquiry-based approaches to teaching and learning differential equations will be exemplified by inquiry tasks developed by this team. Feedback of students and reflections of the CoI on the module are discussed.

12.3.1. Finding the Structure and Contents of the Module. In the first meeting, two weeks before the start of the module, the CoI agreed that the module in the New Biology curriculum would be structured toward quantitative mathematical modelling. This means that students would explore mathematical models with digital tools. It was planned that they would carry out mathematical explorations with real enzymatic data (Moss et al., 1996), pharmacokinetic data (Mas et al., 1999; Heck, 2007) and with a published preditor-prey model of immune response (Mayer et al., 1995). These explorations were meant to convey to students that there is often more than one mathematical model for a phenomenon possible. They were included in the module to sensitise students for the quality of a model and to convince them that understanding of a model is not the main goal, but understanding of the modelled phenomenon and the mathematics needed for that purpose.

The CoI's view on modelling instruction was that students learn most efficiently when it is done in a progressive way: students first get acquainted with simple models, such as exponential growth, and improve them by changing or adding details before they construct their own models. The duration of this module did not allow for students to make their own models. The module would give students an orientation on system biology and systems medicine with the hope and expectation that they would start to appreciate mathematics as a powerful means to explore processes of change in biomedical contexts.

The CoI planned the structure and contents of the module in a backward direction. Looking at the desired end point of the module, the CoI discussed what would be the mathematical concepts needed in the hypothesised learning trajectory. Many discussions were about pedagogical questions like how to promote conceptual learning through an inquiry approach and how to deal with alternative conceptions. The structure and contents of the module crystallised, based on classroom experiences, and ended with five parts:

(1) basic mathematical functions and numerical differentiation;
(2) basic growth models;
(3) chemical kinetics and quantitative pharmacokinetics;
(4) basic concepts and methods of dynamical systems;
(5) applications in a biomedical context.

The mathematical focus was on main concepts of the theory of dynamical systems like direction field, stability of an equilibrium, asymptotics of solutions of (systems of) ordinary differential equations, and more importantly the concept of solving a differential equation algebraically, numerically and graphically. The lecturers explored how to use R for studying dynamical systems in a uniform and consistent way so that students would not get lost in the pool of different specialised R packages for studying differential equations. Concretely, this means that they inspected textbooks, similar courses, and examples of R use on Internet, and discussed own R scripts written in the process of finding a unified approach of R use. The principal lecturer wrote a reference chapter, based on the textbook of Soetaert et al. (2012), before the start of the module. It summarises the basics of R, regression analysis in R, and investigation of differential equations with R. It served as a guideline for the CoI to prepare R-based tasks and instructions, and it helped students look up short explanations of R use.

12.3.2. Implementing IBME. The CoI adopted the following conceptualisation of IBME formulated by Dorier and Maaß (2014):

> IBME refers to a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work.

The CoI members designed tasks to promote student inquiry by paying attention to

- underpinning mathematical methods;
- representing mathematical concepts;
- providing evidence and argumentation;
- exploring/evaluating multiple methods for solving a single problem;
- motivating learning by real biomedical examples.

They enriched lectures with inquiry-based tasks in which students were invited to express own ideas. An example is the task of inventing methods for computing a numerical derivative of a quantity from data only (Figure 12.7). Sometimes small-group work with worksheets in lectures preceded plenary discussions. An example is the worksheet task shown in Figure 12.8.

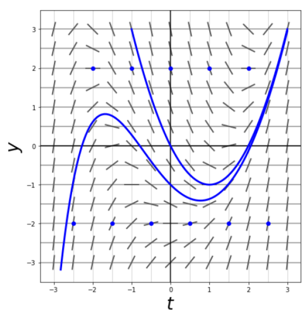
Given are the following values of a function $y(t)$ in the neighbourhood of $t = 1$:

t	0.7	0.8	1.0	1.1	1.2
$y(t)$	0.741	0.819	1.000	1.105	1.221

What is the best approximation of $y'(1)$?
(exact answer = 1 because the function $y(t) = e^{t-1}$ has been used.)

Try several methods and compare the results with each other.

FIGURE 12.7. An inquiry task used in a lecture.



You see above the direction field that corresponds with the ODE

$$\frac{dy}{dt} = t^2 - y - 2$$

and two solution curves.

Sketch solution curves through the blue grid points.
What can be said about the behaviour of solutions?

FIGURE 12.8. A worksheet in which students sketch solution curves and conjecture about asymptotic behaviour of curves.

Regarding tutorials, CoI members rephrased exercises taken from previous courses like Basic Mathematics for Psychobiology to make them more inquiry-based (cf., Dorée, 2017), included visual explorations of mathematics via ICT tools and JavaScript-based simulations, and introduced mathematical programming in a biomedical context.

All elements of IBME mentioned above and linking of mathematical content to real biomedical inquiry come together in the final teaching unit about enzymatic kinetics. In this unit, students explore the Michaelis-Menten model and its effectiveness in explaining real data taken from a research paper (Moss et al., 1996).[4] It is an example of guided and structured student inquiry, meaning that students follow directions and hints in a structured teaching-learning path based on professional practice of parameter estimation, but conclusions are predominantly based on the investigation carried out by an individual student or pair of students. In Table 12.1, the subtasks are typified according to the 5E learning cycle model of Bybee et al. (2006).

Subtask	*Activity*	*E-emphasis*
1	Giving meaning to kinetic variables	Engage
2	Defining and understanding the Michaelis-Menten model as ODE	Engage
3	Estimating the initial concentration of the substrate	Explore
4	Transforming data to a linear model	Engage
5	Computing reaction rates and drawing the Lineweaver-Burk plot	Explore
6	Estimating parameters via the Lineweaver-Burk plot	Engage
7	Doing a numerical sensitivity analysis of kinetic parameters	Explore
8	Using the Eadie-Hofstee plot and the Hanes-Woolf plot	Elaborate
9	Doing nonlinear regression with the Michaelis–Menten formula	Engage
10	Doing nonlinear regression using the differential equation	Elaborate

TABLE 12.1. Subtasks in the enzymatic kinetics teaching unit.

This teaching unit goes beyond what students learn in traditional courses with respect to critical thinking, in particular about the use of evidence and the relationship between evidence and explanation.[5] The use of evidence is addressed in the regression subtasks 6-10 in which students determine the quality of various regression methods (linear and nonlinear) by graphical comparison of computer results with the data. This shows that data analysis involves decision making and exploration to come to scientifically underpinned answers and conclusions. The relationship between evidence and explanation is addressed when students are confronted with an unexpected result that needs an explanation for making progress. For example, students do not get a straight line in the Lineweaver-Burk plot (subtask 5-6) with a simple numerical differentiation method, need an explanation for this (subtask 7), and must explore other differentiation methods or other linearisations (subtask 8).

12.3.3. Learning From Research Literature. The outer layer of the three-layer model is developmental research inquiry and this commonly involves studying research literature on the subject of interest. The most recent literature review about teaching and learning of differential equations was published by Lozada et al. (2021).

[4]The first tasks were inspired by inquiry activities created by Coolidge (2008).

[5]Most textbooks that discuss enzymatic kinetics mention only the Lineweaver-Burk equation as a linearisation of the problem of estimating parameters in the kinetic model. They do not discuss the side effects of this linearisation on the statistical analysis and its results, other possible linearisations, and the use of values found in a linearised setting as initial values for a nonlinear regression method. In the PLATINUM teaching unit we pay attention to all of this in student tasks.

CoI members benefited from the MAA Research Sampler (Rasmussen & Whitehead, 2003) and the review by Rasmussen & Wawro (2017) about students' conceptual understanding of differential equations, equilibrium solutions, bifurcation, and graphical approaches. In these reviews, it is pointed out that the concepts of direction field and solution of a differential equation have many facets that affect student understanding. Cited research papers gave CoI members food for thought about student misconceptions and helped them create teaching units for introducing new mathematical concepts about dynamical systems (e.g., worksheets like in Figure 12.8).

Inspecting research literature about teaching and learning differential equation, the CoI came across and got interested in the publications about the Inquiry Oriented Differential Equations (IODE) course (Rasmussen & Kwon, 2007; Rasmussen et al., 2018). In this course, based on Realistic Mathematics Education, emphasis is on student reinvention of mathematical concepts, teacher inquiry into student thinking, and student inscriptions and their role in the development of the mathematics. CoI members discussed this approach in early meetings and they found this conceptualisation of IBME attractive and shared the idea that students should be routinely invited to explain and justify their mathematical thinking, their solution strategies, and actions in open-ended activities. This is reflected in the basic mathematics module through the added short questions like "Why?" and tasks like "Explain your reasoning" or "Compare with the previous result" to prompt students to think more deeply about the mathematics that they applied, to provide arguments for the choices that they had made in applying mathematical methods, and to explore the effectiveness of several techniques by experimentation and comparison.

Yet, after ample discussion, the CoI came unanimously to the conclusion that they would not adopt the IODE approach because of the guided reinvention and emergent modelling principles. These principles are at odds with the usual way of teaching in biomedical courses where main concepts are discussed in class by lecturers, but not reinvented by students. Also, differing mathematics background of students would complicate the reinvention process.[6] The size of the student population was too large[7] and the student-teacher contact time was too short for this approach.

12.3.4. Feedback From Students. The CoI collected student feedback via a questionnaire with Likert scale statements, rating questions, and open text fields for remarks. The feedback addressed the students' appraisal of the module (instructional design, instructional materials, learning activities and their inquiry nature), the use of the SOWISO environment, the use of R and RStudio, and the points for improvement of the module. Main results are listed as mean values in Table 12.2 for the study years 2018–2019 and 2020–2021.

The Likert scores confirm that students in 2018-2019 had a neutral or positive opinion about the instructional design of the module and the use of SOWISO. However, they found the level of the mathematics module too high, and programming in R difficult and hardly contributing to better understanding of mathematics. The following comment supports this conclusion:

[6]For an IODE class at a school of engineering, consisting of 25 academically strong students, Habre (2020) reported that reinventing knowledge was demanding and in some cases required the intervention of the instructor to control and lead the discussion. Our annual group of about 150 first-year students in biomedical sciences is far more heterogeneous and probably less strong in mathematics.

[7]As far as we learned from research literature, RME approaches have been implemented and researched only in classrooms with 15 to 30 students (many authors of papers on such approaches do not mention or are vague about the number of students involved in their research studies). The feasibility and effectiveness for large student populations is thus unclear.

I found that the R tasks made the content just more complicated. I liked the pen-and-paper tasks most, but the exercises in SOWISO were also fine. In case R tasks must be embedded in this course, I think more attention must be paid to them.

Statement	μ_{2018}	μ_{2020}
The goals of the mathematics module were clear to me.	2.7	3.7
The structure of the course was clear to me.	3.2	4.0
I found the contents of the mathematics module interesting.	2.7	3.5
Through the extra attention paid to applications I saw the usefulness of the mathematics module.	2.9	3.9
I learned a lot in the mathematics module.	3.3	4.0
I had enough preknowledge for the mathematics module.	3.1	3.8
I still have not well understood all parts of the math module.	3.8	2.8
The level of the mathematics module was too high for me.	3.7	2.6
In general I found that the mathematical exercises were clear.	3.2	4.1
In general I rated the level of the mathematical exercises as good.	2.5	4.1
The working of SOWISO was clear and I could work well with it.	3.5	4.5
The feedback in the SOWISO excercises was good.	3.5	3.8
I learned much from the short tasks in the lectures (e.g., inventing a numerical differentiation method and practising with direction fields).	3.6	4.0
I appreciated that the use of R was addressed in the lecture.	3.7	4.2
The working of the RSTUDIO was clear to me; I could work well with it.	2.5	3.4
The R tasks helped me better understand the mathematics.	2.1	2.6
I disliked the spreading of the mathematics module over a long period and I prefer a module taught in consecutive weeks.	4.1	–
I liked that the module was in a short period of 7 weeks. I prefer this compared to a module spread over a long period.	–	4.3
I prefer separate courses in mathematics and statistics.	4.3	3.8
In some mathematical problems you had to explore things by yourself. This type of 'inquiry-based learning' had appeal for me.	2.7	–
I prefer tasks in which I am instructed about the expected outcomes and what to do.	–	4.1

TABLE 12.2. Main results of the questionnaire in the study years 2018–2019 and 2020–2021 consisting of 5-point Likert scale (1=*strongly disagree*, 5=*strongly agree*) statements.

Other points of criticism of students after the first run of the module were about the work load, the course pace, the combination of statistics and mathematics in a single course, and the spreading of mathematics over a long period of time.

Students were asked to mark the lectures, the tutorials, and the mathematics module as a whole on a scale from 1 (*very poor*) to 10 (*excellent*). The median values of these marks were all equal to 7 (meaning *generous pass*). One of objectives of the module was that students would enjoy learning mathematics in the biomedical context. The following comment of a student illustrates that lecturers play a crucial role herein:

The lecturers really did their best to let students pick up that the course was not all about the exam, but that the subjects taught were really interesting. This motivated me to do my best to better understand the course materials.

12.3.5. Reflections of the CoI. As a CoI, we were happy with the pass rate of 70% and details of students' experiences which showed us where students were positive about the module and where they were less satisfied. We agreed with students that improvements were needed concerning the use of R and RStudio, the relationship between the statistics and mathematics parts of the courses, and the large amount of contents within the mathematics module.

As module designers and lecturers, we considered the R tasks as opportunities for students to explore mathematical concepts by inquiry. But many students were in fact still coming to grips with the use of R and RStudio as tools to carry out tasks rather than for learning mathematics. An instrumental approach to digital tool use in mathematics education helped us understand students' difficulties.[8] We concluded that many students were not far enough in the process of instrumental genesis, i.e., in developing suitable utilisation schemes and techniques in order to transform tools as artefacts into instruments suitable for a task or activity.

We illustrate the complexity of instrumental genesis of using R and RStudio with the 'simple' task of plotting the graph of a function. Many abilities, of both technical and conceptual character, can be distinguished: having developed essential graph sense (delMas et al., 2005; Heck, 2012, Subsection 4.2.5), being familiar with structural components of graphs (cf., Kosslyn, 1989), being able to interpret the result of graphing and see how improvements can be made, realising that one variable is plotted against another in a two-variable graph,[9] having basic knowledge about the user interface of RStudio (writing and running R scripts, the concept of workspace, etc.), being able to enter a syntactically correct plot command,[10] knowing how to define a mathematical function in one variable in R and understanding how it can be applied simultaneously on a sequence of values, knowing how to create a sequence of values, and knowing options in the plot command and how to specify them.

So, one must take many things into account for the instrumentation scheme of plotting the graph of a mathematical function in a command-driven software environment like RStudio. Students cannot be expected to figure this out all by themselves and lecturers must carefully introduce students to the basics of programming in R for doing mathematics. This is called instrumental orchestration in the instrumental approach. Referring to the taxonomy of instrumental orchestrations (Drijvers et al., 2013), we applied whole-class technical demonstrations, discuss-the-screen, and explain-the-screen orchestrations in lectures and tutorials. In addition, we used individual technical support, individual discuss-the-screen, and guide-and-explain in tutorials.

The applied instrumental orchestrations without doubt had helped students familiarise with the technology and learn to do mathematics with it, but they had not been optimal because of the process of double instrumental genesis in which we as lecturers were involved. This means that we were on the one hand developing schemes for use of R in doing mathematics and on the other were developing schemes for use in teaching our students how to use R for learning and doing mathematics. We also had not realised that the instrumental genesis of students during the statistic part of the

[8] An instrumental approach in mathematics education focuses on the interactions between students, teachers, and artefacts. This approach analyses mediations attached to the use of a given artefact and instruments developed by the subjects from this artefact along instrumental geneses (see Trouche, 2020a,b).

[9] In case of a plot of the function $y = f(x)$, a collection of points of the form $\big(x, f(x)\big)$ must be computed for various choices of x.

[10] In order to plot $y = f(x)$ one must create two sequences of values x and y, and provide them in the right order in the plotting command plot$(x, y, \dots)$. This ordering is opposite to the usual ordering in scientific language: in science for example, one calls a graph of a quantity y with respect to another quantity x a y-x diagram, whereas in mathematics it is called an x-y diagram.

course had a different orientation than the one needed for the tool use in the mathematics module. In statistics tutorials, students had copied and adjusted R commands and scripts to carry out a computational task, while in mathematics tutorials they were expected to write R scripts for doing mathematics. As a result, many students stumbled over technicalities in R and RSTUDIO, lost oversight over what they were trying to achieve, and forgot to think about the mathematics addressed in the computer tasks. This affected the performance of students in IBME tasks the most. As a CoI, we conceived a plan for revision of the learning trajectory for the use of R and RSTUDIO toward a better promotion of the students' instrumental genesis.

The revision would be combined with a reduction of the contents of the mathematics module to essential concepts and methods, and better tuning of the learning trajectory to the mathematical abilities, programming skills, and scientific content knowledge of the students.

12.4. Work of the CoI on the Redesign of the Module

The redesign of the basic mathematics module taught in the study year 2020–2021 concerned the tutorials in particular. Two 2nd-year students and one 3rd-year student in biomedical sciences had taken the module in previous years and were employed not only as teaching assistants (TAs) for the tutorials but also as module developers in the two months before the start of the module. They made the R instruction more student friendly and accessible. The main idea behind the student partnership for module design was that the TAs could still see the teaching materials through students' eyes. This was considered helpful for restructuring and changing the wording of the tasks in such ways that students could better deal with the mathematical concepts and methods taught. The partnership also offered new opportunities for getting feedback from students on the module. The CoI expected that students would give more feedback in interviews to TAs, who took the module earlier, than to lecturers. In this sense the student partnership gave an extra dimension to the inner layers of the three-layer model of inquiry.

In the rest of this section we report on the redesign of the module, online teaching, feedback of students on the module, and reflections of the CoI.

12.4.1. Redesign of the Module. Weekly tutorials were split into two sessions, one with a focus on learning mathematics concepts and another focusing on learning to use R and RSTUDIO for mathematical computations, simulations, and data analysis. The idea behind this splitting of contents and nature of tutorials was that it would make it easier for the teaching assistants in the R-based tutorials to organise interventions that assist the process of instrumental genesis, i.e., turning a computational environment into a mathematical instrument.

The TAs focused on improving the support of students for learning to program in R and helping them carry out basic mathematical tasks such as working with mathematical functions, carrying out data analysis, and solving (systems of) ODEs numerically and graphing the solutions. They used the reference chapter from 2018 about working with R to create teaching units for use at the beginning of each R-based tutorial. Students learned in this part the basics of R relevant for the mathematical topic of the lesson through practising with R in a trajectory of many small tasks. In the second part of the R-based tutorial students carried out an IBME task that needed the R abilities that they had just acquired. A worked-out solution of each IBME task was made only available to students at the end of the tutorial to compare their own solution with the lecturers' solution.

12.4.2. Online Teaching. Because the module had been realised in the format of ICT-supported instruction in SOWISO, it was rather easy to manage teaching activities in the context of a lockdown due to the COVID-19 pandemic. Lectures became Zoom meetings and breakout rooms were used to organise small group activities (e.g., the IBME task on numerical differentiation shown in Figure 12.7), the blackboard was replaced by a pen tablet for the lecturer, and polls were added to Zoom meetings for engaging students during the lectures. Students stayed muted during the lecture except when asked for a direct reaction, but they could ask questions at any moment via chat. We structured the student-lecturer interaction in this way in the hope that students would still experience membership of a class with a low threshold for interrupting the lecturer by asking a question. To this end, a teaching assistant continuously monitored the chat, immediately answered simple questions, and interrupted the lecturer for answering questions that seemed interesting for the whole class.

Tutorials became MS Teams sessions. Main reasons for using MS Teams instead of Zoom were that students could use this platform also outside the scheduled contact time, have private meetings with peers whom they liked to work with, and could share application screens or digital images with each other (e.g., SOWISO screens) and pass control of applications to others (e.g., giving a TA control over the Rstudio environment). Especially, the screen sharing was effectively used in the interaction amongst students in a meeting and between student and TAs. Screens of SOWISO exercises with worked-out, but not fully understood solutions, as well as Rstudio screens with scripts that did not work, or perhaps not in the intended way, were shared and discussed.

All tutorials were organised as follows. First, all students and TAs convened in a Teams meeting started inside the main module channel. In this meeting, the tasks of the particular tutorial session were introduced. This could be a digitally handwritten solution of an important mathematical exercise or a demonstration of how to work with R or Rstudio. Next, the students moved to the channel for the working group they were assigned to and started there a private meeting with a small group of peers. The TAs remained in the main meeting. When a student or group of students needed help, they came back to the main meeting to discuss their problem with a TA, left a message for help in the chat if no TA was available at the moment, or directly invited a TA to join their private meeting. Two or three times during the tutorial, the TAs passed by in the private meetings of students to ask how things were going.

12.4.3. Feedback From Students. We refer to Table 12.2 for results of student feedback in the study year 2020–2021. All results point to a more positive appraisal of the module regarding design, quality of the instructional materials, and implementation of the module compared to earlier study years. This is also reflected in the marks given (on a scale from 1 [*very poor*] to 10 [*excellent*]) by students for the module as a whole, the lectures, and tutorials: median values increased to 8, 8, and 7.5, respectively.

The teaching assistants held online semi-structured small-group interviews to explore how students had experienced the module. Subjects were the structure and content of the module, the guidance during the tutorials, the R tasks, the links between theory and practice, and the IBME tasks. The interviewers worked in teams of two with one of them taking notes during the interview. Together they wrote a summary for discussion with the rest of the CoI in an online meeting.

Interviewees told that the transfer from school mathematics to the mathematics module was fine because it started with subjects with which they were familiar. But once the subject of differential equations started, it became more difficult to keep up with contents and pace in the lectures, especially for students with Mathematics A background. The support of teaching assistants during the tutorials was good: screen sharing to explain R code or a mathematical concept helped students move on with tasks that were at first not well understood.

The students mentioned that the mathematical level in the tutorials gradually raised and was very high when differential equations, chemical kinetics, and quantitative pharmacokinetics were studied. Pure mathematics exercises were easier than the assignments on biomedical applications. Yet the students liked that there were both types of assignments and said that “easier does not imply nicer” and that “application tasks require more insight, but add more to understanding than the pure mathematics exercises.”

The discussions about the R tasks revealed that the students considered the learning curve from copying and adjusting R scripts, which was the approach in the statistics part of the course, to writing R scripts themselves in the mathematics module as steep. It helped that the R assignments were mostly done in small groups so that students could help each other. Mathematical exercises were mostly done individually.

The students mentioned that the links between theory and practice were good in general. The lectures connected well to each other, but occasionally less well to the tutorials in the same week. Some of the students wondered whether they could learn enough for the exam from the examples in course notes, the lectures, and the tutorials. This indicated an assessment-driven study behaviour directed to minimisation of time investment for passing the exam.

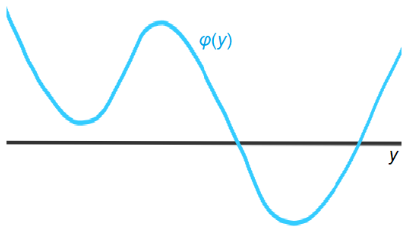
IBME tasks led to mixed reactions of students. Some students found this style of working difficult because the inquiry tasks were open, with a variety of methods that could be applied, and with no single correct result, but with an outcome that is subject to own evaluation. Without a worked-out solution they found it difficult to reflect on their own results or attempts made. Some of these students simply gave up on the inquiry tasks or did not spend much energy on them because they did not expect that more effort in these tasks would lead to higher exam results. Students who liked the IBME tasks mentioned that working on these tasks led on the one hand to new insights in applications of mathematics in biomedical sciences, and on the other hand promoted understanding because they had to think about every line of R code and look at what happened or should have happened. All students suggested that TAs would present worked-out solutions in class so that they could compare them with their own (intermediate) results instead of having to wait until next week when worked-out solutions would be made visible in the SOWISO environment.

12.4.4. Reflection of the CoI After the Revision of the Module. The CoI was pleased that the restructuring of the module and the redesign of the R tutorials had a positive effect on the students’ appraisal of the module. On the one hand, it interpreted the student feedback about availability of worked-out solutions and the other comments of students as an indication that many students were still focused on getting correct answers instead of concentrating on the mathematical concepts and inquiry into mathematics in a biomedical context. On the other hand, it noticed that progress in this direction had been made. Yet, the CoI still identified room for improvement in several dimensions.[11]

[11]The CoI consisted of several persons, each with own experiences and personal reflections, and with own ideas about teaching and learning. But there were actually no strong disagreements or difference of views within the CoI and always could CoI members come to a common understanding on the evaluation of module. This is why we write sentences starting with “the CoI . . . ” instead of presenting individual reflections or letting one member of the CoI speak for all of them.

The CoI still struggled with the amount of content in the module of seven lesson weeks. It sympathised with the students' suggestion to introduce new mathematical concepts in smaller steps, especially out of consideration for students with Mathematics A background. This challenged the CoI to think deep about which important and instructive subjects could still be dropped from the course without lowering the ambition level too much. Also the number of contexts of applications of mathematics was reviewed by the CoI. For example, the question was raised whether chemical kinetics is necessary in an introduction to Systems Biology, but we decided to keep it (at least another year). However, the need to free space for more effective promotion of conceptual learning became evident from the following example. In the exam, students were asked to draw a phase line based on a given graph and do the same for a specific differential equation. The conceptual task was:

For the differential equation $\frac{dy}{dt} = \varphi(y)$ with the below graph of the function φ



(i) How many equilibria are there for this ODE based on the above graph?
(ii) Draw the phase line for the ODE based on the above graph.

The procedural task was the same, but for a concrete function $\varphi(y) = y^2 - 2y$. The success rate for the conceptual task was significantly lower than for the procedural task.

The CoI intended to elaborate on making the learning curve for working with R less steep. Three types of improvements were considered:

- paying more attention in the mathematics module to programming;
- increasing the students' computer skills during the statistic part of the course before the mathematics module starts;
- explaining the differences between the use of R in statistics and mathematics.

The CoI experienced that inquiry-based teaching and learning, especially in the R-based assignments, would benefit from discussion of IBME tasks during the tutorials, in which students are encouraged to reflect more and deeper on their work. Inquiry into students thoughts and ideas is possible only through effective interaction between students and the lecturer. The lecturer can only effectively interact with students when (s)he is able to keep an overview of what all groups of students think and do, and can choose individual, group, or whole classroom discussions according to what seems best at the moment. It is a big challenge to do this with a group of 150 students, even with the help of technology.

The lecturers had difficulty in motivating all students for doing inquiry. For example, about 30% of the students dropped out as soon as the lecturer announced in the lecture the IBME task of inventing and evaluating numerical differentiation methods shown in Figure 12.7. These students actually undermined the didactical contract by avoiding to deal with the IBME task and others may have used the task not in a way actually intended by the lecturer. This phenomenon was also encountered in the German case study described in Chapter 14. The German authors concluded that it is indicative of a fundamental principle in teaching-learning contexts, namely that no teaching can force learning. The CoI wondered whether this is related to lack of motivation for learning only, and not to the difficulty of assessment in an IBME course too. After running the revised mathematics module, the CoI found it promising that many students expressed that they had enjoyed the module and had learned much about the role of mathematics in biomedical sciences.

12.5. Concluding Remarks

In this section we share experiences for lecturers in higher education based on the lessons learned by the CoI at the University of Amsterdam. Firstly we recommend lecturers to join a community of inquiry or to start one, because it helps develop deeper thinking about higher education and sharpen the vision on instructional design of mathematics and science, and because it makes inquiry in teaching and learning easier and more doable. Collaborative inquiry helps lecturers develop an inquiry stance in practice and fosters critical alignment to the teaching-learning-practice. It offers opportunities to discuss and question established ways of teaching, to seek for new ways of classroom activities and mathematical learning of students, and to develop student tasks according to a new vision on teaching and learning. More brains and hands make work easier and help achieve more. Especially involvement of teaching assistants in a community of inquiry is strongly suggested because they have a complementary perspective on instruction. For example, they are more able to view the instruction materials as students and can help in inquiry into student learning. The following quote of a TA in this case study illustrates what was learned from interviewing students:

> All in all, I think we have collected information with which we can do something. Broadly speaking, the course was very structured and the degree of difficulty for students was okay. Some small details have come up several times, so we could do something about that. An example of this is that many students found the hints for R assignments unclear or that they led them astray. Some students suggested to include short pieces of code as a hint rather than a thought or question that should help a student further. This kind of suggestions we could discuss in the next meeting

Within a community of inquiry more informative evaluation instruments can be deployed that go beyond dealing with day to day issues arising in practice. We recommend lecturers to collect data via research-based or own questionnaires because they allow deeper analysis of student results and their course experiences than institutional evaluations can do.

Working in a community of inquiry, especially when it is organised around a particular course, also helps maintain ambitions and keep going on because one does not want to break promises made to colleagues in the mutual engagement. It seems best to get a group of lecturers together and first think of small changes in the instruction and try them out. Changing and/or extending existing tasks to tasks that promote student thinking and engagement, or making existing tasks more suitable for use by students with identified needs are good starting points for discussion.

Working together at international level helps avoid a narrow view on instruction and educational settings. One quickly realises through discussions that different views on instructional design are possible and equally valuable, and that lecturers in other countries have similar difficulties and challenges in teaching and learning. Our experience was at least that setting up a community of inquiry in the framework of the PLATINUM project enriched our work at all levels of the three-layer model of inquiry, increased our joy in instructional design and teaching our students inquiry-based activities, and increased the quality of our work.

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Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project
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CHAPTER 13

The First Experience With IBME at Masaryk University, Brno

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13.1. Introduction

This study provides an insight into the emerging community of inquiry at Masaryk University in Brno, Czech Republic. First, we present the historical and institutional background that determines the conditions in which our teaching team operates. We briefly summarise the possible ways of professional development intended for academics at our university (except for pedagogy students preparing for the future career of primary or secondary school teachers). We believe that the relatively low institutional support of academics in this field dramatically influences the limited professional knowledge of didactics and pedagogy among our team members. Furthermore, we describe our community, which is characterised by considerable fragmentation—both in terms of courses taught and belonging to individual faculties and university workplaces. The physical distance between the buildings where our teachers work is also one of the limiting factors for the functioning of our community. Due to this fragmentation, it is also difficult to characterise the functioning of the community as a whole. Individual sub-teams differ in their approach to designing and implementing IBME tasks in teaching, monitoring, and evaluation. Therefore, we present three different examples of approaches to IBME education, which in our opinion, demonstrate this diversity. Their analysis is presented in three separate sections.

13.2. The Historical and Institutional Background

In this section we picture the pedagogical education of university teachers in the Czech republic and the professional development of lecturers at Masaryk University.

13.2.1. Pedagogical Education of University Teachers in the Czech Republic. While young academics are well prepared for a scientific career during their studies in the Czech Republic, they are usually not prepared for a teaching career at all (Čejková, 2017). Traditional universities combine science and research with teaching, but the attention of academic staff is focused more on science and research at the expense of teaching. This is most often explained by the fact that research performance is the subject of continuous evaluation; it is tied to both the financial remuneration of academics and their career growth, while a system for monitoring the quality of teaching is not present at universities. However, as mentioned by many authors, excellence in research does not necessarily lead to excellence in teaching.

The topic of teachers' professional beginnings is widely addressed in Czech pedagogy, but almost exclusively in the context of primary and secondary education

(Šed'ová et al., 2016). Historical roots of this situation in the Czech Republic are uncovered by Vašutová (2005): the system of pedagogical education of university teachers nearly disappeared in the 1990s. In the post-revolutionary period, we began to rely heavily on a personal approach to teaching and individual responsibility based on renewed academic freedoms. However, due to new requirements and direct and indirect pressures on higher education institutions, the scope of activities of university teachers has demonstrably expanded, performance started to require more time, and its complexity increased. So a new generation of university teachers is not well prepared for their university career (Vašutová, 2005), mainly because they were educated only in the field they now teach and lack teacher education. This shortcoming in undergraduate training is not compensated even when novice lecturers start teaching; there is usually no systematic work with PhD students and novice academic staff.

In the absence of institutional support mechanisms, some academics work independently on their professional development in teaching, even though they do not receive many external incentives to improve their teaching performance. As mentioned by Čejková (2017), insufficient pedagogical education of beginning university teachers is most often replaced by the use of personal contacts. This is realised in various ways—from direct requests for advice through informal interviews and sharing of resources to the observation of colleagues, and so on (Pataraia et al., 2015). Although learning from colleagues is an important socialising element, some authors point out possible risks. Hativa et al. (2001) argue that an unplanned and uncontrolled socialisation process can lead to the acquisition of fragmented pedagogical knowledge and unsubstantiated assumptions about what teaching practices are desirable and effective. Observational learning of beginning teachers can lead to an undesirable effect in following inappropriately chosen patterns. In addition to using the services of more experienced colleagues, beginning university teachers use another source of knowledge, which is similarly common and similarly risky, namely their own experience as student. This is another crucial source playing a key role in the preparation and implementation of teaching (Oleson & Hora, 2013). Similarly, beginning teachers acquire professional skills by trial and error in their own teaching (Hativa et al., 2001). The is no assurance whether they achieve the required quality in this way.

13.2.2. Professional Development of Lecturers at Masaryk University. The problems described in the previous section are also present at Masaryk University (MU). The first systematic attempt at MU to educate new lecturers in teaching competencies is relatively recent; in 2017 the Pedagogical Competence Development Centre (CERPEK) was established within a framework of a local project. It aims at increasing the level of pedagogical competencies of beginning university teachers (having less than 5 years of practice). CERPEK offers a two-semester study programme "Development of Pedagogical Competences," which consist of four parts:

- The laboratory of pedagogical competences (covering topics such as quality of university education, preparation for teaching, communication, students' evaluation, feedback reflection, use of modern technologies, etc.);
- video-reflection on recorded teaching (adjusted to participants needs);
- teaching workshops;
- the mentor programme.

Participants of this programme first attend a twenty-hour intensive seminar, then they learn through online courses. During the year, they have to work out various tasks, including preparing a model lesson to be recorded. As a result, they will receive feedback from course instructors. The programme should help participants identify their strengths and weaknesses. Attention is paid to various topics—how to plan teaching well, how to effectively pass information, what tools to use, etc. However, one member of our group participated in the course and he described the experience as not very beneficial. In his view, the workshops have been too general and the individual topics were not integrated into a unifying framework.

The programme is prepared by a team of specialists in the area of pedagogy, andragogy, and psychology as a general didactics course without subject specificity, so it does not focus at all on teaching mathematics. The capacity of the course is quite limited—about 30 places are offered each year. Up to this date, less than 100 participants graduated in this program (which is a very small fraction of a total more than 2600 teachers employed at the university), only seven of them were from the Faculty of Economics and Administration and only one was teaching mathematics. This situation may change in the future if the centre manages to raise additional funds to expand its offer. CERPEK offers no support for inquiry-based teaching and learning yet. We tried to establish collaboration between the Centre and our CoI, but the COVID-19 pandemic complicated the situation. In 2019, some arrangements were done to prepare together the organisation of the Workshop on Inquiry-Based Education. PLATINUM community experts were asked to hold the workshop for the CERPEK Centre, but it had to be postponed until the epidemic situation improves.

Until CERPEK is able to provide sufficient systematic support to beginning university teachers, some activities organised from the bottom, by young academics themselves may be of great benefit. For example, the youngest member of our CoI greatly appreciated participation in the course "DUCIT Teaching Lab" primarily aiming at future lecturers of the Faculty of Informatics. If the course is not full, young lecturers of mathematics (mainly doctoral students) from other faculties may enrol in this course. According to our colleague, the course helped him to answer important how-to questions such as "How to structure the lesson," "How to motivate students," "How to ask properly," and "How to give and get feedback," but also questions connected directly to mathematics teaching.

There are no such activities organised for older teachers, but the motivated ones seek other possibilities to enhance their teaching skills. For example, we can mention several two-day courses for academics held by the university language centre such as How to Start Your Term Effectively, Communication with Students, Feedback and Evaluation, Intercultural Teaching, or five-days summer school Academic Skills in English. However, there are no official courses or university programs focused on inquiry-based pedagogy.

13.3. The Community of Inquiry at Masaryk University

In this section we introduce and characterise the Community of Inquiry (CoI) at Masaryk University, describe how this community works, and present the first steps of the MU CoI in introducing inquiry-based teaching and learning in some courses.

13.3.1. Characteristics of the MU CoI. Our Community of Inquiry was established in 2018 in connection with the participation in the PLATINUM project. We are all lecturers; there are no didacticians or researchers in mathematics education among us. The group is made up of around ten people, but this is changing over time—some members left the university, but new people joined the group. The majority of CoI members are from the Faculty of Economics and Administration (some are teaching statistics courses, and others are teaching mathematics courses). Two members of our CoI are also teaching the course Mathematical Analysis 1 at the Faculty of Education, and one of them works at the Support Centre for Students with Special Needs (Teiresias Centre) as well.

Team members involved in the cases described in this chapter form a group of two experienced lecturers (Mary and Marge) with more than twenty years of practice, three teachers with an intermediate experience (Lenny, Patricia, and Luke) working as course instructors, two post-docs (Hanna and David) involved as observers in the lessons, and one PhD Student (Tamara). The names of the participants are pseudonyms so that their identity is not revealed since we publish their opinions using excerpts from their reflective narratives.

Previously, most of us did not even know the ideas of Inquiry-Based Mathematics Education (IBME), but gradually within the project, we got acquainted with the IBME principles and, as far as possible, started to include them in teaching. Some of us inclined to these principles intuitively before participating in the project but lacked the theoretical support, as illustrated by perceptions of two teachers:

> It took me time to understand the term 'inquiry.' However, though I didn't know this term, I applied at least some principles of inquiry in my teaching since I started to teach. I didn't know the term 'procedural teaching,' but I always struggled to suppress teaching algorithms in favour of understanding. However, still, I need to move from students' understanding based on my explanation of the underlying ideas to their investigation based on thinking-provoking tasks. This is for me one of the main objectives of the PLATINUM project. I wish to collect either existing or develop new inquiry-provoking tasks, which could be implemented into our statistics courses. As we are rebuilding the course of Statistics 1, we have an opportunity to incorporate IBME principles systematically. For that reason, I appreciate any source of IBME tasks for probability and statistics, or possibly links to any developed curriculum with IBME element. (extraction from the reflective narrative of Mary, June 2019)

> I have several experiences with tutoring students on every level of education, from primary school to university. I always tried to lead them to solve mathematical problems on their own. I only push them the right way by questions and showing similarities to already-figured problems. But I could never imagine how to use this approach in classes with more students and with such a full schedule. So I am very happy for this project and for the opportunity to be part of it. I consider it very inspiring to get ideas on how to provide this way of teaching from so many people with the same interest. (extraction from the reflective narrative of Tamara, March 2020)

> I had supposed that there would be many courses focused on teaching students how to teach. Still, there were only a few such courses, and these were newly established. On the other hand, it was at least some progress in attitude to teaching. In these courses, I first met with Inquiry-based tasks, using GeoGebra and an interactive blackboard. During my PhD study, I participated in the conference STAKAN, focused on teaching statistics; some contributions also involved inquiry-based principles. This occasion motivated me to try to implement some of the activating elements into lectures from the beginning of my teaching career. (extraction from the reflective narrative of Patricia, March 2020)

13.3.2. The CoI Meetings and Discussions. Before we engaged in the PLATINUM project, the regular cycle of practice evolution applied in our teaching included the following steps: plan for teaching, act in the classroom, reflect on experience, feedback to regular planning. Now it was transformed into a teaching inquiry cycle by introducing systematic observation, analysis, and reporting. As described by Goodchild et al. (2013): "in the inquiry cycle, systematic observation and analysis inform the reflective process which provides possibilities for re-planning in better informed ways leading to a more knowledgeable design of teaching" (p. 398).

Moreover, the level of cooperation and communication between the teachers has grown significantly.

Our group of inquiry is rather heterogeneous; its members come from several faculties, operate in various workplaces, and teach many different subjects. This makes the coordination of the group activities somewhat tricky. We meet on an irregular basis as a whole group, sometimes together with the people from the Brno University of Technology community (e.g., during the workshop Teaching Introductory Statistics with Technology in February 2019 or during the workshop with Barbara Jaworski and Simon Goodchild in December 2019). As we do not have a strong background in the theory of mathematics education, we try to use some scientific resources and the broader PLATINUM community's support. The frequency of meetings of the smaller groups connected to a specific course is at least once a month, but short informal discussions happen more often, during lunches, via email or social media, phone calls, etc. The meetings last mostly about one hour; they are usually unstructured—we discuss our experience from tutorials, topics for future seminars, and assessment tasks. The frequency and intensity of the group meetings involved in statistical courses has increased lately because these courses are being redesigned completely.

13.3.3. Implementing IBME in Some of Our Courses. Inquiry-based activities were introduced in the courses Statistics 1, Mathematical Analysis 1, and Mathematics 2. All courses are taught through lectures (2 hours per week) and seminars (2 hours per week).

- The course *Statistics 1* is obligatory for all bachelor's programmes at the Faculty of Economics and Administration. It is a large course with about 450 enrolled students divided into 20-25 seminar groups taught by 8-10 instructors each year.
- The course *Mathematical Analysis 1* is taught at the Faculty of Education of Masaryk University. There are three seminar groups, each of which have 20-25 participants, so in total 70 students of the bachelor's programme Mathematics for Education. Most of the students are in the 2nd semester of their studies. After they successfully finish their bachelor studies (3 years) and the following master studies (2 years), they could start their professional career as mathematics teachers in secondary schools.
- *Mathematics 2* is an obligatory course for first-year students in the Master program of Economy, Finance, and Management. There are about 90 students enrolled in the course each year. The course syllabus covers selected topics from calculus and linear algebra, such as constrained optimisation (especially linear programming) and differential equations.

These courses represent three different cases of IBME application ranging from including short inquiry-based tasks to each seminar (Statistics 1), over team project assignments in Mathematics 2, to whole IBME teaching units in Mathematical Analysis 1. The goals for which we use IBME tasks in individual courses differ: in Statistics 1 and Mathematics 2, we usually aim at introducing new concepts and arousing interest in the topic under discussion; whereas the goal in Mathematical Analysis 1 is usually the repetition of basic concepts and ideas, and deepening of students' understanding. Activities in Mathematics 2 are focused on applications, and they are intended to bridge theory and practice. The ways of monitoring and evaluating the education process also varies: observations by other teachers were applied in the first two courses, whereas in Mathematics 2, the teacher reflected on student presentations and reports from a feedback questionnaire. While observers made a structured record of what was happening in class in Statistics 1, observations from teaching were passed on orally to the teacher by the observer in Mathematical Analysis 1. An example of feedback and the subsequent reflection of teachers can be found in the following sections. We start with a newly designed Statistics 1 course. Next, Luke and his colleague Lenny report on their experience as instructors of the seminars in the Mathematical Analysis 1 course, and finally reflections from Mathematics 2 are summarised.

13.4. Statistics 1: The Experience of Tamara and Patricia

During the planned revision of the course, almost two dozen IBME teaching activities were created. Some of them are based only on a moderated discussion of students (or with the use of aids like dice, lottery equipment, or multi-coloured cards), others need the use of software, for example, to solve assigned tasks using simulations, etc. Three examples of these short activities are shown in Figure 13.1

IB tasks for Statistics course, 1st semester

Exercise 1 (Flashcards [**week 1**]). *We have 3 larger flashcards (nominal, ordinal and quantitative) data and various examples of data on smaller flashcards. Students are divided into smaller groups and have the task to assign variables to the correct data type. Evaluation follows*

Instructions. *Teachers get cards sorted according to the data type. Some may be controversial, depending on how it is taken - specifically, e.g. US presidents - we can sort them in time, but sometimes we look at them as nominal - depending on the context. Discuss with the students.*

Exercise 2 (Questionnaire [**week 2**]). *The teacher will bring the printed questionnaire to the lesson in sufficient amount and let the students to fill it. Then, instead of collecting it, teacher announces that the answers do not matter, and say that he/she is interested in the opinion on the questionnaire as a tool of data collection. Whether the questionnaire looks professional, questions are formulated correctly, etc.*

Instructions. *The questionnaire is of course all wrong. The mistakes were made intentionally: missing some motivation, explaining why they should answer, the overall composition of the questions is meaningless, etc. Discuss.*

Exercise 3 (Random generator [**week 3**]). *Work in pairs. Select* **random** *ten people from the list of people. Use a 10-sided dice to generate random values.*

Think about whether your chosen people selection process is really random, i.e. whether each person has the same probability of being selected.

Then randomly select 10 men (women) from the list.

Instructions. *Students in pairs have a randomly sized sample of 10 from the list of people. They have a 10-sided cube. Then the same task, but with limitation (limited to men, etc.).*

Activity follows with a discussion of how they have proceeded and whether their process is truly correct.

FIGURE 13.1. Activities for the course Statistics 1.

All tasks were originally prepared by one of the teachers and their preparations were no subjects for debate within our CoI (only after the realisation of the activities). To evaluate the benefits of the inquiry-based tasks from the students' perspective, we asked them to fill out the feedback form after each IBME unit. We also included questions focusing on achieving learning objectives and questions associated with students' engagement. The evaluation process is at the time of writing not yet finished.

Referring to the three-layer inquiry model described in Chapter 2 (see also Jaworski, 2019), the connection to higher layers of inquiry was realised through observations of lectures and seminars. Two colleagues, in the role of observers, were taking minutes to capture the structure of lessons and timing of the tasks. An example of an observation report on the activity in week 2 captured by Tamara is shown in Figure 13.2. Before the COVID-19 pandemic, they were present in classrooms so it was possible to add comments on students' engagement and their inquiry development. After switching to distance learning, lessons were taught via distance learning platforms. Some of them were recorded, so it is possible to reflect on them even without an observers' presence.

Minutes	**Observer's comments**
14:00 - Lesson started with introduction to main topics of the lesson: random selection, data collection.	
14:02 Starting the IBE activity. The teacher gave students a task to fill in a questionnaire.	
14:08 - The teacher announced a change of assignment. Students had no longer answer the questions, but they should decide if the questions were correctly formulated or how to make them right.	It was easier for students to find errors and reword particular questions after answering the questionnaire. Students were more open to share their opinions. As usual, there were a couple of very active students, that answer often and some with any response. Students were very creative and had various suggestions.
14:13 – Teacher read each question aloud and opened the discussion. After discussion to every question from questionnaire teacher summarized the proposed solutions and joined own perception. There where an example of a manipulative question, incorrectly chosen intervals for age, scale without explanation of extremes or double-barreled question, etc.	
14:34 – The teacher explained why they did this activity and gave some recommendations on how to make a questionnaire right.	

FIGURE 13.2. Observation report from Statistics 1: activity on data collection.

The observations followed by discussions within the group of course tutors revealed that two or three of the activities missed their goal and would be disposed of. Though, most of them were found useful and would be kept for the future use (some after slight adjustments of formulations in task assignment for better understanding). The teachers agreed that the lessons with IBME units are more demanding than traditional ones, but most of them welcome the challenge. Especially the younger of us say it is more fun to teach this way:

> I believe that inquiry-based tasks can help make students be more active, be involved in seminars and better understand the discussed topic. Many students are afraid of maths and stats courses, and they need to be encouraged, and inquiry-based tasks can help them to get more confidence. On the other hand, these tasks are usually time-consuming. Therefore a good option for me is to involve only short inquiry-based tasks

to each seminar. I think students were more communicative and able to create their ideas (extraction from the reflective narrative of Patricia).

But not all tutors are in favour of the activities. Some of the tasks are perceived as problematic, especially those not having a unique solution. Teachers also report that sometimes it is not possible to keep discussion under control; so it gets too broad or off-topic. That may be limiting students who need a more systematic approach to learning and confusing students with weak foundations of underlining theory. One of the tutors even declared the intention to quit teaching this particular course and the mentioned issues may add to the reasons for his decision.

13.5. Mathematical Analysis 1: The Experience of Lenny and Luke

Mathematical analysis 1 is a course offered to students at the Faculty of Education. Its main topic is differential calculus: real functions of one variable (7 weeks) and real functions of two or more variables (3-4 weeks). Students of the course attend lectures (2 hours per week led by the lecturer who is not a member of the CoI) and seminars (2 hours per week led by us—Lenny and Luke, the main authors of this section).

Students are introduced with topics during lectures on a more theoretical level. During seminars, we (Lenny and Luke) should continue, add more practical information and guide students to understand these topics. The content of sessions is chosen by us, not by students, and given by a strict curriculum of the course. We usually prepare one document for each session including the most important facts, examples to compute during instruction or at home, and the recommended resources to read. This file is shared with students before the session starts. During sessions, we discuss the key terms and try to activate students to come with their answers, comments, questions. We also guide them during practical solution of examples and give them feedback. Students are supposed to solve the problems by themselves or in groups, not only wait for explanation of an instructor.

The lecturer is very close to retirement, and we decided that we would not ask him to modify his lectures to be more inquiry-based. But he agreed we can change seminars and add inquiry-based tasks or units. We started to plan this modification at the beginning of January 2019, so we didn't have more than two months for the design and development phase as we planned to use the first IBME tasks at the beginning of March. That's why we focused on particular topics and didn't modify all seminars' sessions, but only three of them.

We prepared two smaller IBME tasks[1] and one IBME unit based on two worksheets[2]. We designed IBME activities to enable students to:

- understand key theoretical terms such as the limit and the derivative of a real function of a real variable,
- work with applications for plotting graphs of 2D functions (GeoGebra, Wolfram Alpha, etc.) in order to interpret the above mentioned fundamental concepts of the course geometrically, and to
- understand the relationship between the first derivative and monotonicity, and the second derivative and convexity/concavity.

All IBME activities were organised in the same way, going from a rather closed format to an open format. Students got detailed instructions and were asked to do small consecutive subtasks within groups of 2–4 members. They were encouraged to use all reachable resources and work with applications for plotting graphs of 2D

[1]We selected one to be described in Section 13.5.1.

[2]The IBME unit is discussed in more detail in Section 13.5.2.

functions on their cell-phones, tablets or laptops. These teams worked separately, we observed their activities and help them individually in case they got stuck or didn't find the answer to the particular question they posed. After they finished, we started the overall discussion and asked the students to come with remarks and findings. This part was very open: we only moderated the discussion and tried to activate students to response and give their own informal explanations. Finally we invited students to summarise and formulate conclusions most relevant to the topic.

13.5.1. IBME Task—Limit of a Function.

The task instructions: Make groups of 2-4 people. One of the group specifies limit conditions or requirements on continuity of an unknown function. The others try to find an example of a function which meets the requirements. You can change the roles then. Examples of requirements:

(a) Find a function $f(x)$ such that $\lim_{x\to 3} f(x) = 5$.
(b) Find a function $f(x)$ such that $\lim_{x\to 3} f(x) = 5$, but $f(x)$ is not continuous at $x = 3$.
(c) Find a function $f(x)$ such that $\lim_{x\to 0} f(x) = -\infty$.

Goals: We designed the task to enable students to repeat the knowledge that they had received during the previous theoretical lecture. We intended to give students the opportunity to

- recall the concept of limit and continuity of a real function, and
- get back to their own resources and read through the notes they made about the topic during the previous lecture.

Because the task was designed as a game between students, we expected them to be motivated enough to come with requirements that would not be easy to meet, for example limits at infinity, infinite limits, or limits at the point of discontinuity. When giving a solution, students were supposed to justify their proposal and it was open in which way they did so—whether they explained it verbally, computed that or demonstrated it on the function graph drawn by themselves on a sheet of paper or plotted by GeoGebra, Wolfram Alpha, etc.

Tutors' experience: Our impression was very positive. All students made groups and started actively working on the task. We recognised it was fun for them. More than half of the groups used the examples we had offered together with instructions. That was not our intention, so we agreed to ask students explicitly to create their own tasks next time when we repeat the session. In the end, we invited students to come with their own cases. It was useful because they made several mistakes and we all started reasoning and explaining all kinds of limits of functions.

13.5.2. IBME Unit—Monotonicity, Convexity/Concavity.

Unit scenario: In this case, we prepared two worksheets and planned the whole 2 hours' session (100 minutes) as follows:

(1) Working on the 1st worksheet on monotonicity of a function, see Figure 13.3 (25–30 minutes),
(2) Practical solution of other examples selected by teachers[3] (20 minutes);
(3) Working on the 2nd worksheet on convexity/concavity of a function, see the Figure 13.4 (25–30 minutes);
(4) Practical solution of other examples selected by teachers[4] (20 minutes).

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[3]Students were asked to specify intervals of monotonicity and local extrema of selected functions.

[4]Students were asked to specify intervals of convexity/concavity and inflection points of selected functions.

Worksheet – Monotonicity of a function

Let's consider the function $f(x) = \frac{1}{4}x^4 - x^3 - 2x^2 + 3$.

(a) Make the graph of the function using GeoGebra.

(b) Specify intervals on which the function increases or decreases; write it down in the following table:

	$(-\infty,\ -1\rangle$	$\langle -1,\ 0\rangle$	$\langle 0,\ 4\rangle$	$\langle 4, \infty)$
$f(x) = \frac{1}{4}x^4 - x^3 - 2x^2 + 3$				

(c) Compute the first derivative of the function.

(d) Determine the slope of the tangent line to the graph of the function intersecting in the following points; write it in the following table:

x_0	-4	$-0,5$	3	5
$f'(x_0)$				

(e) Use GeoGebra to create the graph of the function's first derivative.

(f) Specify intervals on which the first derivative is positive or negative. Write it in the following table:

interval				
$f'(x)$				

(g) Compute: $f'(-1),\ f'(0),\ f'(4)$.

Final task: Draw conclusions from your inquiry.

Figure 13.3. Worksheet 1 on monotonicity of a function, taken from the course Mathematical Analysis 1

Expected prior knowledge of students: elementary real functions of one variable and its properties, ability to compute a derivative of a given real function, tangent line and its slope (understanding the relationship between the slope of the tangent line and the function derivative at the given point); practical experience with GeoGebra or any other application for plotting graphs is helpful, but not necessary.

Goals: Performing activities of the unit enables students to

(1) recall the knowledge and skills they have received during the previous lecture and use it during performing activities described on both worksheets;
(2) make observations, formulate findings and justify them in a smaller group or during the whole class discussion to identify the relationship
   - between the first derivative and monotonicity including local extrema
   - and between the second derivative and convexity/concavity including points of inflection;
(3) get to know/remind the main features of GeoGebra.

Both worksheets are meant as activities to introduce students with methods to search for the intervals of monotonicity and points of local extrema, and for the intervals of convexity/concavity and points of inflection. But they should not stand as the only example how we can investigate these properties. That's why the final question is open and gives students or a teacher possibility to pose difficult questions. These worksheets need to be complemented with other, more complicated examples of functions, that make students think about points for which the function is not defined or the first/second derivative doesn't exist or the first/second derivative doesn't change the sign despite the fact it is equal to zero.[5]

[5]See the unit scenario: the session was supplemented by additional parts in which we focused on more complicated examples and explained both methods in more detail.

Worksheet – Convexity/concavity of a function

Let's consider the function $f(x) = \frac{x}{3-x^2}$ and its first derivative $f'(x) = \frac{3+x^2}{(3-x^2)^2}$.

(a) Make the graph of the function $f(x)$ using GeoGebra.
(b) Compute the second derivative of the function.
(c) Specify intervals on which the second derivative is positive or negative. Fill the following table

interval				
$f''(x)$				

(d) The graph of function f goes through the following points

$$A = [-1, -\tfrac{1}{2}],\ B = [0, 0],\ C = [1, \tfrac{1}{2}].$$

Determine equations of the tangent lines in these points and add their graphs on the GeoGebra canvas.

e) Look at the neighbourhood of the points A, B, C and compare the mutual position of the function graph and the tangent line in these points. Try to research how is this geometric relation connected with the sign of the function's second derivative.

Final task: Draw conclusions from your inquiry.

Figure 13.4. Worksheet 2 on convexity/concavity of a function, taken from the course Mathematical Analysis 1

13.5.3. IBME Unit Monitoring and Evaluation. We asked our colleagues David and Hanna to observe us during the sessions on monotonicity and convexity/concavity of a function. We sent them the worksheets in advance and described how we planned our session. We also specified the goals of the unit and how students should work with the worksheets. After the sessions had been realised we spent a few minutes discussing our first-hand impressions. Later on, we all met together and evaluated our sessions.

Luke's reflection (after the lesson observed by Hanna):

Thanks to Hanna I have an idea about time spent on both worksheets and other information concerning the activity of 18 students who attended the seminar. The first worksheet took us 30 minutes in total. Some students worked individually, and some made groups. I was prepared to give them a paper sheet with a graph of the function in case of problems with GeoGebra. Few people asked me for that, but most of them worked with GeoGebra. The first student finished all tasks after 11 minutes, the last one after 22 minutes. We then spent several minutes discussing. I remember a lot of students were active during the discussion and came to the right conclusions without my help. I felt this IBME task fulfilled my expectations: the students were then able to work independently when solving examples on monotonicity and local extrema of given functions.

The second half of the seminar was dedicated to the worksheet on convexity/concavity, again with the same setting as in the previous case. Most of the students used GeoGebra, one group asked me for the paper sheet with the graph of the function. We spent 40 minutes in total to solve all tasks and to discuss. I recognised a serious problem when students were trying to find the second derivative of the given function. Only 2 of 18 were successful; all others needed my help, so I had to compute it on the board in front of them. This activity took us a lot of time. There was one other computation in the end as the students were asked to find equations of tangent lines for three points. Some of them had problems again, but it was much better and not so time-consuming.

After 30 minutes, I asked all students to summarise their inquiry. Maybe due to the complications with computations they seemed to be tired and weren't very active.

I may add another reason for their inactivity: for most of them the terms convexity/concavity and inflection points of a function were new unknown things, therefore it was not easy for them to understand the relationship between the second derivative and the position of the function's curve and the tangent line in the given point or interval. It was then difficult for me to convince them to interact and try to come with conclusions. After my own summary of the worksheet's results one student finally answered correctly and explained the relationship of the second derivative and convexity/concavity including points of inflection. Then we had about 15 minutes to solve other examples on convexity/concavity and inflection points of given functions.

Lenny's reflection (after the lesson observed by David):

I think both classes were successful. We did not finish everything but students became so engaged by inquiry that I did not want to interrupt them. In the beginning we finished examples on l'Hôpital's rule application. Then I gave them the worksheets on monotonicity and they were able to finish all tasks. After that I selected a function and asked them to find its intervals of monotonicity and local extrema. They worked independently without my help and succeeded.

At the end of the seminar, we started with the second worksheet on convexity and concavity. We did not manage to pass it all and stopped the work after task (c) to specify intervals on which the second derivative is positive or negative. I asked the students to finish the rest at home and we will summarise the activity during the next seminar.

Several students confirmed it was useful to write all three tables (the worksheet on monotonicity) on the board, one below each other. They were asked to fill the missing cells and then quickly find the right conclusions.

David told me after the seminar that it would be better to formulate the inquiry-based questions[6] more precisely so it can help students to come up with conclusions more quickly and easily. We both confirmed it is better to show these inquiry-based questions at the end of the worksheet's activity, so the students can better organise received ideas.

Next time I would choose a more simple example on convexity/concavity so students would not spend so much time on differentiating. They struggled to find the second derivative, on the other hand, they should be able to do so, but for some of them it had been the first experience 14 days ago.

13.5.4. IBME in Mathematical Analysis 1—General Conclusions. It was our first experience with inquiry-based teaching of mathematics. We created a small community of inquiry and worked together when designing, implementing, and evaluating IBME tasks. We appreciate the feedback from David and Hanna, who came with interesting ideas. As we wrote before, all IBME tasks served to repeat key terms from the previous lectures and met this goal from that point of view. Unfortunately, we were not able to add more IBME tasks during the second half of the semester due to lack of time.

The second main goal of seminars was to introduce students with procedures on computing a limit of a function, its derivative, finding local extrema, and so on. These are the skills students should know how to carry out and we tested them during both credit tests. As we struggled with time in the end of the semester, we focused on demonstrating students all procedures mentioned above and didn't have possibility to add more IBME tasks.

During the first phase of implementation we also recognised weak points of the IBME tasks we had designed:

[6]Lenny meant invitations to make conclusions on students' work formulated as the final tasks on both worksheets.

- In some cases (see our reflections in Sections 13.5.1 and 13.5.2) we formulated subtasks or questions too generally or we were not accurate enough, therefore students undertook the inquiry in a manner not directing towards valuable conjectures or gave so many irrelevant details not helping them to generalise their remarks.
- We did not estimate the time needed to work on both worksheets during the IBME unit well. Next time we may save some time if we do not ask for so many subtasks which are complicated or time-consuming but not relevant in a view of our educational goals that we described above.
- Students experienced difficulties when working with GeoGebra on smartphones. For some of them it was the first time they actively use this application and they had troubles entering functions into edit fields or it took a long time for their smartphone to respond and display a graph. We both reflected we should have recommended them in advance to take a tablet or laptop with them as it would have been probably much easier for them to work with GeoGebra on such a device.

Next time we can re-design all IBME tasks and the unit so that we can reach the goals we determined more effectively. Sometimes, less is more. And of course, it is better to use more simple examples so that students are not much occupied by particular activities that are not relevant to the final goal.

There is one added value with regard to the future employment of the students. As they will probably become teachers of mathematics in secondary schools, it was interesting for them to try IBME activities themselves and think about the pros and cons of this type of teaching/learning process.

13.6. Mathematics 2: The Experience of Marge

Within the PLATINUM project, several partial IBME activities were originally prepared for the Mathematics 2 course. However, due to the transition to online mode, most of them were omitted in 2020-2021. They required moderated group discussions, which is difficult to implement now (the university-wide introduced online learning platform does not support break-out rooms). So instead of these short activities, students completed a team semester project this study year. Eighteen groups consisting of three or four students chose one of the given problems (applications of linear programming in economics, finance, or management). The steps necessary to complete the task include building a mathematical model of the problem, choosing an appropriate software, using it to solve the problem, interpreting the results, and answering additional questions. An example of one of the assignments is shown in Figure 13.5.

The tasks were prepared by one of the course instructors in the first two months of the semester. Case studies from selected textbooks of operations research were used for the preparation of the project assignments, but the formulations were adjusted to change the procedural tasks into IBME activities. This means conversion of the sentence "Apply the sensitivity analysis to find the stability region for the capacity constraint" to "Try to guess what happens when the capacity of the resources is reduced by three units." The problems, as well as the way they were assigned to students were discussed with the rest of teachers at one of the meetings in the middle of the semester, so the expected outcomes and the plan for the activity implementation was agreed on:

The activity is designed to arouse interest in the topic and introduce new concepts connected to the area of linear programming. The applied character of the tasks is supposed to help the students build connections between theory and practice. When

Task 1 Production mix of an engineering company

An engineering company is using two processes (grinding and drilling) for the production of five kinds of products. The individual products generate following unit profits (CZK):

PROD 1	PROD 2	PROD 3	PROD 4	PROD 5
550	600	350	400	200

Each of the products has a different processing time using the individual production processes. The times are listed below (in hours). The dash indicates when the process is not needed.

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5
Grinding	12	20	–	25	15
Drilling	10	8	16	–	–

In addition, the completion of each of the products requires 20 hours of assembly line staff time. The factory has three grinding machines and two drilling machines, and the machine operator has a six-day working week with two shifts of 8 hours each day. Eight workers are employed in the assembly, each working one shift a day. Your task is to find out how many products are to be produced in order to maximize the overall profit. Formulate the model, find the optimal solution and find out which resources will be completely depleted and which will leave unused capacity. Next, try to answer the following questions:

1. *How much higher should the price of product 3 have to be to make it worth producing?*
2. *What is the value of the additional hour of grinding?*
3. *What is the value of the additional hour of drilling?*

FIGURE 13.5. Assignment of a team project, course Mathematics 2

deciding on the implementation of the activity, we tend to be open in almost all considered dimensions of inquiry learning: we let the students involve themselves freely concerning exploration, making observations, planning investigations, justifying, and posing questions; the only exception is the closed form of formulations of findings. The cooperation and using tools are essential for the completion of tasks.

Students elaborate on the solution within their groups. Intended working time for the activity is 10 to 15 hours in total, but it can be split between more people. The group work is realised outside the lessons during November; students can consult the teacher every week during special office hours to discuss their progress. In the first two weeks of December, presentations of student solutions take place during the lectures (nine presentations of ten minutes length each week). It is realised as a videochat, one student is selected in each group to speak for the team and share the computer screen with a POWERPOINT presentation. After the presentations, the teacher generalises the findings obtained from the problem solutions and makes students familiar with concepts of the duality in linear programming, sensitivity analysis, shadow prices, and other topics that could help answer the questions asked in the tasks.

13.6.1. Evaluation of the Activity. As the teacher was present only at some parts of the whole process (consultations and final presentations), the observation was replaced by student self-assessment in this course. The presentations were not recorded because some presenters feared that the recording would make them nervous. After the presentations, we have sent the students a feedback questionnaire to evaluate the activity. First, students expressed their overall opinion on the activity using a quantitative scale, see Figure 13.6.

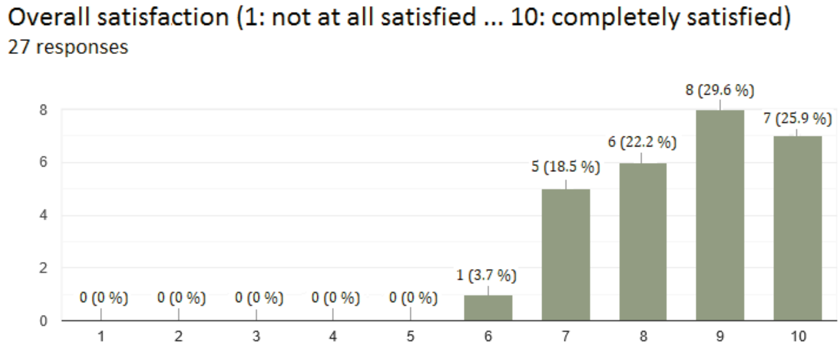

FIGURE 13.6. Students' evaluation of team projects in Mathematics 2.

Students also answered open questions, and they reported on their progress in solving the problems. They should state in particular:

- how long it took them to solve the task;
- what the level of cooperation was in their group
- what difficulties they encountered in carrying out their tasks,
- the means by which they overcame these difficulties;
- what knowledge and skills they think that they acquired during the activity; and
- what they consider to be the greatest benefit of this way of teaching.

Below, we provide the translations of feedback from four teams selected to represent the variety of answers of the whole student cohort (18 teams):

Group 1: I think we spent too much time preparing the presentation. I would consider it more beneficial to practice on easier optimisation problems (more exercises covering typical problems that may appear in the final test—solving problems using graphical method, etc.).

Group 2: The cooperation inside our team on the assigned task was great. It took place via social networks, thanks to which we contacted and discussed on a regular basis. We used written conversations and up to it made about three calls. During the first call we tried to figure out how to model the problem: what variables to choose, and how to insert them in the objective function and the constraints. This was the hardest part for us. That's why we agreed to have two days to think it over, and then to meet again and agree on further action. We tried to look for similar examples and possible solutions on the internet and consulted with our friends and the teacher. That helped us a lot to complete the task.

We are not sure if the project task would help us in further study or in writing final theses, although it is undoubtedly good for better understanding of the theory covered by the course, maybe also for improving our critical thinking.

Group 3: We tried to come up with a solution in the first video call, but we didn't figure it out. The first video meeting resulted in various suggestions on how we will be able to work on the solution, as well as tasks for each member of the group (to watch the lecture, get acquainted with the Solver add-in in Excel and conduct a search on the issue). During the individual work, we shared suggestions or relevant links to articles and videos so that all members have access to them, and then we analysed and discussed them. After the second video session, based on the knowledge gained through self-study and discussions, we managed to model the problem. We implemented the solution with the help of the Excel add-in Solver.

When preparing the presentation, we came across an additional issue, which was to write an objective function and constraints of the problem using a double sum notation. Again, we followed a similar procedure as in solving the problem itself. We first shared the information we found relevant, links to articles, explanations from the textbook and other research results. Subsequently, we had another video call using Microsoft Teams and together we worked it out. The total length of video calls was approximately 3.5 hours. Together with other activities, we spent about 10 hours working on the project. We consulted the teacher to check the solution. The problem seemed complicated at first, but after getting acquainted with the Solver add-in by Excel, we changed our minds. Writing a double sum seemed to be the biggest problem for us. However, we also consider this as the biggest benefit, because this part of the task combined the theoretical knowledge that we gain in class with an application on a real particular case. Another benefit is that we had the opportunity to get to know another useful Excel add-in we had no previous experience with.

Group 4: Our project task showed us how we could make decisions in the future if we want to invest our money in various financial instruments and at the same time, maximise profits. We realise that this type of problems does not capture the real world with 100 percent accuracy, but we think that such types of problems are an excellent tool for learning mathematics through real-life problems and thus arousing greater interest in mathematics. We were thinking about how we could make this project task closer to reality, and we thought of taking into account the risk factor and solving this task for more scenarios.

As can be seen, the opinions varied significantly. Some students did enjoy the task but did not find it very useful (Group 1 and two other groups). They found it too difficult and not connected to the final exam. On the other hand, some students suggested the inclusion of additional constraints in the problem leading to a more complex problem structure with higher application potential (Group 4). Six more groups mentioned that motivation by the real application made them more engaged in the activity. The majority of students considered the activity beneficial and declared that it helped them to increase the level of their understanding of the topic in general (see Groups 2 and 3). Students of Group 3 and five more groups found another benefit of the activity in getting acquainted with the recommended software tool for solving optimisation problems. Although it was not planned as a goal of the IBME activity, it helped to arouse more interest. There was almost total agreement on the usefulness of working in teams. In general, students appreciated the collaboration; only in one case a team reported a non-responding team member, but after the intervention of the teacher this student started to participate in the team work. It was difficult to check whether the tasks took the time intended for their elaboration. With some exceptions (Group 3), the teams didn't report on their work progress properly, only some of them vaguely reported that the problem was too time-consuming (Group 1 and three more groups).

At the end of the semester, the lecturer together with the course instructors discussed the results and students' feedback and the conclusion was to keep the activity in the slightly modified form for the next year. If it is possible to transfer it from distance to in-person teaching, we will be able to better evaluate the activity. The only major modification considered for the next course is the inclusion of a simplified version of the project problems in the final exam to involve also the students motivated only by the successful completion of the course. However, we are aware of the fact that the activity is time consuming and its implementation in the course with an extensive curriculum can only be incorporated because most of the process takes place outside the class.

13.6.2. Concluding remarks by Marge. I have started introducing IBME elements into my courses Mathematics and Mathematics 2 more systematically last year.

The activities have been accepted rather positively by students, and my enthusiasm for other efforts grew. This enthusiasm, however, slows down during this semester, because I feel very limited in finding a way how to squeeze IBME activities into a very condensed course, taught on an online platform. In any case, I welcome the involvement in the PLATINUM project because it allows me to prepare my lessons more systematically, to reflect on my teaching, and to exchange experiences with other colleagues. However, I do not expect that the project PLATINUM will help me change my own lessons radically. I guess that one of the main barriers for more intense using of inquiry in teaching is the students' mindset and their expectations. In the opinion poll, students repeatedly report that there was "too much theory and not enough exercising." Even if they enjoy the activities, they usually perceive the time devoted to building conceptional understanding as "lost" or "inefficient."

I am aware of the usefulness of IBME approaches in teaching mathematics but I consider the state of mathematics education at all levels of schools in the Czech Republic to be deeply underdeveloped in this respect. Elementary schools focus on preparing students for typical problems in secondary school entrance mathematics tests and a similar situation is at the next level of education. Thus, if the success rate of elementary schools is measured by the percentage of pupils in high school admissions and the success of secondary schools by the percentage admitted to higher education, it is no surprise that schools focus on training students for standardised problems. And this is naturally the same thing what students expect from their mathematics teachers when they enter university: to train them for the success in standardised exam tests. So, driven by these expectations, students are not able to fully appreciate the benefits of pedagogy aiming at deeper conceptual understanding. In my opinion, a good design of assessment and exams testing rather conceptual than procedural knowledge could break this status quo. It is of great importance to improve the situation also at the lower levels of education.

13.7. Summary

The functioning of the community of inquiry at Masaryk University is special as the individual team members work in several workplaces and participate in the teaching of many different subjects which makes the coordination of our activities more difficult. Meetings and IBME discussions are organised on different levels:

- observations in the lessons;
- small teams of people teaching the same course: rather informal discussions on a regular basis (approx. every two weeks). We don't take minutes, but we share material in the university information system;
- irregular meetings of the whole MU team, usually together with BUT team (structured, recorded); and
- events for a broader community, including people outside MU.

Within our community, IBME elements were introduced into large courses on mathematics or statistics for non-mathematicians (mainly future economists or teachers at primary schools). We described three different cases of IBME implementation. It turned out to be quite challenging as our courses have comprehensive curricula and a tight time plan. Another problem turned out to be the spatial fragmentation of individual workplaces of the university, when, for example, due to the need to move between buildings, it was not possible to immediately provide feedback from the observer to the teacher after the monitored hours. An equally important problem that we have not yet been able to solve, was the resistance of some teachers to the use of

IBME principles in teaching. As our experience from the course Mathematical Analysis 1 has shown, the inclusion of IBME only in a part of teaching (seminars) without alignment with the overall objectives of the subject, the method of evaluation, etc., is limiting the benefits of this approach. Nevertheless, even the small progress matters. Quoting Jaworski (2008, p. 313),

> in an inquiry community, we are not satisfied with the normal (desirable) state, but we approach our practice with a questioning attitude, not to change everything overnight, but to start to explore what else is possible; to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to provide answers to them (Wells, 1999).

To conclude, let's go back to the issue of insufficient professional development at our institution. Our group's specific feature is that our educational background is either mathematics or statistics; we have no mathematics educators among us. The majority of us have not even been introduced to the basics of the theory of pedagogy. Participation in the project helped us a lot to approach the educational process more systematically, to become aware of our teaching methods, to share the experience and collaborate as members of an emerging community of inquiry. So we may say that it adds to filling the gaps in this area. Our experience with IBME units can be valuable for other colleagues at our university if we continue disseminating IBME ideas.

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CHAPTER 14

In Critical Alignment With IBME

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14.1. Introduction

The aim of this contribution is to describe the professional growth of the Leibniz-University-Hannover-group (LUH-group). The four authors of this chapter are the core members of the LUH-group[1] and belong to a working group in the mathematics education department of Leibniz University Hannover. In the beginning, a number of people from other departments were interested in joining the LUH-group, but either transferred to other universities (in Germany staff changes between universities are quite common), or didn't find the time to attend meetings on a regular basis. The LUH-group conducts research in the field of university mathematics education and, with regard to teaching, offers courses in mathematics education for prospective mathematics teachers at secondary school level. This means all LUH-group-members are mathematics education researchers as well as mathematics teacher educators. The reported professional growth is connected to our involvement in and reflection of a developmental research project called Leibniz-Prinzip (see Section 14.2).

We take this project and observations that we made in connection with it as starting point and develop from this a reflection on central theoretical foundations of PLATINUM regarding its concept of *Inquiry-based Mathematics Education* (IBME). Potentials and goals of cooperative development addressed in the concept of *Community of Inquiry* (CoI) will be questioned with regard to their implicit assumptions, prerequisites and conditions for success. Drawing on our local conditions and experiences, we will critically examine these implicit assumptions, prerequisites and conditions for success, which can be understood as forms of personal and institutional specifications of our current prerequisites and potentials for further development. For us, the idea of CoI functions as a counter-horizon[2] against which restrictive conditions and potentials for further development will be fleshed out in more detail (see Section 14.4). The initial idea of CoI articulated in the three-layer-model (see Chapter 2) hints at an interrelatedness and mutual enrichment of developmental research and professional growth. The three-layer-model indicates that the members of the PLATINUM project, in their activities of fostering IBME, are simultaneously involved in different but interrelated

[1]Because we consider *Community of Inquiry* to be an analytical concept, we refer to ourselves (as people) with the term *(LUH-)group*. The LUH-group, of course, can be analysed concerning its characteristics regarding the analytical concept. Any critique in this contribution refers exclusively to the analytical concept of "Community of Inquiry," and not to the work of PLATINUM groups published in this book.

[2]The term 'counter-horizon' refers to a horizon of interpretation that opposes typical horizons referred to in a field of practice. We detail which typical horizons we oppose in Section 14.4.1. The counter-horizon is regarded as an alternative outlook among a number of more common or widespread interpretations.

CoIs. These CoIs can be described with reference to the positions of their members, their inquiry interests and the objective of their inquiry activities, see Table 14.1.

Layer	**Positions**	**Inquiry into...**	**objective**
inner layer	students and teacher(s)	mathematical activity	learning mathematics
middle layer	teachers	teaching and learning practices	professional growth
outer layer	teachers and researcher(s)	IBME	developmental research

TABLE 14.1. Positions of members, the scope of inquiry and the objective of interrelated CoI.

Concerning the positions addressed in the three-layer-model, all group members within our LUH-group simultaneously occupy the positions of teachers and researchers. Having this double-responsibility can be regarded as typical for German universities and university teacher education specifically (e.g. Adler et al., 2005).[3] In this situation, university teachers[4] are often said to be "in a double role." We would argue, though, that the understanding of the positions in CoIs needs to go beyond interpreting them as roles: Common role concepts harbour the danger of (1) subordinating positions and their scope for action to institutional goals and conditions, (2) personalising contradictions and barriers and (3) demanding a professionalisation of persons that aims at a mere satisfaction of role requirements. All three aspects entail an ignorance of contradictions in the institutional-social contexts of reference. Such a mode of thinking, which personalises contradictions, can furthermore be a symptom of an inadequate theoretical analysis of positions (for a reflection of our positions as teachers and researchers see Ruge & Peters, 2021). Thus, challenging common role concepts in teaching-learning relations and professional development might be a starting point for questioning and extending concepts of learning and development. Such possible extensions can be found, for example, in Engeström's (1987) concept of expansive development and in Holzkamp's (1995, 2013) understanding of agency. The concept of expansive development primarily addresses institutional-systemic dimensions of development.[5] Holzkamp's understanding of agency opens up ways to explore the personal possibilities to act within or upon conditions. The relation between personal possibilities and underlying conditions is conceptualised with reference to their societal-mediatedness and historical specificity. This understanding of agency is an

[3]However, staff members in Germany are usually employed on fixed-term contracts that privilege research activities over teaching, both in terms of allocated time and of criteria for promotion and further employment. This creates tensions and contradictions when trying to fulfil both responsibilities.

[4]Unless otherwise specified, we use the term 'teacher(s)' to denote university teachers, 'student(s)' refers to university students (in our case teacher students), and 'pupil(s)' is used for secondary school students.

[5]Engeström proposed an analytical tool for the description of activity systems and further developments of the object of an activity. The activity system is described from the point of view of an individual or a subgroup and integrates the community, its division of labour and rules in the analysis of the development of the object. The development of activity systems is judged by the development of the object, the formation of a new or expanded object (Sannino & Engeström, 2018). Therefore, the focus is on the systemic level and suitable to describe developmental processes of organisations or institutions.

important point of reference of our research activities in the LUH-group (for details see Section 14.4.2) and serves us as basis for reflections and debates.

A central topic of the reflections and debates within our group was the course "Introduction to Mathematics Education," in which three of the team members were involved as teachers. The participants of this course are prospective mathematics teachers at secondary school level who are still at the beginning of their university studies. The focus of the course is on fostering reflective agency (see Section 14.2), that is based on Holzkamp's understanding of agency. In the development and implementation of the course, phenomena arose that we reflected and debated on within our group. The phenomena can be understood as manifestations of conflicts and areas of tension that are typical within our context. In particular we address phenomena that are described in the literature under the headings of *theory vs. practice* (Terhart, 2000), *teaching-learning short-circuit vs. guidance* (Holzkamp, 1995; Huck, 2013), and *autonomy-antinomy* (Helsper, 1996). Our reflections on and debates of these phenomena led us to rethink the concepts of CoI and IBME and their initial framing in PLATINUM. We enrich, differentiate and modify them, to rearticulate the potentials that we claim these concepts hold for the further development of theory and teaching practice (see Section 14.4). To us, these aspects are strongly connected to each other. This is reflected in our mode of participation in PLATINUM, which can be described as a constant back and forth between further development of theory and further development of teaching practice. Both contribute to each other. We want to detail this reciprocity of development in theory and teaching practice, which characterises the process of our joint professional growth and forms the core of our development.

We structured our contribution as follows: In Section 14.2, we first describe the context of the teaching project of our group – the developmental research project Leibniz-Prinzip and the course "Introduction to Mathematics Education," which was developed in this project. We briefly describe the course and its overall goal of fostering reflective agency, before we provide a sample task with a description of the particular content to be inquired into and outline experiences with student reactions and solutions. In Section 14.3, we describe the above-mentioned selected phenomena and contradictions (theory vs. practice, teaching-learning short-circuit vs. guidance, autonomy-antinomy) in the context of this teaching project. We describe our engagement with these phenomena with reference to the theoretical foundations of the concept of reflective agency and then reflect on consequences for our development as LUH-group, i.e., the process of our professional growth. In formulating these descriptions and our interpretations, we consider ourselves in the position of teachers and researchers in the local context of the LUH-group. In Section 14.4, we contemplate the process of our professional growth against the background of the concepts of IBME and CoI in PLATINUM. On a meta-level we reflect on our experiences as teachers and researchers within our local PLATINUM project. Within the global PLATINUM group we also consider ourselves researchers and take up this position for our formulations of a further development of the theoretical foundations of the PLATINUM project. We will present reflections which, among other things, point to the necessity of both the conceptual concretisation of the three-layer-model and the consideration of societal and professional aspects. Our conclusion suggests a restriction of the goals of IBME and an expansion of the concept of critical alignment, which is described in (Jaworski, 2006) as

> ... critiquing and trying to develop, improve or enhance the status quo, alongside enculturation into existing social norms. However, the significance of normal desirable states is just that they are desirable within the social practices in which they have developed.

> It is hard to operate against such practices, or to challenge them in practice. [...] I see the term "critical," in "critical alignment," as indicating a key concept for avoiding the perpetuation of undesirable states. (p. 191)

Within this contribution, we seek to detail our critical alignment. From our point of view, Section 14.4 covers the core of our case description.

14.2. Context of the Teaching Project of the LUH-group: The Course, the Concern for Reflective Agency, and the Sample Task

Our professional practice as mathematics teacher educators is situated at the beginning of the first phase of teacher education, which in Germany has a three-phase structure: The first phase is the university study programme, the second phase is preparatory service, and the third phase is in-service training. The university phase is commonly considered to be "more theoretical" while preparatory service in schools and seminars and further in-service training are thought to be "more practical." Traditionally, the German educational system insists upon an academic education of teachers, particularly for teachers of secondary schools.[6] Prospective secondary school teachers typically study two subjects at university and complete mostly the same courses as regular Bachelor students of the respective subject. Additionally, they have to take courses in educational sciences and subject matter didactics (e.g., mathematics education). These courses are also taught at university, stressing theory and critical reflection as opposed to being a mere how-to guide to methods and practices.

In this context arises a specific phenomenon with relevance to teacher education which is broadly discussed in German mathematics education research called the double discontinuity. The term *double discontinuity* denotes a situation where prospective teachers perceive a disconnectedness between the discourses of university mathematics and school mathematics, which they encounter on their way from school to university and back to school (see Winsløw, 2017; Hefendehl-Hebeker, 2013). The phenomenon is generally regarded as a problem of the educational system, as secondary school teachers who cannot make sense of the university discourse in a school setting are assumed to be less professional and less capable than those who can draw connections between school and university discourses. Its handling, however, is often located in the sphere of responsibility of university teaching. What Winsløw (2017) calls "*compartmentalisation of teacher education*" (p. 79) adds to this general impression of disconnectedness: Many German universities' teacher education curricula are organised in a way that promotes disconnectedness between the different subjects taught at university (i.e., subject 1, didactics of subject 1, subject 2, didactics of subject 2, educational sciences). This organisational separation mirrors differences between subjects that exist on the level of disciplinary cultures. In the following, we will subsume the *double discontinuity* and the *compartmentalisation of teacher education* under the term *phenomena of disconnectedness (of teacher education).*

14.2.1. The Course and the Concern for Reflective Agency. This general context, as expressed above, is simultaneously the locus and the target of the teaching project on which we will now report: the creation and further development of a new mathematics education course for first year students.[7]

[6] In German terms, our students are prospective teachers for the following school types: "Gymnasium," "Gesamtschule," "Berufsschule."

[7] In Germany, the federal states regulate the general structure of teacher training programs, which includes the distribution of credits among disciplinary, didactical and educational teaching units as well as the main content-related objectives. In Lower Saxony (the federal state Hannover belongs to), the regulations are specified in an Ordinance on Master's degrees for teaching professions

The course "Introduction to Mathematics Education"[8] was established in 2015 within an ongoing local reform project called *Leibniz-Prinzip*, which aims to improve teacher education at Hannover University by promoting reflective agency as the overarching educational goal of teacher education (see Dannemann et al., 2019). In view of the above-mentioned phenomena of disconnectedness, a major concern of our course development was and is to create connections between the mathematics taught in the first semester of university studies, the mathematics typically taught in school, and mathematics education theory and concepts. In the context of our research and development activities in the Leibniz-Prinzip project (see Khellaf et al., 2021), we formulated the following course goal, which represents our interpretation of the concept of reflective agency (see also Ruge et al., 2019), and which serves as guideline for course development and for the design of activities:

> In the first phase of teacher education, which takes place at university, explicit engagement with different discourses and views that are commonly present in institutions relevant to the teaching profession and with their justification strategies shall (be promoted and) lead to an enrichment of available perspectives on questions relevant to the teaching profession, foster reflection in students and ultimately enlarge their repertoire of possible responses to profession-specific situations. In addition to cognitive aspects the development of learning environments shall take into account affective-motivational aspects as well as the specific nature of scientific experience (Bachelard, 2002).[9]
> (Khellaf et al., 2021, translation by author)

The goal was formulated to be applicable to any teacher education subject – therefore it does not specify relevant discourses related to the respective subject-matter. In the case of our mathematics education course, relevant discourses are those of school mathematics, university mathematics, mathematics education (research), educational sciences and possibly other discourses present in society, which may involve for example common beliefs and everyday knowledge. This goal was furthermore created by and for teacher-researchers and phrased in a very general manner. It refers only very abstractly to student activities (e.g., "engagement") and doesn't yet define any concrete tasks. Therefore, it leaves a lot of leeway for the design of concrete tasks that aim to address students' prior knowledge and to foster students' development of interests and reflectivity. In order to facilitate task design based on this very general course goal, we also developed two principles of task design (Ruge et al., 2019) which suggest concrete ways of realising the course rationale by describing actions that should be promoted by inquiry tasks:

(Nds.MasterVO-LehrM; `www.schure.de/20411/mastervo-lehr.htm`) that includes several annexes. These documents are based on agreements on teacher training made by the Standing Conference of the Ministers of Education and Cultural Affairs which all German federal states partake in (`www.kmk.org`). Innovative teaching interventions in teacher education that affect the compulsory programme in degree courses must stay within the boundaries of current regulations.

[8]The course consists of a weekly 45 minutes lecture plus 45 minutes of exercise class over the course of one year; its completion awards 4 ECTS. In 2020, the course has been completely digitalised and its structure became more flexible. In the first semester students are introduced to basics of didactic theory and practice text comprehension and academic communication. In the second semester the focus lies more on the comprehension of didactical questions and problems pertaining to specific pieces of mathematics and students engage in mathematical communication and the development of learning material.

[9]Bachelard makes a distinction between everyday life experience and scientific experience. Our interpretation of the concept builds on that distinction and acknowledges a difference between common knowledge and academic knowledge, as well as the cognitive and affective-motivational dimensions in relating common to academic knowledge.

(1) First design principle: understanding and comparing different perspectives and pieces of knowledge. Through the familiarisation with academic knowledge and theories and their comparison with everyday knowledge, students can encounter new perspectives and ways of thinking. In the process, cognitive conflicts can arise, which can motivate further investigations, and the direction of an ongoing investigation might need to be changed as the goal of the investigation is reformulated in accordance with the new insights. Tasks that promote the investigation and comparison of academic views have to be sufficiently rich and leave enough room for students to creatively engage with relevant perspectives.
(2) Second design principle: questioning one's own perspectives and knowledge. Typical problems and tasks are often strongly connected with typical ways of solving them, to the point where it becomes difficult to even imagine alternative possibilities and ways of acting. Actively imagining alternative scenarios with different possibilities can therefore cast light on current societal restrictions that may promote certain traditional approaches and ways of thinking. Such an activity can furthermore result in insights that motivate further investigation into societal restrictions. Tasks can promote such questioning of traditional views and habits by bearing strong resemblance to a typical scenario but then giving some incentive to reformulate the problem situation in different terms (than the usual ones).

In summary, the two design principles presented above aim to foster inquiry into different bodies of knowledge and their connections to each other. They inspired the creation of tasks whose solution requires switching between bodies of knowledge present in different but related discourses. One such inquiry task, that we use in our course, is "the graph sketching task," which aims to realise the second design principle.

14.2.2. The Sample Task: Graph Sketching. The task is introduced by a fictional school scenario (two pupils discussing an idea), in which mathematical questions are raised:

> An upper high school class reviews the topic of inflection point. One pupil draws on his desk neighbour's sheet Graph 1 below and comments: "Yo, I always wondered: If a function looks like this, does it have inflection points on the entire straight segment?"
>
> The desk neighbour, visibly amused, adds Graph 2 below and replies: "Look! Can't you do the same with a parabola? If you flatten it on the bottom, like this, wouldn't you also have lots of extrema? Infinitely many even!"

This introduction is followed by two graph sketches (see Figures 14.1 and 14.2), which in the fictional scenario were drawn by the two pupils:

(1) How many inflection points does Graph 1 in Figure 14.1 have?
(2) How many extrema does Graph 2 in Figure 14.2 have?

FIGURE 14.1. Graph 1.

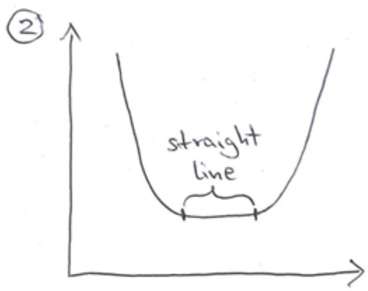

FIGURE 14.2. Graph 2.

The task given to our students consisted of a mathematical task (a) of giving a mathematically correct answer to questions (1) and (2) and a sufficient mathematical justification, and a teaching task (b) of proposing an adequate teacher response to these questions in a secondary school setting and specifying what didactical goals could be pursued in a discussion of these questions. We will limit our further discussion to the mathematical task (a).

The material[10] students are provided with in order to solve the task is a chapter on graph sketching from a German mathematics textbook for upper secondary school (Freudigmann et al., 2012, pp. 38–67). Although they are told that the schoolbook excerpt contains all necessary information to answer the mathematical questions, they are allowed to consult other sources if they like, such as lecture notes or other textbooks.

The textbook chapter we provide contains various types of information, such as exercises and examples, but students are expected to focus on the definitions and theorems of the chapter. Among the theorems are four which specify algorithms for finding extrema and inflection points on differentiable functions with specific properties (e.g., Figure 14.3: sufficient condition for the identification of extrema). These algorithms represent standard techniques to solve schoolbook exercises that ask to find these points of interest on functions that are typically given in algebraic form. The answers to questions 1) and 2), however, cannot be found through the application of these theorems: The theorems are formulated as unidirectional conditional statements "if A then B," where B postulates the existence of an extremum or inflection point; but in the cases of graph sketches 1 and 2, the sufficient condition A does not hold. The questions can instead be answered by looking at the definitions of extremum and inflection point: There are no inflection points on Graph 1 but infinitely many extrema on Graph 2.

Theorem: Second sufficient condition for the identification of extrema
Let the function f be arbitrarily often differentiable on an interval $I = [a; b]$ and let $x_0 \in (a; b)$.
If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a local **maximum** $f(x_0)$ at x_0.
If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a local **minimum** $f(x_0)$ at x_0.

FIGURE 14.3. Example Theorem from Freudigmann et al. (2012, p. 52, translation by authors).

The presentation of the graph sketching task fulfils two purposes. Firstly, it caters to student demands for more practice-oriented activities in university teacher education: The fictional scenario is perceived as realistic in the sense that it might actually arise in school, and the task can presumably be solved within school mathematics as a schoolbook is the only material provided and declared to be sufficient for solving the task. Secondly, the presentation aims to lay the groundwork for the achievement of design principle 2 (p. 258) by providing sketches of graphs for which the standard solution procedure for this type of problem (i.e., checking necessary and sufficient criteria for extrema and inflections points according to the schoolbook theorems) does not work. The fact that the standard strategy for solving graph sketching tasks turns out to be unsuccessful and the possibly surprising task solution may give an incentive for reflection. Ideas about tangents of differentiable functions in the transition from

[10]All relevant definitions and theorems from the textbook are given in our IO3-materials on the PLATINUM website, `https://platinum.uia.no`.

school to university have been investigated in many ways, see for example Biza and Zachariades (2010) and the literature cited there. These studies focus in particular on the question of the relationship between ideas from geometry, for example in connection with tangents to a circle, and calculus, for example in connection with tangents as limit of secants. The task we have developed has other foci, for example, in that it aims at the relationship between procedural and conceptual knowledge of extreme value determinations, its different institutionalisations in schools and universities and, in particular, at issues of the didactic contract (Brousseau et al., 2020), that is, in this case, the adoption of responsibility for one's own mathematical actions.

14.2.3. Experiences With Student Solutions and Reactions. In arbitrary settings, the failure of standard strategies in itself will not necessarily provide sufficient motivation for reflection, as alternative solutions might be readily available and sufficiently plausible in the sense that they will not appear in any way noteworthy or problematic and will therefore not raise any further questions. In our specific case, however, thinking of the intended solution of checking the definitions proved difficult for our students (low solution rate, even in exam situations) and for many, the answers to mathematical questions 1) and 2) came as a surprise (we were told this in classroom discussions). The classroom experiences we made so far suggest that the graph sketching task can provide motivation for mathematical reflection on the significance and role of definitions in solving mathematical tasks or on the concepts of extremum and inflection point including aspects that are relevant for future teachers.

To give an example: Images of coastal roads and motorcycles are often used in German mathematics schoolbooks to illustrate the concept of inflection point. One imagines a mathematical curve to be a road on a map. While driving along this road, a motorcycle will lean to the left when the driver takes a left turn, and to the right when the road turns right. The point(s) at which the motorcycle is perfectly upright (perpendicular to the road's surface) while changing direction is said to be an inflection point of the curve/road. The fact that Graph 1 (p. 258) has no inflection points even though the motorcycle would be upright everywhere on the straight segment can motivate an investigation into the differences between Graph 1 and common school curves and give rise to discussions about didactic properties of commonly evoked imagery or about the nature of points in mathematics.

Furthermore, discussions about the reasons for the difficulties the students experienced can arise, which might lead up to a discussion of societal restrictions such as different didactic contracts (Brousseau et al., 2020) at school and university or differences between the mathematical discourses at school and university (e.g., emphasis of different mathematical techniques in teaching; strong focus on algorithmic procedures in typical teaching units on graph sketching in school). Such topics are not only relevant for the professional development of prospective mathematics teachers, but important for raising awareness of similarities and differences between the mathematics taught in school and practices of university mathematics.

Student difficulties that have the potential for such discussion and reflection include an initial avoidance of the intended difficulty of the task and subsequent mathematical discovery. Some students, for example, make the mathematical mistake of considering sufficient conditions to be necessary conditions as well (in logical terms, they derive $\neg A \implies \neg B$ from $A \implies B$), concluding from schoolbook theorems such as the one shown in Figure 14.3 that no extrema or inflection points can be found on Graphs 1 and 2. Other students undermine the didactic contract by arguing for an interpretation of the task instruction, that renders it solvable through an application of the schoolbook's theorems: They claim the pupils in the introductory scenario must have made a mistake in assuming that their functions were really straight on the straight-looking sections, because the functions clearly have to be polynomials, and polynomials are never straight on open intervals (in this case the students also ignore the fact that constants are also polynomials). In both these cases of student difficulties (the mathematical mistake and the incorrect interpretation of the task instruction), the "artificially created" applicability of the standard theorems to the graph sketching task can motivate reflections and discussions about mathematics, about didactic contracts and about the differences between mathematical discourses in school and university. We will say more about students' handling of the graph sketching task in Section 14.4.

A last point we want to comment on is our idea that the task presentation (introductory scenario, schoolbook as material, graph sketching as topic) is successful in taking into account student demands for practice-oriented tasks in teacher education as affective-motivational aspects (see course goal in Section 14.2.1): We have met students who deemed the fictional scenario introducing the task plausible enough to become worried about their suitability for the teaching profession after experiencing the unexpected difficulty of the graph sketching task. This is noteworthy to us as our course has, in the past, met with repeated and at times fierce criticism by students who deemed its contents and tasks too theoretical, too far away from actual school practices and therefore irrelevant for prospective teachers ("a waste of time"). We will come back to this criticism in the next section, where we will reflect some of the contradictions and other relevant phenomena, we have encountered in our teaching project.

14.3. Phenomena and Contradictions of the Inquiry Teaching Project: Reflections Against the Background of Concepts Underlying Reflective Agency

The previous section concluded with the observation that students are at times quick to argue that the topic or the proposed activities of an assignment have nothing to do with school practice. Discussions of this point with students have in some instances become quite emotional, as students voiced indignation about having to work on some purportedly pointless task. Student calls for more practice-oriented course content are abundant in student evaluations of our course (though admittedly more so in older ones). From these experiences arises the question of how to reconcile conflicting visions (normative views) held by students and teachers of the desired learning outcomes and of the involved processes and activities in a teaching-learning situation. Some didactic choices, it appears, can lead to emotional reactions of resistance from the side of students although they might appear reasonable from the informed point of view of the teacher.

In the case of the graph sketching task, the graphs do not correspond to graphs of functions that are typically[11] taught at schools. From a didactic point of view, however, whether a mathematical problem might appear in school or not does not determine its relevance for teacher education. In our course, the graph sketching task illustrates that argumentation in school mathematics differs from that in university mathematics, and it illuminates specific differences between mathematical and teaching practices at school and university. This purpose is in line with our course goal (p. 256–258)

[11]The fact that Graph 1 and 2 (p. 258) are not typical for schoolbooks does not contradict the point made before that the fictional scenario is reasonably plausible. We will again point to the possibility of interpreting Graph 1 as a road and to the fact that the topic of extrema of constant functions arises in our schoolbook, albeit as a marginal mention.

which can in turn be further justified on the basis of psychological and didactical theories (Ruge et al., 2019). But how can these didactical considerations be relayed to students? As the reference to the work of Bachelard (2002) in our course goal indicates, we believe that it would be very difficult or even impossible for our first-year students to understand our course rationale. They have little knowledge of the specificities and practices of the teaching profession (they derive most of their impressions from what they have seen as pupils in their own school days) and are not yet familiar with the academic discourses underlying above didactic deliberations. This is why they will sometimes make demands that, from an informed perspective, seem counterproductive to successful[12] teacher education. But can or should students be ignored in didactical decision-making?

Despite the didactical considerations that support the graph sketching task, students' concerns cannot be ignored. Firstly, successful teaching depends on the acceptance of the teaching-learning-scenario by the learners (Rihm, 2006). In other words, didactical insights suggest that affective-motivational aspects should be considered in the creation of any teaching-learning-scenario. Secondly, there is institutional pressure to accommodate for student wishes to some extent: Student evaluations, for example, feed student opinions about our course back to the student council, who can then cut funds for our teaching staff if demands are not met. If evaluations are very bad, the faculty also has a response protocol with the intention to bring the evaluations up to an "acceptable" level. For the teacher this means that s/he has to manoeuvre between obtaining students' cooperation by catering to their expectations and articulated needs and insisting on certain didactical choices that appear necessary in order to be able to reach certain insights in the learning scenario.[13] In a broader view, a teacher in such a situation is dealing with an instance of the problem of theory and practice. This multi-faceted phenomenon has been broadly discussed in German educational sciences (e.g., Terhart, 2000) and is a recurring topic in discourses central to teacher education. The problem has to do with the way teacher education has been institutionalised within the German education system; it is connected but not identical with the phenomena of disconnectedness already mentioned in Section 14.2. The problem of theory and practice can be characterised as follows:

- on the level of didactic theory, certain philosophical frameworks insinuate a fundamental difference between theory and practice;
- on the level of implementation, a split between theory and practice is observable in typical institutional implementations of German teacher education in the following forms:
  - division of teacher education between two institutions (university and school/seminar) which are separated in terms of location and (institutional) structure, and
  - official division of responsibility for "academic/scientific education" (at university) and "practical education" (in schools / at seminars) between these institutions.

[12]Successful from the point of view of German institutions concerning teacher education and/or from our point of view as teachers.

[13]The fact that students do not necessarily react positively to teaching interventions such as proposed by IBME, or even reject them as demanding or even chaotic, is a thoroughly understandable reaction and must therefore be taken into consideration when designing interventions. Irrespective of this, student protest can lead to the termination of IBME-oriented teaching activities by the university administration, as described in the case study by our colleagues from Agder (see Chapter 11).

The phenomenon is typically associated with the disconnectedness between the (more "theoretical") discourses present at (German) universities and the (more "practice-oriented") discourses among teaching staff in schools (Schrittesser & Hofer, 2012, see Section 14.2: double discontinuity). A widely discussed problem that arises before this background and that affects practically all teacher education subjects is the above-described lack of tolerance for the theoretical nature of university teaching on the part of students, and motivational problems in connection with this (see for example Wenzl et al. (2018), for a commentary on this phenomenon). An orientation in favour of practice and against theory is also discussed in research in which dominant didactic currents or movements, such as competence orientation or an application orientation (in the sense of the modelling cycle), are not only problematised in an exemplifying manner with regard to their limits, but are themselves identified as expressions of insufficiently reflected institutional-societal contexts. Certain institutional-societal phenomena are addressed, for example, by Brousseau's notion of metadidactic shift (Brousseau, 2002, p. 261). Against this background, Gascón (2011) formulated the following critique of competence approaches: The shift leads to suggestions of teaching practices, in which the intention is to teach students problem solving, by trying to teach them how to learn problem solving by themselves (p. 36). Concerning the theoretical concept of 'competence' as it is proposed in educational sciences, it can be argued that curricular proposals in terms of competences (general, functional, technical, interpersonal, intellectual, etc.) actually turn the pedagogic problem into the solution: Making students acquire competencies is equated to teaching them competencies.

The other phenomenon mentioned before, of students undermining the didactic contract and avoiding dealing with the graph sketching task, is illustrative of another related phenomenon that we would like to draw attention to, namely the "problem" of tasks being used in a way not actually intended by teachers, especially those tasks that can be considered open and grant students a certain degree of freedom. This phenomenon is indicative of a fundamental principle in teaching-learning contexts: No teaching can force learning. Ultimately a triviality, this insight is recognised in principle by all learning and teaching theories. Conceptually, however, it is often relativised to some extent, especially in the way of not recognising the subjectivity of the learner and her/his agency. This happens particularly often in teaching and learning settings at typical educational institutions that are embedded in administrative structures, where the possibility of effectively planning, steering and controlling learning processes in order to move them in the intended direction is implicitly insinuated. In common traditional teaching-learning settings, learning efforts are often feigned or there is a reduction of deep learning to rote learning, mutually recognised by teachers and learners. Teaching without learning can naturally also occur in inquiry situations.

In the area of educational theory on inquiry teaching, in fact, we can find views that imply that the success of inquiry activities can and should be ensured by selecting tasks and managing classroom discussions particularly skilfully. What skilful means can be determined experimentally. This tendency in traditional approaches to inquiry teaching has led Holzkamp, among others, to formulate that such views are ultimately just particularly sophisticated attempts at manipulation with the aim of getting the learners to where the teacher, for whatever reason, wants them to be. They therefore merely represent a special variant of the otherwise widespread *teaching-learning short-circuit* (see Holzkamp, 1995). In response to such fundamental criticism of inquiry-approaches, Huck (2013) argues for the idea of inquiry-based teaching and learning by highlighting its conceptual focus on the importance of understanding subject-specific connections and the relevance of the learner's own thoughts and use of their "practical" insight in learning a new topic. He does recognise the importance of guidance by a teacher but shifts the focus of attention to the fact that learning always includes the participation and engagement of the learner in the process offered by the teaching-learning activity. Letting learners make their own experiences and include their own insights in the teaching-learning activity stands in conflict with a one-way conceptualisation of teaching-learning.

This view is mirrored in our course goal (p. 256–258), which is centrally based on the subject-scientific-approach and its theory of learning by Holzkamp (1985, 1995): In our understanding of reflective agency we tried to conceptualise the promotion of actions or ways of thinking in our teaching as the creation of a space in which our students can enlarge their space of *action possibilities*.[14] By doing this, we strengthen the self-determination and agency of our students on a conceptual level and hope to consequently also achieve this in the realisation of teaching-learning scenarios. In this sense, we consider inquiry learning as an offer to "optimise" teaching-learning-scenarios in this direction, but it can no more force learning than other teaching concepts. In particular, there is no trick that guarantees that students take certain learning steps.

The issues just reflected on are also addressed in another strand of theory underlying our conceptualisation of reflective agency: *structural theory*. As example we can take the concept *autonomy-antinomy* (Helsper, 1996): Every teaching-learning relationship requires the recognition of a certain autonomy of the learner, since learning requires its own mental processes independent of the teacher. On the other hand, teachers in institutional teaching-learning relationships are required to ensure certain learning outcomes (see footnote 7). This antinomy is regarded, in structural theory, as constitutive of (institutionalised) teaching-learning relationships that cannot be bypassed.

In view of these theoretical reflections, we conclude regarding the previously described observations that we cannot avoid such student reactions, but rather have to understand them as possible and somewhat adequate expressions of the configuration addressed in the graph sketching task. In this sense, the avoidance of the task or the undermining of the didactic contract by the students should not be seen as a deficit but as a specific expression of agency that can be the starting point for reflections.

In the following, we want to relate our observations and their interpretations to our own development process (see Chapter 2 and 10). Regarding our professionalisation as teachers, we would conclude that it is too simplistic to value a teacher within IBME just based on the degree to which they (can) ensure that learners develop a practice-relevant, coherent and deep conceptual understanding of mathematics. Actually, if we would assess our own development process from this simplistic perspective, we would conclude that our efforts have been rather unsuccessful and that we are far from having achieved the goal of becoming successful IBME-teachers. But we would still claim a professional growth in PLATINUM: To us, the central point of all observations shared above is that the contradictions inherent in them are not resolvable just by us taking up an inquiry stance or optimising our inquiry activities and teaching practices (further and further), rather we have analysed and elaborated the contradictions for ourselves in order to be able to work within and upon these. One important aspect of professional growth within a complex setting such as ours is to come to terms with and accept

[14]The term 'action possibility' refers to an analytical category. The analytical categories of the subject-scientific approach "conceptualise the mediation between the vital necessities of sustaining the societal system as a whole and these necessities on the subjective level of the discrete individuals" (Holzkamp, 2013, p. 20).

the fact that many contradictions cannot be resolved, and partly lie far beyond the scope of teaching anyway. Accordingly, professional growth cannot consist in trying to resolve all contradictions, but in finding ways to come to terms with them—e.g., by locating, classifying and interpreting phenomena relevant to one's teaching practices. We have dealt with the contradictions by generating more information about them, by finding out what we can expect and by trying to move in this tension-filled field as smoothly as possible. We did not strive for a definite resolution, but are content that we can grapple with the contradictions in a more reassuring way.

So far, we have outlined the contradictory nature of our (institutional and societal) context in which our teaching practices are situated. In the next section, we discuss extensions and concretisations of PLATINUM concepts that allow us to integrate implications from the observations of this section with regard to our professional growth as researchers within PLATINUM.

14.4. Reflecting on Issues Regarding IBME, the Three-Layer Model and CoIs, and How They Underlie PLATINUM

Up to this point, we described phenomena as well as our interpretations of these with a focus on our goal of developing and establishing inquiry-based activities in our teaching. Now, we will reflect about our observations against the ideas formulated within the teaching-learning conceptualisation of IBME (14.4.1), the three-layer-model, and the conceptualisation of CoIs in PLATINUM (14.4.2). From our position as researchers within the global PLATINUM-group, we will also reflect on our experiences as teachers and researchers within the PLATINUM project, and we will argue for the need of a conceptual concretisations of the three-layer-model that accounts for societal and profession-related aspects, among others. These deliberations will lead us to a reformulation of the potentials the CoI-concept entails (14.4.3). To summarise and generalise: In this chapter we propose a deliberate approach to the constraints in the conceptualisation of IBME and indicate a restriction concerning its goals and an expansion of the concept of critical alignment.

14.4.1. IBME as Counter-Horizon for Thinking About Teaching and Learning. The conception of IBME (see Part 2 and 3 of the book) includes many statements about the kinds of learning activities that shall be elicited by teaching. Theoretical conceptualisations in which teaching practices are defined through their learning outcomes have been criticised as "short-circuit of the conceptualisation of teaching and learning" (see Holzkamp, 1995). They bring to attention only those kinds of learning activities and practices that are in alignment with pre-defined "learning goals" but leave little space for *critical* alignment within this narrow interpretative horizon of teaching and learning practices. In view of this, we ask: Is the notion of inquiry-based teaching and learning yet another expression of such a one-way conceptualisation in which teaching leads to learning? Or can we conceptualise IBME in a way that goes beyond such simplifications in its description of the relations between teaching and learning?

We argue that it is possible to take the idea of IBME as alternative outlook: IBME can offer a counter-horizon that opposes and challenges one-way conceptualisations of teaching-learning situations, as long as it is not regarded as a concretely achievable goal.[15] Thus, understanding IBME as a counter-horizon demands restraint in the

[15]To us, IBME is not a list of supposedly favourable "learning outcomes." To formulate goals of IBME in reference to a list of learning outcomes would be a step backwards towards a short-circuit of the conceptualisation of teaching and learning. So-called "learning goals" that are predefined by

setting of goals (in deviation from what is typically associated with IBME). In the following, we will explain in more detail in what way the notion of IBME holds the potential to oppose typical understandings of teaching and learning—we will rearticulate the potential of IBME.

In our institutional settings the typical or traditional horizon for interpreting phenomena in teaching-learning-situations are framed by a language of thinking in accomplished learning outcomes and "customer-satisfaction," which mingle with the ideal of fostering critical thinking. This horizon is rife with contradictions concerning the student as well as the teacher position: Students shall, on the one hand, align (in an uncritical manner) with pre-defined learning-outcomes and, on the other hand, be critical thinkers. Teachers are, on the one hand, considered to be autonomous in their teaching practices and committed to the subject-matter while, on the other hand, they are judged with respect to "customer-satisfaction" (which manifests in the questions of institutional evaluations and surfaces in students' wishes as well). In such a paradoxical framework, we are unable to express the relation between teaching and learning, between teachers and learners adequately. The counter-horizon IBME challenges aforementioned takes by offering a frame for inquiry into research and teaching practices that provides concepts to envision an extension of our possibilities of acting within and upon this paradoxical framing. Instead of limiting the understanding of the object to be studied—in our context, mathematics and mathematics education—to fixed learning outcomes and instead of subordinating teaching to "customer-satisfaction"-criteria, IBME takes into account the agency of both teachers and learners equally and articulates their ability and responsibility to engage in a critical manner with the subject-matter to be studied. Instead of restricting our understanding of teaching-learning-relations by pressing teachers and learners into predefined roles that limit their ability to engage with the object to be studied, the conceptualisation of this relation as a Community of Inquiry, in our interpretation, breaks with these narrow conceptions, in that it allows to ask for the learning opportunities an inquiry activity creates and the potentials we can create within and in trying to move beyond current restrictions.

To illustrate this point, we recall the above-mentioned phenomenon of students misinterpreting the instructions of the graph sketching task, thereby allowing themselves to apply standard criteria (see Section 14.2). The phenomenon sparked discussions in our LUH-group on how to deal with this situation. First, we need to acknowledge that the phenomenon took place in an inherently contradictory teaching-learning situation: On the one hand, the graph sketching task is designed to induce reflection and is open to further reflection. On the other hand, the task is embedded in an institutional setting in which, for the students, solving it is a matter of fulfilling external requirements, and in which, for us as teachers, it is tied to expectations that we plan our teaching in order to achieve predefined outcomes. From the standpoint of predefined learning outcomes those students clearly failed and their behaviour can be judged undesirable. Alternatively, the students' activities can (and should) be seen as a strategy to maintain or expand their agency: The "undesired" reinterpretation of assignments, for example, maintains agency by allowing students to deal with the task. The way they do it, of course, undermines the institutionalised didactic contract. Since we, in the position as representatives of the institution, cannot simply tolerate such reinterpretations, a conflict arises between us and the students. We can,

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a teacher or the curriculum are actually teaching goals. A learner can, of course, formulate concrete goals for her or his learning process, and these can be closely connected to teaching goals. But if we strive for open inquiry, predefining learning outcomes actually run counter to this very ambition.

however, take up this conflict as a starting point for a joint reflection together with students on the inherently contradictory teaching-learning situation. In such a joint reflection, we would first of all acknowledge the students' compliance in delivering a solution. Secondly, we would recognise the fixating power of the school discourse, and the limitations it imposes: The simple solution of looking at the definition appears to be absent from a lot of students' space of possibilities. In this example case of the graph sketching task, our understanding of IBME in combination with insights into the didactic contract and knowledge of the various institutional positions enabled us to point to the potential for reflection on different mathematical discourses, on the nature of our teaching-learning-setting and beyond.

It should be noted that we understand these briefly described horizons and our following reflections not just as individual viewpoints that need to be changed or that shall be fostered. These horizons manifest in structural arrangements, narratives about teaching and learning and theoretical concepts. Therefore, the sustainable further development of teaching practices is not a simple matter of personal adaptation to, say, an inquiry stance in teaching practices or of becoming 'skilful' in selecting tasks and managing classroom discussions. It cannot be obtained, in fact, without altering structural arrangements as well as the conceptual understanding of teaching-learning phenomena present in teaching-learning-settings. Consequently, one goal of developmental research in this area can be the articulation of current restrictions in the form of structural arrangements or taken-for-granted perspectives within current theoretical conceptualisations. Developmental research should not be reduced to optimisation-concerns that limit its potential through the self-subordination to given restrictions.

14.4.2. Reflections on the Relationship Between Professional Growth and Developmental Research. The way the three-layer-model expresses the interrelation between theory and teaching development (see Section 14.1) differs from traditional models of developmental research that typically envision it as chronological four-step process consisting of research, followed by development, then design and, as last step, implementation (see also, Bauersfeld, 2000) arguing against this R-D-D-I model) and it goes beyond a dialogue between mathematics education researcher (on the one hand) and teacher (on the other hand) (Jaworski, 2004), in which researchers are perceived to be responsible for theory development and teachers are charged with the development of their professional (teaching) practice; the development of theory and professional practice, instead, constitutes a shared task. In our context in particular, theory development and development of teaching practice are strongly connected to each other. We see both aspects as being part of our professional growth with/in our group. How our practice as teachers motivates and guides our engagement in theory development was illustrated above, when we relayed our experiences with the graph sketching task. Theory development, for its part, can be seen as being part of our professional growth, because theories hold the potential of broadening our horizon of thinking about teaching-learning-relations. We would like to point out that we understand our involvement with the background theories of the PLATINUM project as critical alignment. In consequence, the critique we offer should not be understood as rejection, but as a critical questioning for developing the theory further in accordance with our experiences in our local context.

If we take a closer look at the three-layer model, the (further) development of theory in mathematics education (outer layer) and the further development of teaching practices (middle layer) are split and also separated by the objectives of inquiry (see Table 14.1, p. 254). We are going to have a closer look at the interrelatedness of these two layers. In order to articulate the potential we see in further elaborating the theory behind the three-layer model and in strengthening the links between the middle and the outer layer, we need to make a theoretical excursion to explain our understanding of professional growth.

Inspired by the subject-scientific theory of learning (Holzkamp, 1995, 2013; Dreier, 1999; Ludwig, 2003) we conceive professional growth as extending one's own space of action possibilities in teaching-learning relations with/in a Community of Practice. This situates our further development within a Community of Inquiry (Jaworski, 2004). Within the subject-scientific approach, learners, teachers and researchers are perceived as "*producers of the life conditions to which they are simultaneously subject*" (Holzkamp, 2013, p. 20). The approach stresses the significance of these life conditions (specifically, teaching-learning conditions as well as the conditions of doing research in mathematics education) and it underlines the possibilities of the subject—learner, teacher, and researcher—to influence these life conditions in alliance with others. The analytical category *action possibilities* [Handlungsmöglichkeiten] refers to possibilities and hindrances to act in and on specific conditions from the standpoint of the subject.[16] Central to thinking in terms of action possibilities is the *twofold possibility* [doppelte Möglichkeit] to either reproduce restrictive conditions or to realise the possibility (however small[17]) of extending established practices and altering structural and socio-political conditions. This distinction is analytical and not to be mistaken for an either-or-relation. The introduced concepts can guide reflection processes about contradictory situations or persisting conflicts with regard to their structural and socio-political conditions. However, contradictory structural constellations are not generally assumed to be removable or resolvable. Structural and socio-political conditions are integrated into subjective reasoning in the form of societal-mediated meanings that constitute a person's space of action possibilities. The societal-mediated meanings that are grounded in these conditions constitute a space of available action possibilities. The space of action possibilities available to a subject is not fixed but can be extended. In consequence, we conceptualise professional growth as an extension of the space of action possibilities that is available to a professional.

The subject-scientific theory of learning emphasises the social dimension of this extension process. In alliance with others, it is possible to seize more opportunities for actions and participate in changing conditions that are constraining one's envisioned practices. Dreier (1999) relates this to community processes:

> the fundamental human duality between acting within the existing limits of social practice and extending its scope of possibilities is grounded in a similar duality of modes of participation [in a community], i.e. of participation in the reproduction of the current state of affairs or of contributing to change it so that participants may extend their degree of disposal over the social practice. (p. 6)

We regard the notion of CoI as a sociocultural construct (Goodchild, 2014) which, as a framework, accounts for such activities that tackle shared socio-political conditions.[18] Jaworski (2004, 2006) points out the risk that community processes could

[16]Action possibilities include, by definition, both opportunities and constraints.

[17]This follows the basic assumption that in antagonistic class conditions, the attempt to gain more control over conditions is always accompanied by the risk of getting in conflict with the agents of power and provoking restrictions.

[18]We share the conviction that research in mathematics education needs to integrate social theory and cannot disregard broader societal conditions in the interpretation of phenomena that can be found in teaching-learning. Otherwise it would conceal the socio-political dimension of mathematics education by reducing didactics to the development and implementation of teaching strategies. To adequately capture these phenomena, it is important for us to be equipped with a theory that provides

hinder further development with/in a community: An unquestioned alignment with and participation in the practices of the local community could lead to a reproduction of undesirable practices. She therefore emphasizes the importance of a critical alignment with teaching-learning practices with/in a community. For us, such an inquiry stance towards one's own practices includes inquiry into learning, teaching and research. Our reflections and debates in the LUH-group can, in this context, be understood as supporting an ongoing *(self-)understanding process* that takes place between the community members and the socio-political conditions in which the professional work is situated. This includes the "reflection on social requirements and conditions in an attempt to (re)establish self-understanding in individual situations of action and to be able to act in a competent [/professional] manner" (Ludwig, 2003, p. 1, translation by author). Therefore, "seeking (self-)understanding" denotes the attempt to gain knowledge about and to trace one's own personal and structural entanglement in contradictory situations, which can consist in the (unwitting) participation in community practices which run counter to one's own interests and desired practices. By striving for (self-)understanding, we attempt to gain more disposal over our research and teaching practices. Rihm (2006) points out that in our routinised daily work, we often 'interpret' situations within the horizon of the typical space of action possibilities of our daily practices. This means we unquestioningly accept quite a number of aspects of typical ways of working in our community. 'To understand,'[19] on the other hand, means to gain knowledge about and to trace one's own structural entanglements in contradictory situations (and possibly to reconstruct participation in community practices that are contrary to one's own interests). Seeking understanding, therefore, means to widen one's own view and to transcend the horizon of our everyday entanglement. By calling into question one's own reasoning, understanding goes beyond a reflection of current conditions as parameters that set boundaries for the exploration of (the range of available) options. It entails questioning one's own interpretations of phenomena related to teaching and learning.

Seeking understanding to gain more disposal over our researching and teaching practices is what we understand as critical alignment within our group. This kind of (self-)understanding goes beyond a merely introspective and individual way of progressing (Rihm, 2006): The intertwining of our perspectives allows us to take a meta-standpoint that makes it possible to recognise the interrelation of different practices prevalent in society and the group (different research, teaching, and learning practices). This reflexive distance does not only allow us to identify supporting and obstructive conditions and to question our own interpretations, but also to recognise potentials of altering conditions (Häcker & Rihm, 2005, p. 375).

Within the LUH-group, we cooperatively try to widen our viewpoints and do not distinguish between researcher and teacher as fixed positions. Regarding the middle and outer layer of the three-layer-model, viewpoints on how teaching and learning are related to each other are important issues for developmental research and teaching practice alike. Our reflections within the LUH-group are of importance for our professional growth as teachers as well as researchers and cannot be assigned to one specific layer.

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a language for characterising human actions within social conditions. For this purpose we rely on the subject-scientific approach.

[19] 'Interpreting' is then not opposed to 'understanding.' Rather, 'understanding' simultaneously suspends 'interpreting' in itself and transcends it (Holzkamp, 1985, p. 395).

14.4.3. Reformulation of Potential of CoI. To us, an inquiry stance towards teaching and learning means thinking in alternatives and potentials.[20] In consequence, we do not take current conditions and approaches to teaching-learning mathematics (education) for granted, but scrutinise them for obstructive elements and possibilities to think beyond the narrow horizon of current practices. This objective can be related to an emancipatory objective of academic work that is also of key importance for building up a professional knowledge base for teaching (Langemeyer, 2020). In teaching-learning relations, we often act in a restricted manner, in a modality of alignment with or subjection to given obstructive structures.[21] But education can also be thought of as a cooperative activity directed towards extending each participant's space of action possibilities, which also includes extending each participant's control over restrictive teaching-learning conditions. It entails the possibility of overcoming obstructions to teaching and learning in alliance with others. Research can provide concepts for reflection, concepts that promote the process of seeking self-understanding for the professional task of teaching. Linking research activity in mathematics education and the teaching and learning of mathematics (education) with each other within a Community of Inquiry has the potential of developing and building on theory that integrates several standpoints of the teaching-learning relations. These standpoints are anchor points for the reflective task of decentring from one´s own viewpoint and jointly developing a meta-standpoint. The process of decentring can be described as a combination of zooming out and zooming in (Busch-Jensen & Schraube, 2019).

The strength of a conceptualisation that locates CoI on all layers of the three-layer-model (inner, middle and outer layer) lies in its ability to draw attention to the possibility of engagement in terms of a critical alignment, that calls for inquiry into the subject-matter as well as inquiry into conditions that obstruct teaching-learning processes and, thus, also restrict inquiry into the subject-matter.

14.5. Concluding Remark

The presented reflections can be understood as our critical alignment with PLATINUM concepts of IBME and CoI, their potentials and limitations. Our reflection resulted in a further development of theory (expansion and differentiation) that is based on our experiences as teachers and researchers in the LUH-group, our participation as researchers in the PLATINUM project, and our theoretical stance towards mathematics education. We have presented our reflections in this contribution from the perspectives of two different positions: teacher and researcher. These two positions are of course not independent of each other, since that would in our context imply a "splitting of our identity." Rather, the two positions are dialectically connected. Unfolding their nonlinear relationship in a linear text was a great challenge for us and led us to make an analytical distinction between issues that we considered to be more of relevance for the position of a teacher in a developmental research project and issues that we thought to be more relevant for the position of a researcher entangled in the practices s/he is inquiring into. The present contribution documents what remains an ongoing discussion of how to grasp and categorise these issues.

We regard our puzzlement concerning theoretical approaches and making sense of them as relevant personal experiences within PLATINUM. To us, working with and developing theory further is not only a cognitive task, but also involves affective-motivational aspects. We acknowledge both the cognitive and affective-motivational

[20] Even if these are not yet realisable under the given conditions.

[21] Preservation of the status quo, or safekeeping one's own position at the cost of the (re-)production of restrictive conditions.

facets and their relatedness in our practice of theory development. In alliance with each other, we take our personal experiences and sensitivities in teaching practice and theoretical work as a starting point for further development. This entails supporting each other if one struggles with opposition to her or his teaching practices or theoretical stance and discussing and classifying doubts. The emotional support of the group is essential, but to work in alliance with each other, to us, necessarily involves a deliberate decentring from one's own viewpoint.

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CHAPTER 15

Two Decades of Inquiry-Based Developmental Activity in University Mathematics

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15.1. Introduction

The authors of this chapter work at a Mathematics Education Centre (MEC) at Loughborough University (LU) in the UK. We teach mathematics and mathematics education and we do research in mathematics education. This case study discusses research and development activities in which the MEC has been engaged for over 15 years including inquiry-based activity which is now related to the PLATINUM project. Our learning from these has been important for our individual development in both research and teaching.

Our teaching activities have involved collaborations between colleagues in the MEC, the Department of Mathematical Sciences (DMS) and the Foundation Studies Programme (FSP). During these 15 years, we have worked within a university culture of mathematics teaching and learning influenced by both the national milieu and the local policies of our university itself. In particular, we can point to

(1) the impact of a culture of university education that permeates practices in the UK. For example, issues related to school education and the preparation of students for university study;
(2) three government-initiated Research Excellence Frameworks[1] (REFs) over the 15 year period assessing research across all departments of all universities. This impacts the amount of Government support flowing into the university;
(3) at LU, the reorganisation from Faculties to Schools; the ways of organising lectures and tutorials; the domination of research over teaching; recent moves to make teaching development more important (e.g. the Teaching Excellence Framework, or TEF[2]).

Mathematics teaching at university level in the UK and beyond has followed a traditional path for many years with the main elements comprising large cohort lectures together with some forms of tutoring (Alsina, 2001; Pritchard, 2010). LU has largely followed this pattern. Towards the end of the millennium, university mathematics teachers became aware that students entering university seemed no longer well qualified for the content and pedagogy of university mathematics (Hawkes & Savage, 2000). This was largely attributed to changes to the curriculum in schools, where, for example, mathematical proof was not required. This raised issues about the kinds of (extra) provision that could be needed. For example, many universities introduced some form of 'bridging course:' at LU, a one-year Foundation Studies Programme was

[1] www.ref.ac.uk
[2] www.gov.uk/government/collections/teaching-excellence-framework

introduced in which students developed their skills and understanding in mathematics and science courses in preparation for entry to a bachelor degree programme. The demands of the REF have resulted in confirmation of a point of view that, in universities, research is of higher importance than teaching. In recognition of this position, the UK government instituted the TEF which also now assesses teaching quality. This has resulted in more focus on the nature and quality of teaching in universities.

15.2. Chapter Structure

15.3. History: we set the scene, positioning the growth of our inquiry-based activity in the context of both national educational and local university initiatives and structures, and influencing political, social and educational perspectives.

15.4. The Teaching Group: a Community of Inquiry focusing on a group of mathematics and mathematics education teachers working together to influence teaching and its development.

15.5. Inquiry-based Tasks in a Foundation Mathematics Course: discussing a developmental research project embracing teacher-researchers, student-partners and post-graduate students in a community of inquiry to develop tasks and teaching units for Foundation Studies students.

15.6. Teaching Engineering Students—a developmental research approach: the development of inquiry-based teaching of engineering students; where successful and where not successful.

15.7. Discussion: focusing on our learning as set out in the sections above and its relations to activity in the PLATINUM project.

15.3. History

The Mathematics Education Centre (MEC) at Loughborough University was created in 2002, within the School of Mathematics, and comprised a drop-in centre for mathematics support (The Mathematics Learning Support Centre, MLSC) plus responsibility for service teaching (including science, engineering and economics). In 2007, the MEC became a research centre focused on research into the teaching and learning of mathematics at university level and has diversified more recently to include all levels of mathematics education. The link between the MEC and the DMS is very strong where teaching is concerned (all staff contribute to the teaching of mathematics or statistics) and, in recent years, with introduction of the TEF, more emphasis has been placed on student learning and the development of teaching. However, research in mathematics education is very different from research in mathematics and, for REF purposes, they make a return to different assessment panels. Several initiatives have been undertaken to involve mathematicians with research into developments in learning and teaching. One initiative, the seminar series "How we Teach", was overtly focused on developing teaching (details follow below).

Parallel influences on research and teaching encouraged the MEC to study the development of teaching. Developmental research, often inquiry-based, became one feature of research in the MEC and included studies which pioneered inquiry-based approaches: for example, inquiry into students' use of digital proofs (Alcock & Wilkinson, 2011; Roy et al., 2017), inquiry into the teaching of linear algebra (Jaworski et al., 2009; Thomas, 2012), an innovation in teaching to promote engineering students' more conceptual understanding of mathematics (ESUM—details to follow below). These aspects of the history of the MEC are important as forerunners of inquiry-based research and development in the PLATINUM project: in particular, the three-layer model of inquiry-based practice has its roots in this work together with related research at the University of Agder, Norway. The first author has a long history in inquiry-based developmental research and has influenced the conceptualisation of inquiry in PLATINUM. This was built on developmental research in both the UK and Norway taking place at school level. A key element of inquiry-based learning and teaching at school level was the idea of forming inquiry communities among practitioners, teachers and didacticians. An inquiry community was seen as a group of practitioners who shared inquiry-based approaches to teaching and learning and supported each other in their development (Jaworski, 2008). At university level, a parallel is to form such inquiry groups between mathematicians and mathematics educators. With this in mind, a series of seminars, with the title "How we Teach," was introduced in which one teacher (mathematician or mathematics educator) gave a short talk about their thinking in some aspect of their teaching. The aim was to generate a discussion of teaching amongst colleagues and thus to encourage everyone to learn from the discussion and to develop teaching. The seminars became a regular feature in the MEC (from 2009–2014); they led to warm relationships between those attending and an enhanced awareness of teaching approaches in mathematics. They were pre-cursors of a specially convened "teaching group" to promote developmental inquiry in mathematics teaching and learning and, subsequently to the centrality of "communities of inquiry" in PLATINUM.

In the three sections which follow, we present aspects of our inquiry-based activity which have been important for us and, we believe, important as examples of key processes and theoretical perspectives in the PLATINUM project. In the first, we discuss the Teaching Group, mentioned above. This can be seen as a community of inquiry where we explored or inquired into new approaches to teaching and learning in mathematics. The second is a research project (called Catalyst) in which we worked with former students to design mathematical tasks for their more recent peers. These tasks used digital software and were inquiry based. In the third, we refer to a research project (ESUM—Engineering Students Understanding Mathematics) in which inquiry based tasks and teaching approach were used to improve students mathematical understanding. A reflection follows to address reasons for why these approaches seemed not to be possible when working with another group of engineering students.

15.4. The Teaching Group: A Community of Inquiry

The Teaching Group at LU started its meeting in 2016 and fulfilled the need felt by several colleagues both in the Department of Mathematical Sciences and the Mathematics Education Centre at LU for a forum to meet and discuss teaching mathematics and statistics at university level. This forum was to facilitate meetings and discussion for academics with complementary expertise and teaching experiences so that a Community of Inquiry (CoI—see Chapter 2 for details) could be established. The model of the CoI fitted well the aims of the Teaching Group: participants wanted to share practice and learn from each other and from educational research about the problems and issues they encounter in teaching. One of the contextual reasons why such a forum was initially successful is that, in the UK, training for new lecturers is generally not discipline specific therefore new colleagues joining the department felt that they needed a forum to discuss the teaching of mathematics specifically. Before this Teaching Group took shape, mathematicians and mathematics educators had shared the seminar series called "How We Teach." These were a regular feature in the MEC (from 2009–2014) and brought together educators and mathematicians interested in mathematics learning and teaching. In each seminar one member of staff talked about their teaching and others joined in discussion exploring practices and issues (Jaworski & Matthews, 2011). The Teaching Group similarly included mathematicians, statisticians, mathematics educators, all in the School of Science at LU but it did not consist of seminar presentations and question and answer sessions. Rather the Teaching Group was an informal forum for colleagues to meet and propose topics for discussion connected to teaching and reflect on their own teaching and on the experiences of others. The group met every two months or so for three years. Membership of the group was fluid—with both new lecturers and more experienced staff joining at various times during the group's existence. We followed a community of Inquiry (CoI) model (see Chapter 2 of this book) in the sense that we:

Inquired: we made use of materials such as education books and research papers, we produced teaching material, and we reflected on our own practice. Much of the focus of the sessions came from issues we encountered in our own practice such as formative assessment for university mathematics and the use of guided notes when lecturing. When discussing summative assessment (e.g. the type of questions to introduce in exam papers for engineers) we explored and learned about the use of inquiry-based mathematical tasks with students. We thought about 'inquiry based' tasks as tasks where the students were not asked to perform a procedure in the questions, but were asked perhaps to analyse and investigate a scenario presented to them;

Learned: we inquired into our own teaching through learning about the educational research on teaching mathematics at university level, reflecting on how our own experiences were mirrored or otherwise in the research we read. We also reflected on each other's experiences and discussed what each of us could learn from them. Our learning was helped by the exchange of ideas and resources: educational research discussions and teaching practice experiences were considered together to enrich our understanding of the teaching of mathematics at university level.

Our meetings usually lasted two hours plus lunch time to carry on talking. Topics we covered included assessment (we talked and read a lot about summative assessment of mathematics at university and especially about exam question content and format), feedback to students, student attendance, types of formative questions, computer aided assessment. Each of us was tasked, before the meeting, to read and present a research paper to the group on the topic chosen and the conversation to follow explored what we could take, in our own teaching, from that piece of research. At other times one of us presented something that they did in their own teaching—a mode of giving students feedback, or a format for guided notes—and the conversation that followed revolved around others' ideas on how that item could be suitable or beneficial for their own practice. The predominance of assessment in our meetings was probably due to a contextual factor related to the UK and a local factor related to the institution we all belonged to. The national UK factor is that in the survey of university students' satisfaction that the UK government issues at the end of each academic year (the National Student Survey,[3] NSS) 'assessment and feedback' are the topics which consistently score the least satisfaction across institutions. Therefore, there is great emphasis across universities to discuss assessment and feedback. Together with this external factor our colleagues expressed a general dissatisfaction with how mathematics for non specialists (e.g., engineers) is assessed. Many of the mathematicians and mathematics educators at LU teach mathematics to engineering students, therefore there was a real interest in discussing assessment for non-mathematicians. The dissatisfaction our colleagues felt consisted of the doubts they held that the current exam

[3] www.thestudentsurvey.com

paper format assessed predominately procedural mathematics while—as it transpired from our meetings—they valued conceptual understanding of mathematics above procedural understanding. Therefore we set off to find questions that could be asked in the exam papers which could assess some of the conceptual understanding valued—and those questions, as in the example reported in Figure 15.1, may be inquiry questions. In order to do so we read literature about assessment, about factors facilitating assessment change for staff and students, and discussed examples of questions that could elicit more conceptual understanding.

For the differential equation $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 5e^x$

(1) Find the complementary function.
(2) Find the particular integral.
(3) Find the particular solution with the initial conditions: $y(0) = 2$ and $y'(0) = 0$.
(4) Is there anything special about the solution method if the right hand side of the differential equation was $5e^{-x}$? Give a short explanation (maximum of 2 sentences).

FIGURE 15.1. Example of questions in an exam paper for first year engineering students.

The last item in this question was an attempt on the part of the lecturer to include some tasks that were less procedural in the assessment of their engineering module (see also Chapter 6 in this book where inquiry based tasks are discussed). The lecturer reasoned that asking the students to investigate something (“Is there anything special ...”) instead of asking the student to implement a procedure (“Find ...”) may stimulate students to reflect on the mathematical situation rather than carry out a well-rehearsed procedure. Since then, some questions which are more open ended—and arguably inspired by inquiry based learning—have made their way in this and other assessment.

In April 2019 we stopped meeting due to the increasing time pressures on staff at the School of Science (one of the characteristics of a culture of university education that permeates practices in the UK). Around this period there was much staff turnover in the Department of Mathematical Sciences and it proved very hard to have new colleagues joining the group. The existing group members found the demands of their day to day tasks too high and the time for meeting informally with colleagues—albeit to learn about teaching—disappeared. For colleagues who were still on probation, research outputs had to be prioritised over teaching activities reflecting, when professional progression is considered, the predominant role, in UK universities, of research over teaching, as mentioned above. During the last session we acknowledged that the experience had been positive, and we all expressed the wish to resume the meetings after a break. However, the COVID-19 pandemic has meant that finding time was even more difficult and to this day we have not resumed the activities of this group.

15.5. Inquiry-based Tasks in a Foundation Mathematics Course

In this section we report on a research project (called Catalyst[4]) where three researchers worked collaboratively with students, using digital tools, to design inquiry-based mathematical tasks for the mathematics course of the Foundation Studies Programme (sometimes referred to as ’Level 0’ or ’Year 0’ of the university degree) at LU. This group formed a Community of Inquiry (CoI)—bringing together different

[4]HEFCE (Higher Education Funding Council for England) Catalyst Fund: Innovations in learning and teaching, and addressing barriers to student success A: Small-scale, ‘experimental’ innovation in learning and teaching. Project code: PK20.

perspectives and expertise. The mathematical focus of our activities was matrices and complex numbers.

Our CoI consisted of three mathematics education researchers (all experienced mathematics teachers, one being the teacher of the foundation mathematics course), four first-year engineering and science students (our 'Student Partners', SPs) who had taken the foundation mathematics course in the previous year and two post-graduate students who assisted with data collection and analysis. In this CoI, we met regularly to discuss progress in the project and to create a co-operative environment where the student partners[5] could feel empowered to share their views.

A pre-requisite to our work was the inclusion of the dynamic geometry software AUTOGRAPH[6] whose designer introduced us to the software. Our first task was to decide on the topics around which we would create inquiry tasks. One of the education researchers favoured the inclusion of complex numbers in order to use the software to help students understand complex numbers conceptually (details of tasks and their use with students can be found in Chapter 6). The second topic chosen was matrices and their relationship with linear equation systems. Our aim was to explore these topics with our student partners to create inquiry-based tasks for use in the teaching of future foundation students, using the computer software to facilitate inquiry. The student partners were included throughout the design process: they learned to use the software and created the AUTOGRAPH files that were used in regular teaching of Foundation students a few months later.

Our group meetings were lively events that created a relaxed environment where the student partners could feel free to contribute. Discussions centred around the mathematics of complex numbers and matrices, how they are taught in textbooks and in the foundation course, how else they could be taught, desirable characteristics of a task, how to utilise the software to formulate and present the tasks and what the effect could be on learning. For example, reflecting on our discussions of potential tasks, one student partner noted how his mathematical understanding changed. He wrote,

> Working with the Catalyst project team helped me in understanding the concepts of complex numbers and matrices at a much higher level as the whole team brainstormed and everyone talking about their methods and approach to the same task and seeing the difference between how a lecturer thinks and how a student thinks really gave a good insight into these topics. (SP Reflective narrative, 11 September 2018)

Thus, at one level of engagement in our CoI, we were located in the inner layer of the PLATINUM Theoretical Framework (see Figure 2.1 in Chapter 2) where we inquired together into mathematics.

For the teachers, this often overlapped with issues of teaching and learning of mathematics, the middle layer of the Theoretical Framework, especially when we discussed designing the tasks using the software. The student partners expressed this overlap when reflecting on their participation in the project:

> Initially just from playing around with the different functions on the software, then as we practised we saw more things we could do and it snowballed from there.
> It was almost like 'reverse-engineering' the questions, we would start with a normal tutorial question, see what the answer looked like on AUTOGRAPH and then re-design the question with the visual cue providing the information as opposed to it being stated directly in words. (SP Reflective narrative, 14 November 2018)

and

[5]See Jaworski et al. (2018) for details of the nature of the collaboration.
[6]`www.chartwellyorke.com/autograph/index.html`

> This is what made me realise that using the graphs on Autograph could help people to see what they were trying to solve in order to understand how to solve it. (SP Reflective narrative, 18 November 2018)

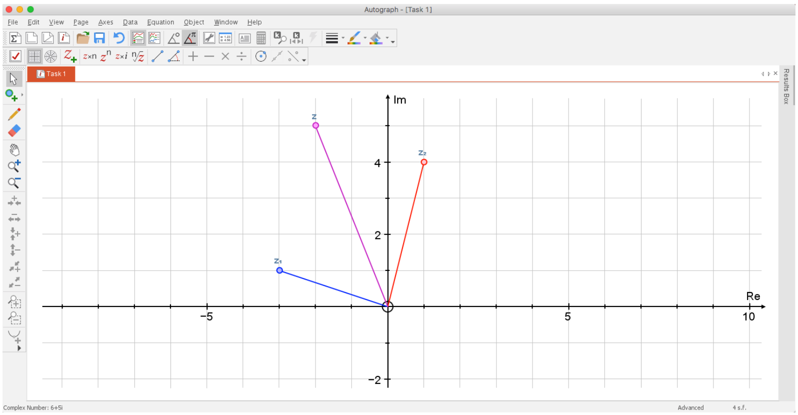
and

> When I think about what I learned throughout the Catalyst project, I cannot help but compare it to the way I originally studied the module during my own foundation year. In the case of complex numbers, I simply needed to understand what the symbols meant and how to manipulate them in a few specific circumstances. This was sufficient to answer the [exam or problem sheet] questions but it quickly became clear that designing them would require a much deeper understanding. (SP Reflective narrative, 11 September 2018)

As this was a research project, the education researchers, together with the two post-graduate students and sometimes the student partners, gathered data (audio-recordings of meetings, narratives and reflections, draft examples of the tasks) and analysed these using mainly qualitative methods, with several publications emerging (e.g., Jaworski et al., 2018; Treffert-Thomas et al., 2019). Thus, at this level of our engagement in our CoI we were located at the outer layer of the PLATINUM Theoretical Framework (Chapter 2, Figure 2.1) where we inquired explicitly into the inquiry aspects of our project with the aim of learning from our engagement and feeding back to inform practice.

As a result of several cycles of activity—designing, discussing, modifying tasks—we agreed on 6 complex number tasks and 5 matrices tasks. The tasks differed in nature but all had a dynamical element, making use of Autograph to either verify a result or explore a relationship further. When designing the tasks one of the student partners commented on the design process as 'reverse-engineering' (see citation above), meaning giving the answer and asking where it came from rather than asking "What is $a + b$?", the latter being a straightforward question with only a correct or wrong answer and not leaving any scope for investigation (an important observation in relation to PLATINUM IO3, see Chapter 6). Once confident in the use of Autograph, the student partners developed some tasks that pleased the teacher of the foundation mathematics course. The student partners formulated questions and produced Autograph files to go with the questions. The Autograph files were used (unaltered) in teaching and the questions were expanded collaboratively by the education researchers to create more context and guidance for foundation students. In addition, the questions (but not the Autograph files) were modified after use in the classroom following reflections and analyses by the education researchers. We found that students sometimes struggled with the wording of questions, in particular with the first (and perhaps easiest) task on addition of complex numbers (Figure 15.2).

Students did not focus on the geometric representation of addition of two of the complex numbers, i.e. the parallelogram (or triangle) law. Students instead decomposed complex numbers into their real and imaginary parts and verified their answers by adding these separately—in essence mirroring addition of vectors. With this task (and Task 2 on subtraction) we noted students' strong adherence to the conventions used in their foundation physics course including reference to the "resulting vector." The following year an adaptation to the terminology—from 'relationship' to 'mathematical relationship' adding also 'how are they connected'—did not produce a different result, students still decomposed into real and imaginary parts and often required a prompt in order to consider the geometric relationship. This led the teacher of the mathematics course to question the nature of the task and consider how to re-design



Task 1

There are three complex numbers labelled z_1, z_2 and z.
z_1 is to be kept fixed while z_2 and z can be moved.
Select z_2 and move it until z reaches the position $6 + 5j$.

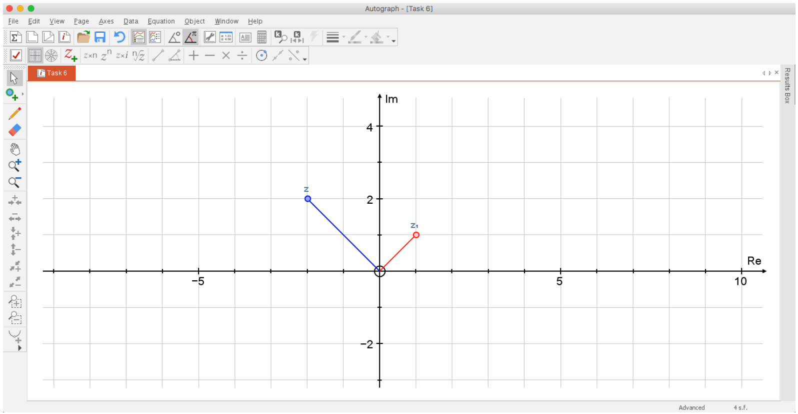
(a) What complex number is z_2? Right click and "Unhide All" to check your answer. The correct answer appears in green.
(b) What is the relationship between z_1, z_2 and z ?
(c) Now calculate by hand: With $z_1 = -3 + j$ and $z = 6 + 5j$, find z_2 such that $z_1 + z_2 = z$.
(d) Re-load *Task 1*. Move z_2 around the screen and notice how z changes as a consequence. What is the geometric connection between z_2, z and the complex number z_1 (which has stayed the same during your movements)?

FIGURE 15.2. AUTOGRAPH file Task 1: Addition of complex numbers.

in order that students engaged in the way that it had been envisaged. This is an example of teaching development (see Chapter 7). On the other hand, it seems that the contextual factors outweighed all others and hence addressing the contexts in which the tasks are delivered is an important consideration for anyone wanting to include IBME tasks in their curriculum. This is an example where research outcomes from the project led the teacher to inquire into her own teaching and make changes to how the tasks were presented in subsequent years. Thus the teacher's inquiry was located in the middle layer of the PLATINUM Theoretical Framework which links the inner and the outer layer (Chapter 2, Figure 2.1).

Multiplication by a complex number $r\angle\theta$ results in a rotation through an angle θ and an expansion of the plane by a factor r. Conceptually very powerful, the teacher's hope had been for students to experience this by working through the AUTOGRAPH tasks on multiplication. For example, Tasks 5 and 6 focused on the squaring and cubing of a complex number, respectively. In addition, Task 6 (Figure 15.3) had the option of a polar grid to visualise the cubing of a complex number, making it easier to 'see' the rotation and expansion.

As this was the last task out of the six, we found that students often did not have time to complete it. In a subsequent year the teacher of the course decided to label each task with a name such as Thelma, Abigail, etc., and laid worksheets out on a table so that students picked a task at random. In many ways that made some tasks more difficult. For example, the task on subtraction usually followed the task on



Task 6

There are two complex numbers labelled z_1 and z_2.

(a) Select z_1 and move it to different positions. There is a relationship between z_1 and z_2 but it is harder to see—so first move z_1 so that z_1 is real. What do you notice about z_2? Try different places for z_1 keeping it always a real number. When does z_2 have a larger modulus than z_1? When does it have a smaller modulus? When do they both have the same modulus? Remember to also try negative values for z_1.

(b) Try to find a relationship between the modulus of z_1 and the modulus of z_2.

(c) Click on the polar co-ordinate icon on the toolbar. Now allow z_1 to take any value, not only just real. Move z_1 and focus on the angle that it makes with the positive real axis. Also focus on the angle that z_2 makes with the positive real axis. Try to find a relationship between the angles as you move z_1 around.

(d) What do you think is the arithmetical relationship between z_1 and z_2?

FIGURE 15.3. AUTOGRAPH file Task 6: Cubing a complex number.

addition and students were able to pinpoint the relationship while when disjoint, they could not. As an experiment, the teacher will try to present tasks in pairs in the next iteration. Here again, the teacher's inquiry into the issues surrounding the teaching and learning of her students was located in the middle layer of the PLATINUM Theoretical Framework (Chapter 2, Figure 2.1).

This project profoundly affected the student partners whose (mathematical) understanding of complex numbers and matrices was greatly enhanced by participating in the design of the tasks. As one student partner wrote:

> It was only through designing the questions that I truly began to recognise and understand the relationships between complex numbers and Argand Diagrams. I believe this is because we went through the process of experimenting with different plots and observing how one change led to another, as opposed to reading and practising specific examples. It occurred to me just how much this process had influenced my understanding of the topics when I came across complex numbers in one of my third-year modules. During a lecture, it was immediately clear to me why the solutions appeared in conjugate pairs whereas many students had to spend some time revising the principle. (SP Reflective narrative, 11 September 2018)

Another student partner who designed matrices tasks also noted:

> I never understood ... until I started doing the project. I thought perhaps other students might be going through what I went through when I was struggling with matrices. This is what made me realise that using the graphs on AUTOGRAPH could help people to see what they were trying to solve in order to understand how to solve it. (SP Reflective narrative, 18 November 2018)

All participants in the project enjoyed working as part of a community of inquiry. The education researchers were happy with the efforts of the student partners in designing the tasks. The student partners learned a lot—about mathematics and about designing tasks for use in the teaching and learning of mathematics. The teacher of the mathematics course acknowledged at one point that the tasks would probably not have come about had it been left entirely to the efforts of the teacher. However, the teacher also expressed some disappointment. Coming to PLATINUM after exposure to IBME activities and thinking deeply about IBME, the teacher of the course wrote after a project meeting:

> In retrospect, and when compared with other tasks that were presented [at the PLATINUM meeting] alongside mine, I began to think whether [the AUTOGRAPH tasks] were more 'hands-on' and 'explorative' than 'inquiry'. I always thought of them as tasks that could raise important conceptual understanding. I had thought less about how much time students would spend on 'inquiring'. I feel quite strongly that it is very hard to devise really good inquiry-based tasks. (Teacher Reflective narrative, 5 June 2020)

In the teacher's view 'inquiry' should involve a period of time reflecting on the different ways of going about finding a solution to the problem given. The AUTOGRAPH tasks were rather prescriptive, certainly allowing for exploration within the AUTOGRAPH environment but ultimately leading to a single (teacher approved) solution.

The PLATINUM project provided an opportunity to see a variety of different IBME tasks raising our own understanding of their potential and scope. Many of us in the PLATINUM project were teaching mathematics at university level but contexts (degree in mathematics, engineering, teacher education, etc.) and level (first year, second year, post-graduate, etc.) as well as topic area (calculus, complex analysis, modelling, etc.) differed greatly. Just as we discussed tasks in our local CoI, the wider discussion in the PLATINUM CoI inspired us to question the goal of presenting a task and what students may do to solve it. The challenge now is to incorporate aspects of the tasks we have seen, shared and discussed in the PLATINUM CoI into new or our own mathematical contexts.

15.6. Teaching Engineering Students

As mentioned above, one of the tasks of the MEC was to lead the teaching of engineering students in mathematics courses. Several members of the MEC were very experienced in this work and had contributed to the writing of the HELM books.[7] Supported by the Dean of the Engineering Faculty, a team of three teachers from the MEC (an inquiry group, CoI) decided to design a teaching/learning innovation: an inquiry-based approach to teaching a mathematics module for a cohort of engineering students using inquiry-based tasks. All three contributed to the design of the project (Engineering Students Understanding Mathematics, ESUM) and one was the lecturer

[7] HELM—Helping Engineers Learn Mathematics—is a set of around 50 workbooks presenting key ideas in a range of mathematics topics. They were produced at Loughborough University by members of the Mathematics Education Centre for use by students in the Engineering departments. They have been widely used in and beyond their original focus at LU and in other UK universities. They are freely available from `mec@lboro.ac.uk`.

for the cohort. Funding was received from the national HE-STEM programme and it paid for a fourth member of the team to act as researcher in the project, observing teaching, collecting data and aiding reflection.

The design of the teaching involved inquiry-based tasks for small group work in tutorials and the use, by the lecturer, of more open questions in lectures, seeking to engage students in participation in both types of session. Use of small group activity in tutorials was part of the innovation. Groups were assessed on a small project tackling inquiry-based tasks. A great deal was learned from the various stages of the project which fed back into the teaching of two successive cohorts. Several publications charted our learning in this project (e.g., Jaworski & Matthews, 2011; Jaworski et al., 2012).

In the style of ESUM, it would have been extremely valuable to repeat this inquiry activity in the teaching of other cohorts of engineering students. For one cohort in particular, the lecturer in their mathematics module was the same teacher as in the ESUM project. Unfortunately, she did not have the support of an inquiry group, or funding for a researcher to collect data etc. However, she hoped it might be possible to use some of the tasks from ESUM and to build some inquiry-based ideas into the teaching.

When a new lecturer was appointed to teach a module for a particular cohort of students, it was common, in their first year at least, to follow the specification of the module material and use the same teaching plan and assessment tools as in the previous teaching. This she did, with the only change being the replacement of 'in-class-tests' with a digital version, using STACK software[8] and the inclusion of some inquiry-based tasks in (otherwise traditional) tutorials. The STACK tests supported an inquiry approach to mathematical questions, providing feedback for students. Otherwise, lectures were conducted in a fairly traditional way following the previous structure of the course.

The STACK tests proved very popular and were used again with a new cohort. However, the lecturer was very disappointed that she had not found it possible to make the module more inquiry based. We present an account of her teaching of the module, with extracts from her own personal reflections.

> Here I am addressing inquiry in the second layer in our PLATINUM model: 'inquiry in teaching mathematics'. This means that I am reflecting on my own teaching, recognising my goals for teaching, the issues that arise in relation to these goals, and ways in which teaching might be developed or improved.

In the previous semester, she had taught a module on introductory mathematics to a cohort of 200 students in the department of Aeronautical and Automotive Engineering. These students had been recruited with a wide range of mathematical experience: some had high level qualifications (grade $A^\star$ in A level Mathematics and Further Mathematics) while some had more basic qualifications (BTEC or A level mathematics grade B or C[9]). So, for example, in addressing the topic of 'Introduction to Matrices' some students had already learned to find the inverse of 3×3 matrices and to solve systems of equations with 3 variables; other students did not yet know

[8]STACK is an open-source system for automatic computer aided assessment of mathematics and other STEM subjects; see `www.ed.ac.uk/maths/stack` for more information.

[9]In the UK, the most common qualification requirements are General Certificate of Education Advanced Level (A level) Mathematics grades A*, A, or B. Some universities admit students to engineering courses with A level grade C, or with BTEC qualifications—a BTEC is a vocational qualification studied at school or college. They tend to be work-related and are ideal for any student who prefers more practical-based learning. BTEC qualifications allow students to continue further study at university or enter the workforce.

how to add or multiply matrices. These differences extended to all topics in the course specification. In her reflections, the lecturer wrote:

> In my first year of teaching this module, I worked with students in a fairly traditional style, presenting mathematics using PowerPoint slides in lectures and helping students in tutorials to work on problems presented in problem sheets related to the topic. With 200 students, I found it difficult to address individuals or to engage with any form of discussion in lectures, or to use explicit inquiry-based tasks; although I had been able to do all of this in earlier teaching of a cohort of 50 students in Materials Engineering (the ESUM project).

In comparing the two cohorts, the size difference (200 v 50) was highly significant; the difference in student mathematical experience was significant in both cohorts, but there were more highly qualified students in the 200 cohort. It was difficult to design lecture material to suit all 200 students, and a different pedagogy was needed for experienced as for non-experienced students. She wrote further:

> I have never used the teaching approach of many of my colleagues of spending a lecture writing out the mathematics from start to finish on a large board (black or white) at the front of the lecture room. I prefer to use PowerPoint because (1) it allows me to face my students as I talk, and to actually look at them and make eye contact (as far as this is possible in a large lecture hall). Also, (2), PowerPoint allows me to animate my slides, building up mathematical formulae and relationships using the whole space of the slide, and emphasising concepts using colour, movement and timing. I talk as I animate and so there is both an oral and a visual exposition of the mathematics.

Every lecture at LU is recorded on the university system of recording all lectures for students to access as they wish. It is encouraged also to save lecture notes and slides on the course VLE (Virtual Learning Environment) page for student access. There should therefore be no need for students to spend their lecture time copying the words and symbols from the slides. Although this is often emphasised in lectures, many students ignore the message and, nevertheless, try to copy everything written. It is as if there is an unwritten rule that what lecturers write in lectures should be copied by the student for future study. The lecturer reflected:

> In teaching, I wish to engage students with the mathematics. As I talk to them, I hope they are trying to make sense of what I say, and I hope that the visual words, symbols, diagrams and animation on the slides contribute to their sense-making. I use a slow clear articulation so that students are not disadvantaged by my speaking too quickly or not finishing my words.

Feedback about this module from some students to their Engineering tutors was somewhat negative: some complained that the teaching was too slow and elementary (despite the inclusion of more challenging problems in the VLE material). Some did not like the slides, saying that there was not enough time for them to copy everything from a slide before the lecturer moved onto the next one. The lecturer commented:

> I taught this course twice in successive years. I will not do so again since I am reducing my working hours in the coming year. However, I can think about what I might do given time and support. I believe that it would be valuable to set up an innovation project as we did in ESUM to institute more inquiry-based activity - this might be possible in the TeStED programme.

An issue in following up the ESUM programme in this way would have been the lack of resources to support developmental activity. However, at about this time, the School of Science began an initiative called the ‘Teaching and Student Experience Development (TeStED) Programme’, which awards time and resources to teaching development. With interested colleagues, it could have been possible to apply to take part in this programme to build on the experiences in ESUM and in a further project in which student partners helped to design mathematical tasks (Catalyst— see section above). Such activity is as yet very small scale, but it is growing as the university recognises a need to promote teaching development.

These reflections above capture elements of the goals and practice of the lecturer. However, there is a tone of sadness: she has not managed to teach in a way that is more inquiry-based. We read some of the issues she faced: the size of the cohort, the very different levels of student mathematical experience and the use of a mode of delivery which students did not like. Implicit is the culture of mathematics teaching in the university: practices such as board writing are common; students are used to copying from the board for later review, they do not think of the value of reflecting on what is being presented during the presentation. In ESUM, the overt questioning approach of the lecturer had been successful to some extent in encouraging students to participate in the lecture, offering (tentative) answers to questions, and even engaging in discussion with peers when some disagreed with what had been said (Jaworski & Matthews, 2011). At the end of the reflections, the ideas for future development, following experience in ESUM, showed that despite negative experiences, she could see ways of achieving more inquiry-based goals.

As a final word here, mathematics teaching to engineering students in the university is delegated to the mathematics department, and engineering colleagues are not involved. It makes sense to us (authors of this chapter) that teaching mathematics to future engineers should acknowledge the use of mathematics in engineering. This would require collaboration between teachers in the two departments, enabling the design of tasks for students that could span the two subjects. In inquiry terms, this could involve modelling tasks in which an engineering problem is addressed through a developing mathematical model. It would, however, require serious reorganisation of teaching which, for the moment, seems unlikely. We refer readers to Chapter 8, which addresses inquiry-based mathematical modelling in a PLATINUM context.

15.7. Discussion

Our concluding section draws together all of the above, addressing how these activities, developments, research and external factors have influenced our own learning and development. In particular we will focus on how the areas of activity we described relate to the PLATINUM project.

We have indicated (above) ways in which our work has related to the three-layer model of inquiry. In the inner layer, we provided examples of tasks that were designed to involve students in inquiry. Particularly in the Catalyst project, research has shown us the important mathematics learning development experienced by the student partners who developed tasks in collaboration with mathematics education researchers. As the Foundation Studies teacher uses these tasks with her students, year by year, modifying them according to what she learns from her data, we see (in layer 2) a clear contribution to development of the Foundation Studies teaching of mathematics.

The Catalyst project embedded clear activity related to the middle layer of the model. Working with our student partners, we learned as they learned. Although the project was very small scale, we see clearly the mathematical learning outcomes of our student partners as they engaged enthusiastically with task design. Their own words are testament to the learning. We ask, how can we use this methodology with larger groups of students (50, or even 200)? We do not have an answer to this challenge, but it is something for us to work on further in our inquiry community.

In Teaching Engineering Students we see (in layer 2) a teacher overtly reflecting on her teaching and recognising ways in which her teaching practice did not, or could not, achieve what ideally she would like to be possible. One thing that this reveals is that it is hard for a teacher to try to engage alone with inquiry into teaching. Comparison with the ESUM project emphasised the value of having a research associate working alongside to gather data and stimulate reflection. The inquiry group in ESUM (four colleagues), designing, teaching and monitoring activity, was supportive both in the design of teaching (tasks and pedagogy) and in reflective inquiry which led to improvements in the course as it developed.

In the third layer we see an overt developmental intention supported by collection and analysis of data related to questions we wanted to address. The ESUM project had been one good example of this in which a CoI designed, taught and evaluated the teaching and learning in the project, with feedback to future teaching. Such activity was achieved also in the Catalyst project. Here, mathematics teachers engaged overtly in research into the practices in which they participated, addressing clear research questions. The Catalyst work is ongoing in the sense that the teacher is still building on what has been done and learned in ongoing teaching/learning development. Both projects have published articles which share learning outcomes from the inquiry activity with interested colleagues more widely.

We believe that essential to the development arising from this work is the inquiry group. When colleagues together explore (inquire into) aspects of their own teaching and learning, development takes place (both for the individual and for the community) and new knowledge emerges. When the inquiry activity is in the third layer, systematic analysis of data results in knowledge which can be shared with the wider community.

We can show the above in a diagrammatic representation of our inquiry model in PLATINUM.

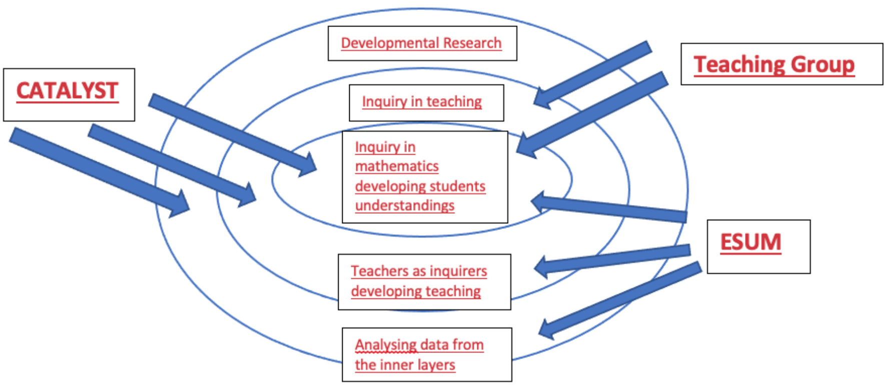


FIGURE 15.4. Linking activities to the 'Three Layers of Inquiry.'

From the model it can be seen that our inquiry activity spans all three layers. In terms of the PLATINUM intellectual outputs (IOs 1 to 6—see Chapters 2 and 5), we have focused on three of them. The inquiry model presents a theoretical perspective of the whole inquiry process (this is IO1). This encompasses our developmental activity through its three developmental stages. The central layer of the model focuses on students learning mathematics together with their teachers using inquiry-based tasks and teaching units. This is IO3. The middle layer of the model focuses on the development of teaching as we have seen in the teaching group and in both Catalyst and ESUM with the inclusion of inquiry-based tasks and teaching units, which is IO3. We see that IO3 relates to both the inner layers. It involves the creation of communities of inquiry through which colleagues work together to learn more about teaching. This is IO2. The outer layer of the model focuses on developmental research in which data is collected from a range of sources and analysed to provide results from inquiry-based practice which can be shared more widely. This relates to all three of our IOs.

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CHAPTER 16

Teaching Inquiry-Oriented Mathematics: Establishing Support for Novice Lecturers

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16.1. Introduction

The professional development of novice lecturers is considered to be of crucial importance in the PLATINUM project. In this chapter, we present what a group of lecturers[1] at Universidad Complutense de Madrid (UCM) (mathematicians and mathematics education researchers), constituted as a Community of Inquiry (CoI), considers valuable for the development of the professional knowledge of mathematics lecturers. With this aim, the group has designed learning tasks and developed seminars within the framework of the PLATINUM Project.

Referring to the three-layer model outlined in Chapter 2, this chapter describes the interaction between the second layer, *inquiry into teaching mathematics* (lecturers using inquiry to explore the design and implementation of tasks, problems, and activities in classrooms), and the third layer, *inquiry into research for a professional development programme for mathematics lecturers*, which takes the results in developing the teaching of mathematics in order to systematise advanced areas and professional development programs at the institutional level.

Focusing on the Inquiry-Based Mathematics Education (IBME) approach, the didactic design certainly has an essential role for the establishment of productive links between research and practice. However, for the didactic design to be effective, it must be considered not only as a by-product of research but also must be incorporated into the programs associated with the professional development of mathematics lecturers.

We present the methodological approach and several materials for professional development. We focus on the challenges and questions that appear in the design and execution of mathematical tasks for teaching matrix factorisation in the subject of numerical methods, both from a theoretical and practical point of view.

The rest of this chapter is structured as follows: Section 16.2 presents the UCM context. Section 16.3 is devoted to the UCM CoI and the fundamentals underlying the approach. Section 16.4 describes the design of materials for professional development in mathematics. Section 16.5 presents the *Matrix Factorisation* case, where the theoretical background is applied to the analysis of acquisition of knowledge and inquiry processes. Implementation results with novice lecturers are presented in Section 16.6. Finally, a discussion and some conclusions are included.

[1]We use the terminology of *lecturer* and *professor* as in UK English for university education. In Spain the equivalent term is *university teacher*. Also, in this paper, we refer to the novice lecturers enrolled in the course on Teacher Professional Development in Mathematics as *participants*, reserving the name *students* for the mathematics learners in the degree programmes experiencing the lessons.

16.2. Complutense University of Madrid

The Complutense University of Madrid, with nearly 72,000 students, is the largest university in Spain with two campuses in Madrid. It was founded in 1499 and is a centre of reference for Latin America. Its 26 faculties offer 320 official degrees (Bachelor's degrees, double degrees, Master's degrees, doctorates, and international degrees) and more than 300 continuing education degrees.

Given the multidisciplinary nature of mathematics, lecturers with very different backgrounds—including mathematicians, experts in mathematics education, engineers, and so on—teach different topics in the Mathematics Faculty or teach mathematics in other faculties.

UCM has a specific teacher training centre where the University Master's degree programme in Teacher Training for Secondary and Vocational Education and Language Teaching[2] is taught in collaboration with the Faculty of Mathematical Sciences for the specific area of Mathematics. This qualification is of an enabling nature to teach at these levels of education.

The professional development of lecturers is not linked to any specific qualification. UCM has a training plan for teaching and research staff,[3] although this is specially designed for lifelong education or continuing university teacher training, and not so much for initial training. Thus, the professional development for teaching of novice lecturers is developed through the faculties that offer studies in each specific subject.

Regarding teaching methodologies for novice lecturers, UCM has a programme for innovation in teaching (called *Innova Docencia*[4]), which encourages (voluntary) lecturers to try innovative teaching approaches and techniques.

Three bachelor's programmes are offered in the Faculty of Mathematical Sciences at UCM (Mathematics and Statistics, Mathematical Engineering, and Mathematics). In the curriculum of the latter there are subjects related to the teaching of mathematics, both at the secondary and university levels. In addition, the Faculty is the headquarter, since its foundation in 2007, of the Miguel de Guzmán Chair,[5] which has as objectives the analysis, research, and teaching of the reality, problems, and perspectives of mathematics education in Spain and internationally. Since its inception, it has promoted research in mathematics education through various research projects.[6]

In the Faculty of Mathematical Sciences at UCM some attempts have been made to connect mathematics education to inquiry-based mathematics education. Reference is often made to problem solving, in which there is a long tradition of research and practice in the field that goes back to the seminal work of György Pólya (1945, 1954). In Spain, Miguel de Guzmán, professor at UCM and ICMI president in the 90s, encouraged teaching and learning in this direction by publishing various books and developing a theoretical framework that gives an essential role to problem solving. The teacher training programmes are under this approach, promoted with the support of the Spanish Ministry of Education and with the collaboration of international experts such as Alan Schoenfeld (1985). More specifically, emphasis was put on reflections on mathematics methods, focusing on the development of mathematical competences and metacognitive skills that can be interpreted in terms of inquiry habits of minds and problem-solving attitudes (de Guzmán, 1995). From the inquiry perspective it is worth mentioning the international study carried out at UCM by Miguel de Guzmán

[2] `www.ucm.es/masterformacionprofesorado`
[3] `https://cfp.ucm.es/formacionprofesorado`
[4] `https://eprints.ucm.es/pid.html`
[5] `http://blogs.mat.ucm.es/catedramdeguzman`
[6] `http://blogs.mat.ucm.es/catedramdeguzman/proyectos-de-investigacion`

and other researchers (de Guzmán, 1998) on the difficulties of the transition from secondary school to university where some mathematical-didactical problem areas regarding epistemological, cognitive, and sociocultural aspects were identified.

The design of materials and resources for professional development of university lecturers of mathematics in the PLATINUM Project at UCM has had important precedents, under the problem-solving approach and based on the Design-Based Research Collective (2003). Since 2009, the research group of mathematicians and mathematics educators has designed and implemented different courses to provide university lecturers and research assistants with educational tools enabling them to better design, implement, and analyse teaching and learning processes (Corrales & Gómez-Chacón, 2011; Gómez-Chacón & Joglar-Prieto, 2010; Gómez-Chacón et al., 2020). The PLATINUM project represents further progress in terms of an international contrast with other IBME approaches and a wider dissemination of practices.

16.3. Community of Inquiry (CoI) at UCM

The starting point for the UCM CoI was in our opinion the collaboration in the ICMI and/or the Miguel de Guzmán Chair described in the precious section. In fact, most of the lecturers that finally joined the PLATINUM project collaborated via these organisations in innovation-related projects: the project leader was already in charge of courses for novice lecturers, and everyone tried to apply new teaching approaches in their lectures. Even though most of them still lacked the theoretical context related to inquiry-based learning, in most cases they tried to encourage students to explore the instructional materials, ask questions, and discuss proposals. Taking these ideas into account, the UCM CoI initially started with eight lecturers. Besides these eight members of the 'core' CoI we founded an extended CoI including several colleagues interested in improving their teaching and PhD students / novice lecturers who want to learn new teaching techniques. In particular, an extended CoI was formed thanks to seven members of *Proyecto Innova-214: ESCEMMAT-Univ1*[7] focused on professional development for novice lecturers in mathematics. These young lecturers participated in the professional development program.

The members of the 'core' UCM CoI learned about inquiry-based learning, trying to apply it to their lectures, collaborated to implement new lessons/assignments, and participated in the general PLATINUM meetings. All these activities were periodically discussed in local meetings, where the members of the CoI presented their ideas and results so that the rest of the members could give feedback. In these meetings general PLATINUM issues were also addressed. Although for most of the PLATINUM project these meetings were held offline, we changed to online meetings due to the COVID19 pandemic. This was not the only difficulty, because the pandemic also forced us to change the teaching mode, making the implementation of some activities more difficult.

In the UCM CoI, Inquiry-Based Mathematics Education (IBME) is considered widely as a way of teaching wherein students are invited to work in ways similar to how mathematicians work (cf., Dorier & Maaß, 2020). Within the PLATINUM project, the development of this perspective is based on the three-layer model presented in Chapter 2 of this book and in (Jaworski, 2020), and it is rooted in the idea of communities of inquiry.

[7] http://blogs.mat.ucm.es/catedramdeguzman/proyectos-de-investigacion/escemmat-univ

Through this CoI, a substantial growth of knowledge and awareness of inquiry-based approaches to teaching and learning mathematics has taken place. These approaches encourage students' engagement, creativity, and conceptual understandings of mathematics.

16.4. Designing Materials for Professional Development

The UCM-PLATINUM team's idea of *professional development* is based on an epistemology of professional knowledge which takes into account the contextualised nature of the teacher's experience (experience knowledge) and the personalised knowledge of the practice (Frade & Gómez-Chacón, 2009; Gómez-Chacón & De La Fuente, 2019). In essence, this 'competence of the actor in her/his context' recognises the subject as the main actor of his development and co-creator of process of her/his professional development.

Our proposals for the professional development aim to provide lecturers and research assistants with educational tools that enable them to better design, implement, and analyse teaching and learning processes in mathematics. In the PLATINUM project, we have prioritised in the lecturers' professional development the sense of belonging to a CoI and the design of mathematical tasks under the IBME approach. For the design of professional material, we have used as strategy the design methodology represented in Figure 16.1. It has been intentionally used to explore the interplay between the layers in the three layer-model presented in Chapter 2. Our conceptualisation of IBME takes explicitly into account this specific nature of mathematical inquiry and the essential contribution of internal inquiry to the development and structuring of mathematics as a domain of knowledge. Following inquiry-based learning epistemological bases, it is not only important to acquire new knowledge, but also to develop analytic and experimental strategies to reach new knowledge.

Figure 16.1 represents the design process for the materials to be implemented in professional development courses for novice lecturers. It is divided into four phases: *Discover*, *Define*, *Design*, and *Develop*. In the creation process of the tasks, possible ideas are defined. These ideas come from the needs identified and the teaching objectives necessary for the development of strategic knowledge in the 'learning to teach mathematics' at the university level. One of the key aspects is the confirmation of problem definition and the creation of a solution through design and class development.

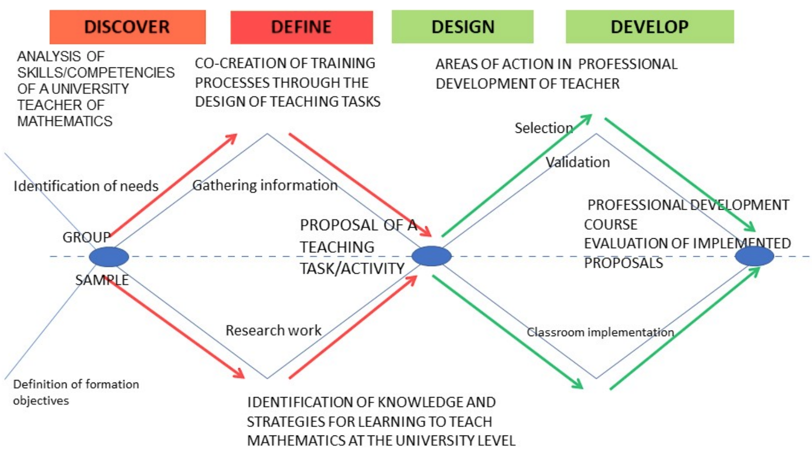

FIGURE 16.1. Design process of creating and evaluating instructional materials in a professionalisation context.

In the proposed conception, research and development are mutually involved. Professional development is viewed from a reflexive position concerning practice. The aim is for new teachers to join in a research project over the practice, in the sense of questioning and analysing it to understand it, and even transform it.

The diagram aims to map the divergent and convergent stages of the design process by showing designers' different ways of thinking. In our case the designer was a team of four PLATINUM members: mathematicians whose fields of research and expertise are applied mathematics, algebra, computer science, and mathematics education, respectively.

16.4.1. Discover. The first phase, *Discover*, marks the beginning of the design. This phase corresponds to a deep contextual dip into the challenge of professional development of novice lecturers at the Mathematics Faculty of UCM. It involved different working sessions of the CoI. These discussions were supported by a review of literature about research in algebra at university level, data from the narrative techniques applied to mathematicians, video recording of discussion group meeting of CoI, and a questionnaire about teaching algebra (for senior and novice lecturers). We used them to understand difficulties of students in algebra as well as to identify needs of professional development in order to integrate the IBME approach. Some of them that will be highlighted in the developed materials are:

(1) concepts of linear algebra—matrix diagonalisation and factorisation, applications such as rigid motions (Linear algebra courses occupy a dominant position in the overall undergraduate mathematics curriculum.);
(2) interconnections between linearalgebra and numerical analysis, linear algebra and geometry, linear algebra and rewriting logic;
(3) inquiry forms and task typologies according to the students from several Bachelor's programmes (Mathematics, Computer Science, Engineering).

The following excerpts from one of the professors and another novice lecturer teaching linear algebra illustrate these aspects in relation to professional development in inquiry approaches:

> My objectives within the PLATINUM project are fundamentally two. On the one hand, to improve my teaching work and, on the other hand, help, as far as possible, to improve the teaching practice of other colleagues. The topics that interest me the most have to do with algebra and especially with linear and matrix algebra: the relationship between concepts such as linear applications and their representation as matrices, the multiple ways of factoring a matrix and its applications, etc. Another issue of interest is how to approach these topics according to the type of student to whom the class is directed. Returning to the PLATINUM CoI, knowing what my colleagues do in other subjects allows me to verify that many of the concepts of linear and matrix algebra appear in different contexts, which I can take advantage of to motivate my students (Professor's narrative, April 2019).

> On the other hand, we work with a programming language, which has a particular syntax and semantics, so we need to go beyond theory and implement the mathematical ideas. Right now, students memorise 'patterns' so they can apply the same structures to similar problems, but they do not really understand why one structure is more appropriate in a given situation, so minor changes in the specification make students fail in their assignments. I think inquiry might be appropriate to take both Mathematics and Computer Science together. If they reason and interact with the concepts, they can relate both fields effectively and reach a deeper understanding. I expect discussions to take place for both theoretical and practical issues. Moreover, the experience in this

Master's course can be replicated, if successful, to Bachelor courses in Logic (Novice lecturer's narrative, April 2019).

16.4.2. Define. The second phase, *Define*, represents the definition phase, i.e., the moment when insights are defined and refined. This phase aims to identify patterns and to reach conclusions based on collected data in the previous phase. The main activities held during the *Define* phase are:

- defining the didactic-mathematical problems to be answered through the design of classroom-directed activities (tasks, units, teaching actions, etc.);
- identifying the approach under which to develop the IBME project;
- selecting the study group, courses, subjects, and levels to carry out;
- anticipating the professional skills to be acquired by novice lecturers.

Table 16.1 shows the main inquiry projects that were designed and implemented in the classroom with undergraduate students from different Bachelor's programmes at the Mathematics Faculty (IO3). Some of them were selected to be considered as base materials to be used in professional development courses. In order to define them, priority was given to the teaching of concepts, taking into account the interdisciplinary nature of knowledge, and how an inquiry approach can help in their understanding. We describe them briefly.

Teaching linear algebra and video games. Affine transformations and rigid motions are the main concepts. An open inquiry project designed for undergraduate students in the Bachelor's programme Videogame Development. It consists of a progressive approach to linear algebra according to three pedagogical principles: the *principle of*

Subject / Field	**Topic**	**Level**	**Connections**	**IBME model/inquiry features**
Algebra	Diagonalisation	Bachelor	Numerical linear algebra, matrix computations, dynamical systems	Flipped classroom methods
	Rewriting logic	Master	Technology Maude language	Semi-guided inquiry
Algebra and Geometry	Affine transformations and rigid motions	Bachelor Master	Technology Interactive geometry system	Open guided inquiry Video-game project
	Isometries and tessellations of the Euclidean plane	Bachelor	Technology Interactive geometry system	Semi-guided inquiry
Numerical analysis	Matrix factorisation	Bachelor	Algebra	Semi-guided inquiry
	Numerical methods for ODEs	Bachelor	Calculus	Semi-guided inquiry
Calculus	Functions, derivation, differentiability	Bachelor		Semi-guided inquiry
	Rolle's Theorem	Bachelor		Semi-guided inquiry
	Elements of Ordinary Differential Equations	Bachelor		'Escape Room' project

TABLE 16.1. Examples of inquiry projects.

concretisation, *principle of necessity* and the *principle of generalisation and formalisation* (Harel, 1989). Specific techniques and resources for the teaching of the concepts of related applications and rigid motions are proposed (see Chapter 7).

Diagonalisation of matrices. This project seeks to relate the concepts of diagonalisation, eigenvalues and eigenvectors to the matrix of a linear mapping with respect to a base formed by eigenvectors, if it exists, using simple examples. In professional development, the aim is to encourage reflection on teaching practice, focusing on the involvement of students in learning concepts such as diagonalisation, so that they are able to guess if certain properties are fulfilled or discard them if they find a counterexample. Using the 'flipped classroom' concept, video, work guides, and evaluation instruments are prepared on the topic.

Specification in rewriting logic. This project aims to formalise the specification of complex software systems through rewriting logic. This formalisation will allow students to prove properties on these systems using different techniques using logic, which requires students to abstract the details and focus on the mathematical properties underlying these systems. Because this project is designed for students in computer science, we design the units using games, interaction between the lecturers and the students, and inquiry.

Matrix factorisation. This project tries to establish connections between numerical analysis and linear algebra. Matrix factorisations are methods for reducing a matrix to a product of simpler matrices, so that complex matrix operations can be simplified by performing them on the decomposed matrix rather than on the original matrix itself. They are fundamental not only in linear algebra for solving systems of linear equations but also have many applications (see Section 16.5).

16.4.3. Design and Develop. The third phase, *Design*, seeks to generate ideas and prototypes for professional development courses. The fourth phase, *Develop*, focuses on the adjustments and further refinements that must be performed to produce more mature prototypes in the medium and long term and offer them in professional development courses for mathematics novice lecturers. The final aim is to produce professional development models and examples of associated practices for inducting these lecturers into inquiry-based mathematics education. So, one of the main activities and goals during this phase is testing, adjusting, and validating the prototype as materials for lecturers' professional development. Some of them were used in the workshop or seminar during the academic course 2019-2020.

Taking into account this implementation, one of the main goals in the *Develop* phase was performing brainstorms with the CoI and end users, defining the essence of the given ideas for teaching using an inquiry approach, comparing them to the core of the mathematical problems and professional competences for novice lecturers.

We would like to emphasise that the model should not be understood as a one-way flow (Figure 16.1). In fact, the designers of the tasks navigated through phases; they intensified or abandoned the use of tools and techniques, and moved back and forth as the challenge and the feed-back with the real context progressed (testing, adjusting, and validating).

16.5. The Matrix Factorisation Inquiry Project

The purpose of these materials for professional development is to support lecturers in developing and extending their range of practices in the subject of numerical methods through an IBME approach, which is taught to students of various study programmes related to mathematics.

Matrix factorisation was chosen as the most appropriate topic for applying inquiry-based learning. In the implementation in the classroom, the students moved between the experimental, theoretical, and algorithmic levels. More specifically, the students began factoring a matrix and had to generalise the procedure to obtain an algorithm and also conjecture a theoretical result. The results of the implementation and the good feeling of the students, as revealed in the questionnaires (Figure 16.3 and Figure 16.4), shows the adequacy of inquiry in this teaching process.

16.5.1. Professional Competences to Develop by Novice Lecturers. The competences to be developed through the collaborative working sessions are

- to learn how to design and elaborate materials on subject matter of the course program that will
  - allow the students to consolidate the concepts addressed in previous courses and topics. In particular, LU-factorisation, making the LU (or $PA = LU$) factorisation of a matrix understood as a natural consequence of Gaussian elimination used to solve a system of linear equations; and
  - encourage the development of mathematical intuition and different representations of the same concept; and
- to encourage reflection on teaching practice, emphasising the involvement and detection of students' difficulties in learning and managing of matrix factorisations that allow them to efficiently solve systems of linear equations, using direct methods.

16.5.2. Teaching-Learning Tasks. The inquiry-based task about matrix factorisation for the students (a documented teaching unit on the PLATINUM website) was prepared by two mathematics professors with wide experience in teaching at university level and, in particular, in the subject of numerical methods. In the design of these materials in professional development of lecturers, other members of the PLATINUM project—the authors of this chapter—participated, too.

In the professional development of novice lecturers we are of opinion that the fundamental issue to be studied is not merely how to present materials better, rather, it is ultimately how students learn and perceive concepts in numerical analysis in connection with linear algebra (previous knowledge in the undergraduate).

Using an inquiry-based approach, we are not proposing another way of teaching numerical methods; rather, we propose to provide a systematic framework for lecturers to become a better learning facilitator, to ask thought-provoking questions, to design lessons that facilitate conceptual understanding of key concepts in numerical methods, to help students make mental constructions of mathematical objects, and to create a lasting effect in student learning of mathematics in general.

16.5.3. Design Process. In what follows we present the design of the materials. We also describe how the experience of the senior professors is presented to novice lecturers, breaking it down into the most significant phases. In addition, novice lecturers are presented with different tasks to make them go through the process of choosing mathematical notions, tasks to be developed with the students, and their timing. According to the process explained in Section 16.4 (Figure 16.1) we describe these phases for the LU-factorisation task.

Discovery phase. We start with the determination of mathematical concepts that will be the focus of the tasks, as well as the discussion of difficulties that novice lecturers identify in the subject of numerical methods.

We focus on the process of developing materials for second year students in various mathematics-intensive Bachelor's programmes (Mathematics, Mathematical Engineering, and Mathematics and Statistics, as well as the dual programme Mathematics and Statistics and Economics). The course is called *Numerical Methods* and includes several topics related to matrix computations. At UCM, it is taught four hours per week during a four-month period, two of them are theoretical, one hour is dedicated to problem solving and one more hour is dedicated to practice in the computer lab where MATLAB[8] is used. It is within the hours of theory and problems that the activity carried out by the team is introduced.

In general, students in the Bachelor's programmes in Mathematics show interest and are motivated by their studies. However, they often have difficulty managing matrices and using them appropriately to solve problems. In particular, there are difficulties in understanding matrix factorisation. The results are felt as abstract, surprising, and unnatural. For these reasons, the factorisations are not introduced as algebraic equations proved by induction on the size of matrix. Instead, factorisation is developed from Gaussian elimination, a technique well-known by the students. Nevertheless, this way of presenting factorisation does not avoid some difficulties in its use and, especially, in its implementation.

Analysis and definition phase. The novice lecturers analysed the different options and approaches according to what was obtained in the exploratory phase and determined what the materials are prepared on (Definition phase in Figure 16.1). The topic of matrix factorisation is chosen, a concept that is often seen by students as unnatural, not at all intuitive, and difficult to implement.

The theme is framed within the methods of numerical resolution of systems of linear equations. The objective of LU-factorisation is to create two triangular matrices—lower triangular matrix L, upper triangular matrix U—such that $A = LU$. This allows to easily solve the given system of equations $Ax = b$, computing the solution of the triangular systems $Lw = b$ and $Ux = w$. The matrix L is obtained by placing on the main diagonal the numbers 1 and the Gaussian elimination multipliers in the places indicated by their indices. The matrix U is obtained as the matrix resulting from the elimination process. Eventually, it may be necessary to exchange two rows to continue the Gaussian elimination process and so instead of the factorisation $A = LU$ we have $PA = LU$, where P is the permutation matrix that accounts for those exchanges.

Materials design phase. Within the framework of inquiry-based teaching, the analysis of the material that the lecturers of the PLATINUM project have implemented is crucial (Design phase in Figure 16.1). With these examples of tasks, the novice lecturers were invited to identify tools to guide their students, and how to encourage reflection on the inquiry process and the acquisition of mathematical concepts. Different tasks are proposed:

Task 1: Reflect on how to approach in a more natural way the matrix factorisations studied ($A = LU$ and $PA = LU$), linking them to Gaussian elimination.

Task 2: Find examples of matrices that are versatile enough to exemplify both types of factorisations.

Task 3: Design of the exercises and their scheduling, so that they can serve to provoke a gradual sequence of conjectures and their eventual refutations or confirmations. Thus, a matrix that admits LU-factorisation is considered, and students are directed to apply the Gaussian method to it, storing the multipliers in L and calculating the product of L by U to reach the original matrix and provoke a first conjecture. After

[8] www.mathworks.com

that, a matrix is chosen as a result of exchanging the rows of the previous one and the calculations are reproduced, seeing that the original matrix is not obtained, which will lead the students to revise their guess. Finally, we start from the second matrix and repeat the calculations but store the multipliers in the place that would occupy the zeros. In this way, the product of the matrix L thus stored by the matrix U is a permutation of the starting one, which leads the students to guess a new result. Once the exercises and questions for this activity have been decided, a worksheet is prepared to be distributed among the different groups of students.

The inquiry nature of the tasks is discussed, focusing on the characteristics of an inquiry-based mathematical task regarding the following dimensions:

(1) How and who initiates the questioning? (*Questioning*).
(2) Which is the nature of the problem? (*Nature of the problem*; *Mathematical knowledge*).
(3) To what extent are students responsible for the inquiry? (*Student responsibility*).
(4) How the lecturer's goals are made explicit? (*Goals*).

These dimensions portray the modalities of inquiry-based teaching and learning that range from teacher- to student-centred. In the Matrix Factorisation project, as it carried out at UCM, the lecturers elaborate the questioning after considering students' concerns. The tasks are formulated as a partially open-ended problem: Students have to cope with an open task and limited material already prepared. It is the design of tasks that breaks with the usual routine of classes in which the lecturer introduces the concepts to perform, later, practices or problems where these concepts appear. In this project, autonomous study is encouraged, as well as the ability to work in groups, to ask oneself questions, and to guess results. In the classroom, students have to perform a series of tasks that test their knowledge and ability to manipulate matrices. In addition, the proposed tasks include questions in which students are encouraged to conjecture results, confirm or refuse them with new examples and adjust their conjectures in view of the conclusions thus obtained (*Inquiry*).

In the problem classes the students were divided into groups of four, trying to encourage both group work and discussion among equals. The teacher supervised and resolved the doubts that the different groups posed, always trying to encourage students to work independently. In short, the aim was for the student to participate actively rather than being just a mere receiver of information. Students are asked to justify their conclusions with respect to knowledge or evidence. Figure 16.2 presents some examples proposed by the lecturers.

Finally, Task 4 is proposed to prepare a survey to ask the students to make explicit what they have learned during the inquiry session.

Task 4: Prepare a sample survey for students to complete after the activity.

After the lecturers participating in the sessions had shared their ideas, they were shown the two questionnaires proposed in experience with undergraduates. Questionnaire 1 (Figure 16.3) was posed to the students after finishing the tasks. A few days later, when the subject corresponding to this task had already been explained, Questionnaire 2 (in Figure 16.4) was presented to the students.

Development phase. The sequence and results of the implementation with mathematics undergraduate students of the Bachelor's programme at the Mathematics Faculty are presented to the group of participants in the professional development workshop. The steps followed in this session are listed below.

First part

We consider a matrix A to which the Gauss method can be applied without permutations. In fact, as we will see, the element that is left on the diagonal at each step (which is called the *pivot*) is the largest among all those that are, in the corresponding column, from the diagonal to the end of the column.

```
A=[8 4 2 0;4 6 1 1; 2 1 4.5 1;0 1 1 2.25]
```

```
A = 4x4
    8.0000    4.0000    2.0000         0
    4.0000    6.0000    1.0000    1.0000
    2.0000    1.0000    4.5000    1.0000
         0    1.0000    1.0000    2.2500
```

We are going to apply the Gauss method to matrix A, **always** choosing the linear combination of rows in which each row is subtracted from the pivot row multiplied by a number (which is called *multiplier*).

We save the initial matrix A in a matrix U in which we are going to make the transformations.

```
U=A
```

```
U = 4x4
    8.0000    4.0000    2.0000         0
    4.0000    6.0000    1.0000    1.0000
    2.0000    1.0000    4.5000    1.0000
         0    1.0000    1.0000    2.2500
```

For the first column, the pivot is 8. Therefore, we will have to subtract from the 2nd, 3rd and 4th rows the 1st row multiplied, respectively, by $\frac{4}{8}=0.5$, $\frac{2}{8}=0.25$ and $\frac{0}{8}=0$. We are going to store these numbers in the matrix L which, initially, is the identity matrix. We will save them in their **first column**, since they are the multipliers used to make zeros in the **first column** of U. We keep the multiplier used to make a zero in the **second row** of U (that is, 0.5) in the **second row** of L, that of the **third** (0.25) in the **third row** of L and that of the **fourth** (0) in the **fourth row** of L.

```
L=eye(4);  % Initialize L to identity matrix
L(2,1)=0.5; L(3,1)=0.25; L(4,1)=0;
L
```

```
L = 4x4
    1.0000         0         0         0
    0.5000    1.0000         0         0
    0.2500         0    1.0000         0
         0         0         0    1.0000
```

What does matrix U look like after having made these linear combinations of rows on it?

U=

FIGURE 16.2. Task example.

Questionnaire 1

(1) How was the experience of facing an open problem, in which you are asked to conjecture from concrete examples?
(2) What aspects did you find less clear, more difficult to understand or realise? Has the use of 'clues' been useful?
(3) How much do you value the sharing of the conclusions?
(4) Have you benefited from the discussion? Has it been useful to you? What for?
(5) What do you think about this way of approaching the results? Do you think it would be positive to include more activities of this nature in the course? What activities can you think of?
(6) Rate the activity globally from 1 to 10, with 10 being the maximum score.
(7) Write below any comments you want to make about the task.

FIGURE 16.3. Questionnaire 1.

Questionnaire 2

(1) Do you think the task has facilitated your understanding of Gaussian elimination en LU factorisation of a matrix?
(2) Please indicate the aspects that you think have been easiest for you (if any).
(3) Now that you know the theory and implementation, would you change anything in the design of the task?

FIGURE 16.4. Questionnaire 2.

(1) Introduction about the objectives, creation of groups of four students, and delivery of worksheets.
(2) Accompaniment of the lecturer. The lecturer supervised the groups during the activity. Problems with comprehending some statements came to light and were clarified. Some clarifications were also needed about what the students were expected to do.
(3) Processes of conjecture and justification in the investigation. It is proposed to conjecture on different aspects (possibility of LU-factorisation and how to obtain it). We worked on the different examples of matrices chosen in Task 2. The greatest difficulty lies in the students' lack of habit inferring general results from concrete facts. For example, this is the answer of one of the student' groups to the first question of Questionnaire 1, which can be considered representative of all of them: "I think it is very useful because you force yourself to ask yourself and try to discover things that you would not normally think about if they were explained directly to you". Another answer to this question was: "It is difficult but when working in a group, ideas that had not occurred to me came up and it was easier to reach the conclusion", which shows that students positively value the contribution of working in a group.
(4) Once the task was finished, a discussion group was organised. First, a representative from each group communicated the group's answer to each of the three questions of the worksheet to the other groups, so that all students would be aware of the conjectures of all their peers. Next, through a debate led by the teacher, the groups assessed, assumed, or criticised the responses and opinions of the other groups. To this end, the lecturer encouraged the members of each group to explain their arguments. Often, these were reformulated taking into account the answers given by the other groups. To this end, the lecturer encouraged the members of each group to explain their arguments. There were different situations of the type "We had thought... but hearing what group X responded we realised that it was not so."
(5) Results of the experience. A synthesis of the answers from the student groups was presented to the novice lecturers, as well as the type of feedback given to the students, where the difficulties encountered were reflected, the mistakes made in the development of the practice were analysed, and guidance on mathematical concepts and processes worked on was given. Some arguments were surprising. For example, one group pointed out the symmetry of the matrix of the first example, an irrelevant detail in this context.

    However, most of the answers were quite reasonable. For example, here we have some typical answers to the first question of the worksheet (about a matrix A which has LU-factorisation with no permutations): "In this 1st part, we come to the conclusion that the product of the transformer matrix L by the upper triangular matrix U gives rise to the original matrix A," "Every matrix A can be decomposed as a product of an upper triangular matrix U and another lower triangular matrix L, such that $A = LU$, where L 'stores' the transformations by rows to triangularise A." And the corresponding answers to Question 2 of the worksheet (about a matrix B which has no LU-factorisation without exchanges of rows): "We suspect that the change in rows 1, 3, and 4 has caused that the only row that remains the same is 2 and therefore the LU product is different from the initial matrix B," "Clarify that the transformations that 'stores' L are not only multipliers, all the elementary operations that are carried out in U have to be reflected in L."

(6) Results of the survey applied to undergraduate students on the evaluation of the proposal. In general, the activity was valued positively by the undergraduate students with an average score of 8.44 points out of 10. There was a general perception that, at the beginning, what was being pursued was not well understood. With the teamwork and the clarifications made by the lecturer, the difficulties and the paralysis that arose from having to guess a result were overcome. Teamwork also contributed to losing the fear of writing something wrong. The pooling of ideas from the different groups was also highly valued; it was thought that it brings out aspects not considered in the first stage (when working in groups), as well as difficulties different from one's own.

The results of the questionnaire showed quite clearly the usefulness of the activity for a better approach to matrix factorisation. Some of the answers to Question 3 of Questionnaire 1 were of the type "Positively because by exchanging ideas you self-correct or reaffirm yourself on your own and everything is much clearer" or "Very positive because having thought about the exercise before and having your own conclusions, sharing helps to identify your own mistakes and, also, see other points of view on how to interpret the exercise." In the same vein the answer of another group to Question 4 of Questionnaire 1 was: "Yes, seeing the way of understanding the exercise by the other groups and the different contributions is useful to better understand the activity. Further, you get to ask yourself doubts that had not arisen before".

The results of the survey show quite clearly the usefulness of the activity for a better approach to matrix factorisation. We present some answers:

> In general, everything is easier with practice, since it is more visual than if the method is taught in a theoretical and conventional way (Student).
>
> I think the method used is very didactic, since you do not learn by seeing, but by doing; and, sincerely, in mathematics many times opportunities of this style are missed, to be able to approach the problem directly, to pose it with the help of some clues, and to obtain conclusions by yourself, or in this case in a group. I find it really interesting, and the best way to teach mathematics; to force you to think from time to time, instead of giving it all thought (Student).
>
> I think it's pretty good, mainly the part about stating a theorem and seeing how we were wrong (Student).
>
> The sharing of ideas with my peers has made me learn new ways to approach a problem as soon as I see it (Student).

The results of the implementation encouraged novice lecturer's reflection on teaching practice, emphasising the involvement and detection of students' difficulties in learning and managing of matrix factorisations that allow them to efficiently solve systems of linear equations via direct methods. Several challenges are formulated in relation to the awareness of mathematical knowledge in the inquiry process, particularly in the transition of processes experimentation, theorisation, and algorithmisation.

16.6. Implementation Results With Novice Lecturers

Valorising professional development sessions, we tried to answer the following question: what characterises the attitude adopted by novice lecturers regarding the IBME approach? Our analysis revealed three areas in which novice lecturers adopted a critical attitude: towards mathematics; towards learning mathematics; and towards teaching mathematics. These areas were found to align with the stages depicted in Figure 16.1— *discover*, *identify*, *design*, and *develop*—and relate to the learning, practice, and teaching of mathematics. In the analysis of the results we have taken into account three conditions: (1) awareness that the lecturers' experiences may have been different, (2) reflection on that experience, and (3) judgement of the aspects to be included in future teaching improvements.

16.6.1. Critical Attitude Towards Mathematics. In adopting a critical attitude towards mathematics in an inquiry-based learning approach two aspects arose: (1) the understanding of the mathematical content to be taught and (2) the nature of student participation.

In the notion of understanding, 'the capacity for abstraction and complex reasoning,' which entails the understanding of theoretical concepts in action and the updating of those that serve as support for the acquisition of new ones, is required. In relation to the subject of numerical methods, a component of an algorithmic nature was highlighted and characterises mathematics as a sequential content. The contributions on the algorithmic nature of mathematics were of interest:

> The first and most important difficulty students have in Numerical Methods is with the concept of algorithm. In particular, retrieving and using in an orderly and sequential manner the elements necessary for any of the methods studied. This translates into the second difficulty, which is to produce code that implements that algorithm. Students do not, in general, find it easy to break a task down into its elementary parts that can then be coded (Novice lecturer – Identify phase).

These observations provoked a discussion in the CoI about the integration of mathematics as content and process. Processes emphasise the dynamic nature of mathematics, how it is created and how it evolves over time. In this case, the understanding of the concept of algorithm and the mechanisms underlying the process of algorithmisation was highlighted.

Regarding the second aspect, the nature of student participation in undergraduate lessons, two views were made explicit. One was by senior lecturers and professors who stressed that students when they participated in Numerical Methods classes under the inquiry approach, engaged with the tasks and developed shared knowledge. It highlights that students' experience of mathematical meaning and connections can emerge in different ways. There may be different reifications (representations) of the same concept or they may relate to the way in which the same reification can lead to different experiences (tasks and activities) in which learners are involved.

However, as a second view, the participants highlighted more the "lack of curiosity to go into deep and abstract concepts." One of the participants seemed to recognise this when he said:

> The lack of curiosity, understanding this as the pupils' natural response to theoretical concepts and their lack of interest in finding out more than what they have seen. It is therefore the teacher's responsibility to sequence.

Mathematics is reified through classroom tasks and discussions, through procedures, representations and transformations, patterns, relations and connections, theorems and proofs. All of this involves precision in language and the development of skills and competences on the part of the teacher in designing tasks from this point of view. Less awareness of ways of developing meaning through participation is evident in novice lecturers.

16.6.2. Critical Attitude on Learning Mathematics. In this section, we consider the ways in which participants adopted a critical stance toward learning mathematics through inquiry. Two aspects stand out in the discussion because of the contrast between the perspective of the professors or seniors lecturers and novice lecturers: learning is fundamentally experiential and social and learning is a matter of student engagement.

Firstly, learning is fundamentally experiential and social on the part of the participants; learning is more individual and in relation to the subject and mathematical knowledge. In the discussion of the results presented on collaborative work as a means of advancing knowledge, an emerging attitude appeared among novice lecturers: awareness of the social construction of mathematical knowledge. In their own words:

> Something to bear in mind, too, is teamwork. After the discussion and considering our own experience, working in a team enriches the students, as they support each other and exchange information and ideas. This helps to better understand and develop concepts. So, it would be good to develop teamwork strategies for activities and that it can be something transversal (and dynamic, i.e. not always with the same team) where there is a forum even in the whole class to discuss and contribute. Collective learning can be very beneficial, especially with abstract concepts and the complexity they bring with them (Novice lecturer).

Two themes were identified in the discussion: meaningful learning and learning as an active process. There is an evolution of learning towards something more experiential, social, and engaged. In the training session we paid explicit attention to the students' learning process as indicative of an (emerging) critical attitude at university level, given the dominant tendency of lectures. By the end of the training session, the shift from a conceptual understanding linked to content towards a reflection on the more holistic learning process becomes explicit. Students contrasted different ideas about learning, and/or offered critiques of their approaches to learning. We consider that in novice lecturers, variation in learning experience is essential to reflect on, contrast, and criticise one's own past and current experiences.

16.6.3. Critical Attitude on Mathematical Meaning and Processes. Several participants indicated that their practice with engineering students focused on the procedural part and on the theoretical-practical connection, which they found to be an essential deficit in the students. Learning in this context turns into memorisation and the application of previously memorised procedures. However, despite this finding, the novice lecturer in charge found it difficult to design tasks that were open-ended and left students a wide margin for exploration. Figure 16.5 shows some of the items in the sequence proposed in response to Task 3 by one participant (Design of problems and their pacing, so that they can serve to provoke a gradual sequence of conjectures and their eventual refutation or confirmation) (see Section 16.5.3).

In Figure 16.5 one can see how closed and guided the inquiry is. The participation in the CoI and the joint discussion helped to raise awareness of these aspects and to see what questions were left for student understanding and creation.

However, in the beginning the lecturers indicated a shift towards understanding, interpretation, and creativity as important parts of inquiry approaches, where meaningful learning of mathematics is crucial. As evidence pointing to this evolution in

(1) Let A be the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. Obtain its echelon form by Gaussian elimination. The transform the above procedure by multiplying with elementary matrices. Do the same with the matrix $B = \begin{pmatrix} 2 & 6 & 1 \\ 2 & 7 & 4 \\ 1 & 0 & 1 \end{pmatrix}$.

(2) Let A and B be the matrices of the previous exercise. Using the above, get the LU factorisation of both matrices.

(3) Let $\left\{ \begin{array}{rcl} x_1 + 2x_2 & = & 1 \\ x_1 + 3x_2 + 3x_3 & = & 1 \\ x_3 & = & 1 \end{array} \right.$ and $\left\{ \begin{array}{rcl} 2y_1 + 6y_2 + \ y_3 & = & 2 \\ 2y_1 + 7y_2 + 4y_3 & = & 1 \\ y_1 + \ y_3 & = & 0 \end{array} \right.$

(a) Write both systems in the form $AX = b$ and $BY = b'$.

(b) Obtain the factorisation $A = LU$ and $B = L'U'$ of the coefficients matrices of both systems.

(c) Solve the systems $LZ = b$ and $L'Z' = b'$.

(d) Using the above solutions, solve the systems $UX = Z$ and $U'Y = Z'$.

(4) Let C be the matrix $\begin{pmatrix} 0 & 2 \\ 2 & 6 \end{pmatrix}$.

(a) Compute the reduced echelon form of the matrix C by Gaussian elimination.

(b) Repeat the process using elementary matrices.

(c) Is is possible to obtain a factorisation $C = LU$? If the answer is affirmative, compute it, otherwise explain why.

(d) The same question with $PC = LU$ for some permutation matrix P.

(5) Compute the factorisation $D = LU$, where $D = \begin{pmatrix} 3 & 1 & 7 \\ 5 & 6 & 4 \\ 4 & 2 & 1 \end{pmatrix}$.

FIGURE 16.5. Items of Task 3 by one novice lecturer.

thinking, most of the participants, in the beginning, did not consider matrix factorisation to be a difficulty in Numerical Methods and that meaningful work had to be done on this concept in the classroom. When asked about this, they were inclined to give answers such as: :

> I have not singled out matrix factoring as a difficulty, because when we talk about difficulties, we tend to focus more on theoretical, deep and abstract concepts; but it is true that many basic matrix processes tend to fail... I consider that the essential mathematical concepts for this subject are: elementary row operations, elementary matrices, Gaussian elimination and systems of linear equations (question posed in the Definition phase).

At the end of the evaluation session, this same participant indicated:

> After the final discussion and after seeing the implementation in the classroom, I have had doubts and, above all, concerns. I have been able to see that when it comes to setting exercises (tasks), I give quite a lot of guidelines and the activity is quite closed. This can mean that the student's creative capacity is not developed and that he or she does not ask questions to be able to continue investigating. Therefore, as a teacher, I should try to guide the student in the exercises, leaving him or her to reach a conclusion that can then be verified or rejected.
>
> To do this, I have to work on the transition from theory to practice. I feel that I am able to do simple enough steps supported by examples when explaining the theory,

> but when it comes to the student working on his/her own, I am not able to reflect this. I could see that in the proposed exercises there is a lack of freedom to develop ideas, which makes the learner too constrained. Now I am beginning to consider that open questions or reflections by the students will help them to develop that mathematical capacity for argumentation and meaningful conclusion.

We see a reflection on the learning process among the participants and an assessment of the value of inquiry in going deep into 'mathematical meaning' and into the mathematical processes involved in the concept. This opens up for teacher training processes the deepening of the nature of tasks under the inquiry approach where the dimensions of questioning (key questions to move towards the solution), mathematical knowledge, responsibility of learners in the investigation, objectives, etc., are all important.

16.7. Concluding Remarks and Ongoing Work

The case study portrays our experience of development of novice lecturers' professional knowledge and practice regarding inquiry-based teaching. We focused on the methodological strategy that we used to establish operative connections between the processes of inquiry at two levels: *Inquiry in teaching mathematics* (lecturers using inquiry to explore the design and implementation of tasks, problems and activity in classrooms) and *Inquiry in the design of professional development programme* for mathematics lecturers.

The proposal to support novice lecturers in the design by sequencing in four phases—*Discover*, *Define*, *Design*, and *Develop*—has been effective. It has fostered the crucial characteristics of inquiry-based teaching: the origin of questioning, the nature of the mathematical problem, students' responsibility in conducting the inquiry, the management of student diversity, and the explanation of the teacher's goals.

Some challenges in order to support professional development of novice lecturers were raised: what do we mean by an inquiry-based task in mathematics? What is specific to inquiry in mathematical work and differentiates it from another knowledge? In the proposal about matrix factorisation, the conceptualisation of IBME took explicitly into consideration the specific nature of mathematical inquiry and the essential contribution of internal inquiry to the development and structuring of mathematics concepts through experimentation, theorisation, and algorithmisation. These actions contributed to the lecturers' knowledge and competences, but also to the formation of habits of mind for inquiry.

Finally, for us, the ideas raised in this project made us search for a strong interplay between research and professional development activities in relation to IBME. We also want to understand better how communities of inquiry among mathematics lecturers at university level can be established, maintained, and extended via the professional development of novice lecturers.

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CHAPTER 17

Development of a Community of Inquiry Based on Reflective Teaching

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17.1. Introduction

In this case study we give an overview of the development of a local Community of Inquiry (CoI) at Brno University of Technology (BUT) in Brno, the Czech Republic, during the implementation period of the PLATINUM project. The key word of this chapter is 'development' and it concerns history, goals, achievements, challenges and lessons learned of the local CoI. Other important key phrases are 'classroom observations' and 'discussion' that contributed to the professional development of the local CoI members and their teaching practice significantly.

The case study is structured in four sections. Section 17.2 briefly describes the history of the local CoI at BUT (BUT CoI). The purpose of this section is to set the background and context for the next sections, including relationships with another local CoI at Masaryk University (MU CoI) in Brno (see Chapter 13). Section 17.3 describes the formation and development of the idea of Inquiry-Based Mathematics Education (IBME) within the BUT CoI. The purpose of this section is to summarise ideas and goals of the community and its individual members as well as approaches developed, applied and adapted to achieve the defined goals in the local context of the university. Section 17.4 brings a summary of achievements of the BUT CoI during the three years of the project (September 2018 to December 2021) and a list of the most significant challenges met by the CoI within that period. The purpose of this section is also to report lessons learned and a self-reflection of the CoI members about successes and setbacks experienced within the PLATINUM project. Section 17.5 contains a brief conclusion of the case study and an outline of potential future development of the BUT CoI.

17.2. Background, History, and the Team of the BUT CoI

In this section we present the BUT CoI's inquiry into people: building up the CoI team by looking for motivated colleagues who would be interested to get engaged in the IBME project. We report on the background, history and the team of the local CoI at Brno University of Technology.[1] The Central European Institute of Technology,[2] where the PLATINUM project was managed, constitutes the key element of a world-class research infrastructure providing state-of-the-art equipment and ideal conditions for basic and applied research, especially in material science. Teaching activities within PLATINUM took place at the Department of Mathematics in the Faculty of Electrical

[1] www.vut.cz
[2] www.ceitec.eu

Engineering and Communication (DM FEEC). The Department of Mathematics takes care of mathematics teaching for two faculties, the Faculty of Electrical Engineering and Communication and the Faculty of Information Technology. Mathematics courses are usually organised for groups with large numbers of students, ranging from 100 to 800 students with various backgrounds.

History of the BUT CoI started a few years before the PLATINUM project was shaped. The story begins with the series of seminars organised by The Centre for Research, Innovation and Coordination of Mathematics Teaching[3] (MatRIC) in Norway in the spring of 2015 (see Chapter 5). Later that year, the local project coordinator-to-be at BUT gained experience in implementing educational projects through the support of Norway Grants (renamed to EEA Grants[4]). During the years 2015 and 2016, about 20 colleagues and PhD students from the Czech Republic were invited one by one to participate in activities organised within these projects. One of them was Maria Králová who became the local PLATINUM coordinator at Masaryk University (MU). Unfortunately, not many of the other participants demonstrated an intrinsic motivation to put effort in improving teaching and learning of mathematics through educational projects, in particular, no colleagues from the Department of Mathematics.

Taking this situation into account, it was necessary to look for interested participants elsewhere during the preparation of the project proposal. Participation in the project was discussed with two PhD students at the Department of Mathematics and one colleague from the Faculty of Mechanical Engineering (FME) at BUT. All three previously took part in one or more events organised within the Norway Grants. They tentatively agreed to participate in the proposed project and the future BUT CoI. After acceptance of the proposal, the participation of the two PhD students as observers[5] in the classroom was confirmed. On the other hand, the colleague from FME could not confirm participation in the project. The CoI wanted to find one more member. Based on previous experience with successful collaboration, a colleague from Tomas Bata University in Zlín[6] was asked whether she would be interested in participating in the project. The colleague took part in several meetings where the preparation of a joint project proposal had been discussed. She agreed to participate in the project and tentatively promised to participate in some events abroad. So, the BUT CoI line-up had four members by the day that the proposal was accepted and all members had already met people at institutes abroad that participated in the PLATINUM project.

The kick-off meeting in Kristiansand (see Chapter 5) brought a promising launch of the collaboration between the two CoIs based in Brno, the BUT CoI and the MU CoI. This collaboration indeed continued to grow during the first year of the project and is referred to as the Brno CoI. Four large gatherings were organised by the Brno CoI, with topics concerning development of inquiry-based mathematics tasks, organisation of project events, in particular the meeting in Brno in June 2019 (see Chapter 5), and work at the three levels of inquiry in the fundamental model of IBME (see Chapter 2). Besides the large Brno CoI meetings, the BUT CoI had about 7-8 meetings with 2-3 members attending. The purpose of these meetings was to discuss organisation of observation in the classroom, organisation of inquiry-based teaching, and development of inquiry-based tasks. In the second half of the first year of the PLATINUM project, one

[3] `www.matric.no`

[4] `https://eeagrants.org`

[5] By 'observer' we mean a colleague who is present in the classroom/lecture hall during the lesson and collects data about teacher's and students' activity.

[6] `www.utb.cz`

inquiry-based task was tried out with students in three tutorials that were observed. A few more tutorials were observed in which no inquiry-based tasks were present. The outcomes of these observations were very helpful for reflecting on the teaching practice (see Section 17.3). Unfortunately, there is no record of observed data from the spring semester 2019 available, only a template and a few notes made by the teacher based on the oral feedback and discussion. Keep in mind that the PLATINUM project was not completely set up as a research project and that the BUT CoI members had little or no experience with doing educational research.

The end of the first year of the project came with significant changes in the BUT CoI team. Although the meeting in Brno in June 2019 had been perceived as successful and the feedback was positive (see Chapter 5), it became clear that two team members were losing their motivation to continue as members of the BUT CoI. One reason was that they needed to focus on their full-time obligations and did not have enough time to spend on their part-time work as a classroom observer. Luckily, one of the PhD students obtained the degree and became a member of the MU CoI, which meant that the contact between this colleague and the BUT CoI was maintained.

We needed to look for new colleagues to join the BUT CoI, who would do observations of classroom activities. It did not help to ask colleagues from MU CoI as they also had a lack of observers. A few former collaborators were asked, but none was available. Eventually, a new colleague outside the academia was recruited to take responsibility for observing the classroom activity. By the start of the second year of the project, the BUT CoI had three members.

The main focus of the BUT CoI in the first half of the second year was put on inquiry in the classroom activities. A template for classroom observations was developed and many more tutorials/seminars were observed in both Brno and Zlín compared to the first year, some of them introducing inquiry-based teaching units (see Section 17.3). Discussions of what happened in the observed tutorials/seminars were intensive, especially at the four in-person BUT CoI meetings that took place during that period. Traditionally, the nature of the meetings was very informal.

In December 2019, one more formal event took place in Brno. The Brno CoI was hosting the visiting professors Barbara Jaworski and Simon Goodchild (see Chapter 5, Section 5.5, list of meetings). The purpose of the meeting was to clarify the concept of inquiry in mathematics education relevant to the local CoIs (BUT and MU) as well as relating this concept to teaching and learning activities. The visitors and the Brno CoI members participated in classroom observations that were part of the meeting programme. No one could know at that time, but this was the last in-person meeting of the Brno CoI, gathering together about ten participants.

After the BUT CoI meeting in January 2020, part of the CoI moved to an annual meeting with former university classmates. One of them was interested in the project, its activities and outcomes. The contact was maintained during spring 2020, and the colleague became a new BUT CoI member in September 2020 at the start of the third year of the PLATINUM project.

During the third year, in which the COVID-19 pandemic affected the CoI's work, the possibility to meet face-to-face was limited. The local coordinator met with other community members only sporadically. However, the pandemic also brought new possibilities of collaboration. It made the communication between the BUT CoI members more frequent and contributed to strengthening relationships between the members. The frequency of PLATINUM related discussions between the local coordinator and other team members increased to once per week or two weeks. It was more challenging to observe online tutorials, but this also brought a new dimension to the role of observers: without their participation, it would be difficult if not impossible to do experiments and inquire with and in a virtual environment (in our case, MS Teams and MS OneNote). Increased fluency in online communication made it possible to keep a former BUT CoI and MU CoI member who moved abroad engaged in the BUT CoI activities informally. On the other hand, absence of the face-to-face meetings with the members of the MU CoI led to a less intensive collaboration within the Brno CoI. Everyone was too busy with coping with their work under new circumstances and had little time left. Only one virtual meeting of the Brno CoI was arranged in 2020. However, representatives of both local CoIs kept in touch and the status quo seemed to be sustainable beyond the PLATINUM project. By the end of the third year of the project, the BUT CoI had four members and this line-up seemed to be stable enough to continue after the implementation period of the project. There was also a chance that the BUT CoI will have one more member by the end of 2021.

We summarise the development (nature, form and activity) of the BUT CoI during the three years of the PLATINUM project.

First year:

- 4 members (1 teacher in Brno, 1 teacher in Zlín, 2 observers in Brno);
- regular F2F meetings of (only Brno) members (once in a month including joint meetings with the MU CoI);
- discussions about inquiry tasks and event arrangements, discussions about the nature of inquiry in IBME;
- little classroom activity (7 tutorials observed in Brno, 1 inquiry-based task in 3 tutorials in Brno).

Second year:

- 3 members (1 teacher in Brno, 1 teacher in Zlín, 1 observer in Brno);
- less F2F meetings but all CoI members (once in a month before COVID-19), regular online and phone discussions (once in 1-2 weeks during COVID-19);
- visit to the Brno CoI by Barbara Jaworski and Simon Goodchild;
- discussions about inquiry teaching units, evaluation, and inquiry-based mathematical modelling;
- more classroom activity (26 tutorials observed in Brno, 4 tutorials/seminars observed in Zlín, 2 inquiry units in 3 of the 4 seminars observed in Zlín).

Third year:

- 4 members (1 teacher in Brno, 1 teacher in Zlín, 2 observers in Brno) + 1 remote member;
- two F2F meetings of (only Brno) members, regular online and phone discussions (once in 1-2 weeks both during and after the teaching periods affected by the COVID-19 pandemic);
- discussions about inquiry-based mathematical modelling, evaluation and IBME activities in virtual environment;
- virtual classroom activity (40 tutorials observed in Brno, 1 inquiry-based modelling task in 3 recorded tutorials in Zlín).

17.3. Contribution of IBME to Reflective Teaching

In this section we present the BUT CoI's inquiry into mathematics teaching and learning: development of teaching approaches, teaching units, observing templates, questionnaires, and other relevant outcomes.

As far as the BUT CoI members remember, we encountered the idea of IBME and developmental research at the MatRIC seminar in Kristiansand in May 2015 for the first time (see Chapter 5 for details). It was Professor Barbara Jaworski who presented the topics at the MatRIC seminar. Later we had the opportunity to listen to similar presentations given by Barbara Jaworski at several more occasions (Loughborough September 2015, Trondheim November 2015, Brno February 2016, Loughborough September 2016, see Chapter 5). It means that we knew the theoretical basis of IBME and the developmental process well. However, we found it difficult to link the theory to our own practice of teaching mathematics at a university. One of the community members met another practice of IBME before the PLATINUM project had started:

> I first met the concept of Inquiry-Based Mathematics Education in 2016 at the meeting of Czech mathematics teachers in Srní, and then in 2017, I attended a one-hour workshop at an event in České Budějovice. The workshop was based on physical manipulation with geometric objects, observing their properties and classifying them. And also, my first idea of IBME was, in fact, joined mainly with manipulations and 'do it yourself' things rather than mental activities.

The PLATINUM CoI started a practical discussion about Inquiry-Based Teaching and Learning at the kick-off meeting in Kristiansand in September 2018 (see Chapter 5). The fundamental three-layer model of inquiry was introduced (see Chapter 2). However, it was not clear to the BUT CoI members how to put it into practice. One of the members reported in a narrative:

> I did not understand what it actually means, hence I decided to focus on reflecting on inquiry-based mathematical tasks and the particular course where I would use those tasks.

A similar situation occurred at the PLATINUM workshop in Madrid (see Chapter 5), with the tiny difference that the BUT CoI started to realise that nobody would tell us what IBME actually is. The answer to the question "What is the inquiry?" has always been the same: "It depends on what you want to achieve." One BUT CoI member recalled how puzzling it was:

> This time, it was more about inquiry in mathematics education. I was asked a tough question: What do I want to achieve? What is my inquiry? I did not know the answer, and I was assured that nobody else could answer it for me. I had no idea what my inquiry question could be, which on top of all that should be researchable.

Another BUT CoI member recalled similar feelings with a different conclusion:

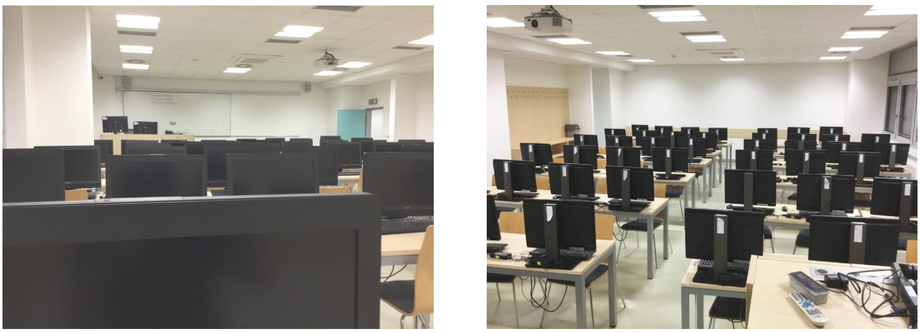
> Though I had no expectations, I must admit that I was a bit surprised that I did not get the answers to my questions yet. It looked like all of us need to reinvent the basis of IBME, namely the notion of inquiry itself, again for ourselves. So what do I think that is inquiry and inquiry-based activities? Now I feel it this way: inquiry-based activities are such that given proper motivation and needed tools, students who are at least a bit engaged are about to act and react in order to solve the given problem.

After the return from the Madrid meeting, it was time for the first observed experiment at BUT. At the beginning of the project, there was a discussion about collecting data and GDPR. BUT CoI decided to do classroom observations without video recording to avoid necessary formal requirements. Observations were to be done in the form of collecting anonymous data in the format of written notes. A template for classroom observations was proposed and agreed.

The classroom setup chosen for the observations was a tutorial with 30-50 first-year electrical engineering students, mostly male, in a computer lab (see Figure 17.1). Each student has an all-in-one computer, a keyboard and a mouse. There is little space for a notepad, textbook, etc. Screens are large, so students can 'hide' behind them. The teacher desk stands on an elevated platform in front of the classroom, 50 centimetres above the rest of the room. One whiteboard located in the front can be slid in the left/right direction. If it is slid to the right, the projection screen covers most of the whiteboard.

The topic of the observed tutorials in the first year of the project was calculus of complex functions in one variable and complex contour integration. One inquiry task was handed out to the students: the complex cosine (see Figure 17.2). The task was intended to be flexible enough to make it possible for the students to choose the level of inquiry that suited them: structured inquiry, guided inquiry, or open inquiry (see Section 6.2.1; Banchi & Bell, 2008; Wenning, 2005). The task was presented in the way of guided inquiry, that is, the teacher provided questions only and it was up to students to find their own approach to the questions. The order of the questions was partially structured, but the students had the freedom to start with any question with the possibility to introduce their own questions. If they did not get an idea how to proceed, the students could use support of the teacher.

The outcomes of the experiment were mostly as expected—students tried to "guess the correct answers"—with a few exceptions regarding students' own questions. However, as one of the observers reported, most students actively engaged with the task: "We were pleasantly surprised that students actively discussed prepared questions." The word *actively* here means that the students actually were working and discussing



Student view Teacher view

FIGURE 17.1. Computer lab setting for tutorials at BUT, DM FEEC.

FIRST NAME & SURNAME

The function of complex variables $\cos z$ is defined as follows:

$$\cos z = \frac{e^{jz} + e^{-jz}}{2}$$

Questions:

(1) What can we feed as input into $\cos z$? (What is the domain?)
(2) What can be at the output of $\cos z$? (What is the range?)
(3) Does the function $\cos z$ have any zeroes (points where $\cos z = 0$)? If so, where are they located?
(4) Does the function $\cos z$ have a derivative? If so, where (on which subset of the domain)?
(5) Is the function $\cos z$ somewhat " related" to the real function *cosx*?
(6) How can we imagine/visualise the function $\cos z$?
(7) My own question (to the topic):

FIGURE 17.2. An inquiry-based task on the complex cosine function.

the prepared questions, which was in contrast to the usual students' practice of copying information written on the whiteboard passively.

Questions and ideas related to inquiry in teaching started to emerge after a few observed tutorials. The observations were literally an eye-opener. The teacher described his experience as follows:

> If you practise the 'traditional way of teaching'—calling students to the whiteboard or calculating yourself—and there is no one observing the classroom activity, you have absolutely no idea what is going on in the classroom!

The observation we considered as particularly important was that when the teacher dealt with a student at the whiteboard and the progress was slow, other students in the classroom got bored quickly. To prevent this undesirable behaviour, a sequence of modifications of teaching activities was made up and tried out. The experience was summarised in the following narrative:

> So I started to call volunteers. It worked for one tutorial. Then they stopped raising hands. So I started to ask students for suggestions and I did the calculations following their suggestions. If it didn't work, we started again with another suggestion. It worked for one tutorial. Then they stopped giving suggestions. So I posed the problem and let them reflect for a few (2-3) minutes. Then I asked for suggestions, and for each suggestion I called a volunteer to the board. It worked for one tutorial. Then they stopped giving suggestions. So I posed the problem and let them work on it for longer time (5-10 min). I walked among the students, gave suggestions and tips and I was available for questions. When a student was done, I asked them to present their solution on the whiteboard.

Another observation, important for the students, was that the teacher should speak loudly with face turned to the students in such a large classroom and had to learn writing in an accessible way (capitals, large symbols). After an observed tutorial, a brief meeting of the BUT CoI usually took place. The purpose was to give immediate feedback to the colleague responsible for the tutorial. Sometimes a longer discussion took place, for instance about the nature of inquiry. One of the observers expressed the opinion that "inquiry is a form of self-evaluation." One of the teachers recalled the struggle with inquiry during the spring semester 2019:

> After about a month of frustration, one day I woke up in the morning and I knew what my inquiry is. There were actually two of them. My primary inquiry is concerning 'weak' students. Why do they do incorrect steps? Don't they understand the algorithm? Don't they understand which objects they are dealing with? Don't they know the properties of the objects? Don't they know what is it good for? My secondary inquiry is about the students without any difficulties. What to offer them that could bring benefit to them and promote inquiry at the same time?

At the end of the semester, the CoI discussed a possibility of students' feedback. The survey was partly closed, asking to rate particular teaching activities, partly open, asking the students to express what was good and what could be improved (see Figure 17.5 for an example of a short questionnaire). Analysis of the responses showed that the most and the least popular/liked teaching activities were what we anticipated: the most popular activity was that the teacher performs all calculations on the whiteboard and the least popular activity was that the teacher calls students one by one (not by names) to the whiteboard to work on tasks. The inquiry task shown in Figure 17.2 was the second lowest rated. One reason for that might be that the students were not used to such type of tasks and teaching activities and preferred to stick to what they were used to.

One of the observers reported about impact that the activity in PLATINUM had on her teaching:

> Thanks to the knowledge from the PLATINUM project I continued trying to change my way of teaching. My main goal was to show students that at least mathematical basics aren't hard to understand and there is nothing to be afraid of. I was focused less on pace in teaching during my classes (sometimes it was necessary to go faster due to big amount of material for one class), but more on students' understanding of concepts, on their ability to connect one topic with another. And even though it was still with mainly basic exercises from previous years, I tried to make it at least a bit inquiry and not only receiving of facts.

Another inquiry emerged after the spring semester had ended. The teacher recalled:

> After a further month of reflection and a discussion with Yuriy (Rogovchenko), I came up with another inquiry. This time, it was an inquiry in teaching arrangements. Given the context—room arrangement, equipment, number of students, topic to be taught—how to arrange activities in the classroom so that the students, I mean, my students, get the optimal combination of learning content and learning experience?

This inquiry indicated one of the directions/primary goals of the BUT CoI for the second year of the project. It also matched the idea of differentiation, defined by Petty (2014) as "the process by which differences between learners are accommodated so that all students in a group have the best possible chance of learning."

Due to the BUT CoI changes, a new observer in Brno was recruited at the beginning of the fall semester 2019 (see Section 17.2). The new colleague recalled that in the beginning she had doubts about her contribution:

> I was aware that I am not a student that would understand mathematics through teaching methods that are applied at universities standardly. I thought that I am not able to contribute to the project.

The renewed CoI discussed what could be its inquiry into teaching and learning, and what kind of outcome of the observations would be helpful for that inquiry. One of the members recalled:

> In the beginning, we wanted to know what the students actually do during the class. Who interacts with the teacher or with the neighbours, who shows any reaction to a teacher's activity and how they react, and what the students do when they don't participate actively in the class.

One of the main purposes of the inquiry into the student activity was to find out how to engage the students in the classroom activity, in mathematics. It turned out that there is a need to study/measure two main types of activity: activity of the teacher and activity of the students. Both types were considered worthwhile to be studied in more detail, as well as interactions between the teacher and the students and amongst the students. A template, available on the PLATINUM website `https://platinum.uia.no`, was proposed that would be convenient to collect the data that was considered as useful: location of the students, who communicates with whom, whether they use PCs or not, what they use the PC for (learning or other activities), whether they use mobile phones and for what. This type of data provided required information—how students reacted on various attempts of the teacher to engage them in the classroom activity—and served as a source/reference during the after-class discussion because it was easier to talk about particular students according to the location they occupied. The template was developed through the semester and further data fields were added: number of sets of tasks, number of tasks in each set, how many times did the teacher walk around the classroom, how many students did the teacher talk to, how many students raised hand when a question was posed, how many times did a student ask the teacher for help, and how many times did a student take a photo of the whiteboard.

Observing the classes and analysing the classroom activity had a lot of consequences to form and content of the teaching activities. It was possible to see which students prefer collaborative work and which prefer to work alone, what resources they use, how they find them, and how they work with them. It was clear immediately after the first tutorial that the teacher must come to the classroom a few minutes before the class and air out the whole classroom. The observers' work had essential impact to the change of teacher's behaviour. The observations indicated that the teacher should possibly

- speak loud and clear, with confidence and face turned to the students;
- walk around the classroom and talk to students even if they do not ask;
- repeat briefly the main points/highlights of the lecture;
- write the list of tasks for the next week on the whiteboard;
- write in capital letters, and as large as possible;
- leave the students a short time to reflect after posing/asking a question;
- look prepared and determined; and
- formulate instructions clearly and in simple terms.

In the middle of the fall semester, the BUT CoI was invited to visit and observe two inquiry-based seminars at Tomas Bata University in Zlín (see Section 17.2). The classroom setup of sessions observed at the university in Zlín in November 2019 was different from the setup at BUT in Brno. The format of the class was a seminar with elements of lecture, tutorial, and group work. Each group consisted of about ten first-year engineering students, both male and female. Students had no computers, which left enough space on the desk for resources and notepads. The teacher's desk was on the same floor level in front of the classroom, attached to the first row of student desks. Two whiteboards located in the front and a projection screen were arranged such that they did not cover each other (see Figure 17.3).

FIGURE 17.3. Student view within the classroom setting at Tomas Bata University in Zlín.

The topic of the inquiry-based teaching unit observed in Zlín was an introduction to derivatives and is available on the PLATINUM website. The main aim of the lesson was to introduce the concept of the derivative of a real function of one variable at a point with an emphasis on understanding its geometrical interpretation. Four inquiry tasks were given to the students during the seminar. The students first worked alone but were encouraged to discuss with neighbours and the teacher, and gradually worked in pairs or small groups. The nature of the inquiry was guided, structured, and scaffolded. The inquiry-based seminars were each time followed by one more inquiry task in the lesson next day.

A brief CoI discussion took place after the two observed seminars. The observers from Brno found the seminars interesting and inspiring in both the format and the content. The combination of lecture, tutorial and group work, together with the more personal approach of the teacher to the students facilitated to maintain the dynamics of the lesson. A positive observation was that all the students actively participated in both seminars. After the observers' notes were shared with the teacher, she could not hide surprise at certain points: "Have I really said/done that?" It suggests that sometimes teachers are so immersed and excited during the teaching activity that they do not perceive/remember all details about what they say.[7]

The visit of Professors Barbara Jaworski and Simon Goodchild to the BUT and MU CoIs (see Section 17.2 and Chapter 5) had significant impact on the BUT CoI, its goals and activities. An important outcome was the decision to split the case studies of BUT and MU (see Chapter 13). Further, all participants had the opportunity to observe an inquiry-based task in a statistics tutorial at MU. Finally, the BUT CoI hosted Barbara Jaworski and Lukáš Másilko (from the MU CoI) as observers in one tutorial. This unique experience brought two important reflections: (1) feedback from outside is very useful regardless of language barriers, and (2) the students behaved and performed better in the presence of guests. The CoI members had an inspiring discussion with Barbara Jaworski after the tutorial. The topics were how to bring more inquiry into the lesson about infinite series, how to arrange activities so that the students come prepared and are ready to discuss mathematics, and how to avoid/replace manual assessment of a large number of written tests/exams. Based on the discussion, the goal for the spring semester 2020 was set, namely to try the flipped classroom principle (Fredriksen, 2020) to create time for student inquiry in learning mathematics in class. Although there was no possibility of using Computer Aided Assessment (CAA) systems for assessment due to the subject arrangements, the BUT CoI considered to at least try to support students' learning at home by CAA. The idea was supported by the classroom observations that revealed that students use computational software like computer algebra systems anyway.

Checking the students' results in the final exam brought a surprising insight: the form of active learning (Freeman et al., 2014) practised during the fall semester did not have any significant impact on the students' results. This was perceived as a failure because the CoI expected that more active engagement of students would bring improvement in results. Based on the chosen goal, the CoI started the spring semester 2020 with a combination of active learning and flipped classroom. However, neither this combination worked as expected. Although the students were told that they must come prepared for the next class and got clear instructions, most of them did not have a look at the specified topic. However, the students seemed to be OK with solving the given tasks themselves in the classroom. One student commented: "Better a tutorial where I solve tasks than a tutorial where I just sit."

The next in-person meeting of the BUT CoI took place in Zlín in March 2020. The CoI met in Zlín to observe two seminars, one of which was planned to be inquiry-based. Classroom setup of these sessions (shown in Figure 17.4) was similar to the setup in fall 2019 (see Figure 17.3 on p. 315). The format of the inquiry class was a seminar with elements of lecture, tutorial and group work, while the other class was close to a standard tutorial with students coming to the whiteboard one by one. Each group had about fifteen first-year engineering students, both male and female in the inquiry

[7]It seems there has not been done much research in this direction yet.

class, and only female in the other class. Students had no computers. The teacher desk was more or less on the same floor level (but not above) in front of the classroom either attached to the first row of student desks or with a small gap between. Two whiteboards were located in the front and a projection screen was arranged such that it did not cover the whiteboards.

FIGURE 17.4. Another classroom setting at Tomas Bata University in Zlín for a seminar.

The topic of the inquiry-based teaching unit observed in Zlín was an introduction to definite integrals and is available on the PLATINUM website. The main aim of the lesson was to introduce the concept of the definite integration and its geometric interpretation. Four inquiry tasks were given to the students during the seminar. The students first worked alone with tablets provided by the teacher but were encouraged to discuss with neighbours and the teacher, and gradually worked in pairs or small groups. The nature of the inquiry could be labelled again as guided, structured, and scaffolded, but the first two tasks actually allowed students to explore a variety of approaches. The content accessed through the tablets was arranged in the university learning management system (MOODLE).

A brief CoI discussion took place after the two observed seminars. Again, the observers found the inquiry seminar interesting and inspiring in both the format and the content, and the combination of lecture, tutorial and group work, together with the more personal approach of the teacher to the students facilitated to maintain dynamics of the lesson. The non-inquiry seminar was also inspiring with respect to its format, but not regarding its content, namely, integration by partial fraction decomposition. In that seminar, the students tended to use mobile phones for communication outside the classroom more often. One of the outcomes of the discussion was the conclusion that there is no universal way of teaching that would be optimal for every student and under all possible circumstances.

The first COVID-19 lockdown happened in the middle of the spring semester 2020. The teacher and students in Brno benefited from the choice of the flipped classroom approach because the students exposed to this approach adapted to individual work at home quickly and were less shocked than the students taught in the traditional way. Although some students did not like the flipped classroom approach before the COVID-19 pandemic, they admitted that it helped them during the lockdown: "I did not like the (flipped classroom) activity, but it was a good preparation for the distant learning." The students said that initially they did not like the flipped classroom approach because they spent time working at home and it took them longer time than working in class.

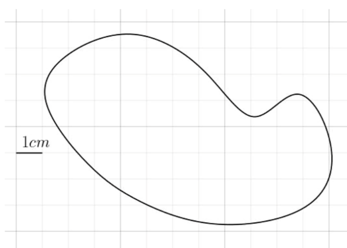
After the spring term 2020 affected by the COVID-19 lockdown, the CoI discussed the teaching experience and students' results. One of the teachers expressed the satisfaction with students' results because 84% of the students passed the exam. That led to the following discussion about the teaching goals which contributed to the change of that teacher's perspective. Teacher 1: "Once you give marks to students, it changes your point of view towards the goals. In particular, no longer will be your goal (just) that the students pass the exam." Teacher 2: "True. If a talented student comes to try to pass without effort and be satisfied with E, I will let them come back again."

Another outcome of the observation visit to Zlín was the development of an evaluation tool for the inquiry-based teaching unit *Introduction to Definite Integrals* observed in the classroom (see Figure 17.5). The purpose of the questionnaire was to evaluate students' understanding of the concept of the (Riemann) definite integral. Forty-four students filled in the questionnaire between 2 and 3 months after the observed seminar. Their answers showed that more than 90% of students had successfully formed an idea of the concept and symbolisation of a definite integral and what can be calculated with it. One third of students were able to combine the idea of the definite integral with a practical task. But most would still prefer a less sophisticated method/procedure to calculate/estimate the area of a general shape. According to the results of Question 3, students (almost 90%) did not understand the underlying concept on which the definite integral is based or did not absorb/recall the terminology used. Concerning the work with tablets, most students responded that they were used to digital technology, which supported inclusion of such activities in the teaching design.

Questionnaire – Definite Integral

1. What do I understand/imagine under the notion/symbol of the „definite integral" $\int_a^b f(x)dx$? What do I expect/imagine that I am able to calculate using the „definite integral"?

2. How would I proceed to find/determine area of the shape inside the curve in the picture?



3. What comes to my mind when I see/hear the term „upper/lower sum"?
4. In some classes/seminars we worked with touchscreen devices. Rate the level of work with tablets and electronic materials.

☐ Very easy ☐ Easy ☐ Normal ☐ Difficult ☐ Very difficult

5. What would help me understand/learn math better?

FIGURE 17.5. The questionnaire related to the teaching unit *Introduction to Definite Integrals.*

After summer holidays in 2020, the focus of the BUT CoI discussions changed towards mathematical modelling. The reason was that BUT was one of the three partners in the project Intellectual Output (IO) *Mathematical modelling teaching resources from real-world problems in business, industry and society* (IO5, see Section 5.4

for the list of IOs). It is important to say that BUT was the partner that had least experience with using mathematical modelling as a learning activity. Two CoI members participated in regular meetings of the IO5 team that included up to nine participants from four universities in four countries (see Chapter 8). At that point, there was a lot to learn from the other partners, but less to contribute: theoretical ideas and personal reflections/opinions without practical experience.

Based on a real-life application, one inquiry-based modelling task was proposed, where a wheeled robot was supposed to be programmed to detect source(s) of radioactive radiation in an unexplored area of polygonal shape. The students would be encouraged to compare different solution strategies and determine an optimal trajectory of a robot to scan at least 50% of the area at least twice. Expectation was that the students would use their knowledge of linear algebra, calculus, optimisation methods, control theory, and engineering to deal with the task. However, that also implied that the task should be presented to students at master's level which none of the teachers in the BUT CoI taught or was going to teach.

Discussion about practical experience in mathematical modelling at the BUT CoI was set aside because of the start of the new semester. Fall 2020 brought another type of inquiry: inquiry into challenges induced by COVID-19 restrictions. One of the former observers who kept in touch while living abroad described the spirit of the COVID-19 restrictions period as follows:

> I've stayed in contact with my former colleagues as all of them were working online anyway, so it was easier and more natural for this time to be in touch even while being in different countries.

At both universities where the BUT CoI members worked, there was an inquiry into remote teaching activities and students' remote learning. Discussions within the CoI were often related to the 'degree of freedom': while teaching arrangements were up to the person responsible for a subject in Brno, there was a whole university teaching policy in Zlín. The teachers in Zlín had to be online during the scheduled classes and had to actively engage with their students for at least half of that contact time. The colleague teaching in Zlín had chosen a blended approach: lecture-like content delivered to the first group of students was recorded while in other parts of the class students worked on tasks with the possibility to ask and discuss with the teacher, but those parts were not recorded. The other groups of students watched the recorded video of the lecture-like content and then again worked on tasks with the possibility to ask and discuss with the teacher. It appeared to be a good practice for the smaller groups of 10 to 20 students in Zlín.

The BUT CoI grew a little at the start of the fall semester 2020 (see Section 17.2). A new colleague in Brno joined the CoI as an observer. She expressed her motivation to join the CoI as well as her feelings about it in a narrative:

> I have spent 16 years on maternity leave with 5 kids. I did not think I would ever get back to university mathematics. I have reflected for a few years on where and how would I find a job. An invitation to participate in the PLATINUM project came unexpected and sooner than I could start looking for a job. The idea of an inquiry-based approach fitted perfectly to my idea of effective learning. My enthusiasm that I can do something that I like so much was mixed with my worry whether I will be able to do that.

One of the CoI discussions was related to development of an evaluation tool for the inquiry-based teaching unit *Introduction to the Derivative* developed and tested the year before in the classroom. The purpose of the questionnaire was to reveal students' understanding to the concept of a derivative. The questionnaire was designed and used in a similar way as the questionnaire on definite integrals (Figure 17.5). Although the results of the questionnaire showed that the students' understanding of the concept was lower than expected, the questionnaire itself proved to be a useful tool: it helped to identify which conceptual details were not clear to the students and how the teaching unit could be improved to fit better the virtual teaching and learning environment (in our case MS Teams). The discussion of mathematical modelling in teaching mathematics was reopened when the derivatives were introduced and practised in Zlín. The teacher in Zlín proposed a modelling task: "Given the volume of 0.5 litre of a liquid, minimise the material needed to make a can that would contain the liquid." This task was tried out in three lessons with groups of 6 to 12 engineering students in a virtual environment. Experience from the three lessons as well as the teacher's reflection is described in Section 8.4.2 in more detail. Introduction of online tutorials in Brno in the second part of the semester brought an opportunity for observers to participate in the virtual meetings and make notes/collect data. Most of the tutorials had a stable proportion of participants: about 20% of all students in the group. That makes less than half of regular participants, compared to the in-person lessons. Sometimes it was difficult to engage students. The teacher reflected on the experience: "It was frustrating to talk to the screen, seeing nobody and hearing nobody, and getting answers to only the simplest procedural questions after two or three invitations." On the other hand, the observing CoI member wrote on the observation sheets:

> I realised that some things are more important in online communication. In the real classroom, it is good if the teacher describes what is going on. The students can see that the teacher is reflecting or preparing to write something on the whiteboard. Similarly, after a question is posed, the teacher can see if the students need more time, or if they are not preparing to answer/reply. In the online tutorial, especially by screen sharing, it is important that the teacher comments on everything. Otherwise, awkward silence can occur.

An advantage of the online tutorials was that the time was announced and a meeting in MS Teams was arranged in advance, so the students could choose to contribute to the content of a tutorial by asking particular questions in the MS OneNote classroom notepad or in the MS Teams team channel. However, only a very limited number of students made use of that opportunity. Most of the students preferred either a private chat with the teacher or to remain silent observers. Yet the students saw benefit in the online tutorials, so the teacher's initial aversion turned into fondness in the end.

17.4. Challenges, Achievements, and Experiences of the CoI

In this section, we summarise experiences of the BUT CoI from five semesters of reflective teaching. We present the BUT CoI's inquiry into community: lessons learned by the teachers and observers. A summary of challenges encountered, the means and measures taken to get over the challenges, and the information about which of the challenges have been successfully overcome, may provide deeper insight into the character and experience of the local Community of Inquiry. Reflections of the CoI members illustrate what they perceived and how they thought about activities and ideas related to IBME and PLATINUM. We also include narratives and ideas related to the impact of the COVID-19 pandemic on education.

17.4.1. Challenges, achievements, and experiences in CoI and IBME development. We present a summary of challenges encountered, how they were overcome, and what was the CoI's experience with IBME.

Challenge 1*: Building up a local CoI.*

Achievement: The working style of the BUT CoI has always been informal and open. Thanks to careful considerations of whom to invite to join the local CoI (and a little bit of luck), the CoI has become sustainable and keeps growing slowly.

Experience: It is essential to find colleagues who have intrinsic motivation in teaching mathematics or mathematics education in general. It is far less important how far they live. In fact, the BUT CoI is institutionally independent as it is based on trust and personal relations.

Challenge 2*: Linking the theoretical model (Chapter 2) to the CoI's practice.*

Achievement: The CoI's understanding of the fundamental three-layer model is more intuitive and less theory-based. However, the CoI members agreed that it is important to include all three layers into the CoI activities. That has been achieved by including the observers into the community and feeding the outcome of the observations back into teaching practice.

Experience: For other colleagues than those into mathematics education research, it might be difficult to understand the theory in full and/or link it to their own practice. However, it is important to never give up and keep discussing within the local CoI as well as with other colleagues outside the CoI, within or outside the institution.

Challenge 3*: Defining goals of the local CoI ("What is my inquiry?").*

Achievement: It was not easy to define goals for the CoI when the structure of the CoI is not hierarchical (see Chapter 3, spiderchart *group of inquiry*). However, the main goal of the CoI members was clear from the very beginning: to develop their own teaching practice through engagement with different types of inquiry. The BUT CoI case study presents the history of that development.

Experience: The development of CoI goals usually reflects understanding of the three-layer model (Challenge 2). It might happen to start with the question "What IBME tasks could we prepare for the students?" and continue through "How can we improve our teaching practice to enhance students' learning/conceptual understanding?" to get to "How do we evaluate the outcome of a lesson where an IBME teaching unit was applied?". It is realistic to expect going even further in the future.

Challenge 4*: Finding space for IBME tasks in a tight teaching schedule.*

Achievement: Heavy teaching load was an issue in the CoI because the two mathematics departments in Brno and Zlín are perceived as 'service teaching' departments. Yet the CoI experimented with inquiry-based tasks in two different classroom setups and in a virtual environment as well.

Experience: Everyone can make it if one wants to. There is always some space to develop and test at least one task that is in agreement with the teaching syllabus. It is easier to include inquiry-based teaching units if the teaching schedule is less tight and one has freedom to choose the teaching format and content.

Challenge 5*: Developing the process of observing in the classroom that would be as simple as possible and provide the information needed.*

Achievement: The BUT CoI found an observing practice that is convenient for achieving the goals of the CoI and that is in compliance with institutional rules and current legislation (GDPR). Observation templates not including personal information were developed that could be easily adapted to any class of up to seventy students.

Experience: It is an advantage if classroom observations can be made regularly. New observations can lead to innovations in the classroom activity: modifications of content, changing teacher's behaviour, introducing new activities to engage more students or to engage them in more depth.

Challenge 6: *Engaging students in the classroom activity including those who actively resist to various forms of engagement because they do not want to be engaged.*

Achievement: The CoI observations suggested that between 10% and 20% of students did not appear to be engaged. The student resistance was overcome successfully by convenient arrangements. The 'proof of concept' was observed during the visits to Zlín. When there was a small group of students in a classroom on campus with an active teacher, it was difficult for the students to resist. The teacher knew each student by name and could recognise in a few lessons if there was any problem.

Experience: Students often do only what they have to. It is therefore important to decide what *we* (teachers) want them to learn/achieve/develop, and to foster it by arranging the conditions to pass the subject so that they fit the teaching goals.

Challenge 7: *Designing inquiry-based activities that encourage students to learn and promote students' conceptual understanding.*

Achievement: The CoI developed and tested two IBME teaching units, one inquiry-based modelling task and one self-standing inquiry task. Another inquiry-based modelling task has been developed, but not tested yet in practice.

Experience: Compact IBME teaching units seem to have largest positive impact on conceptual understanding of the students. Modelling tasks seem to promote collaboration/group work also in a virtual environment.

Challenge 8: *Including digital technology into the IBME activities.*

Achievement: One of the IBME teaching units developed by the BUT CoI was handed out to students through the Learning Management System (LMS) Moodle. This unit implicitly assumes that students have access to convenient digital technology. The unit was tested in Zlín where the students used touch devices brought to the class by the teacher.

Experience: Modern computational software is capable to perform many of the calculations that we traditionally ask students to do with pencil and paper. It is beneficial for both students and teachers to explore the possibilities of such software and incorporate its use in educational activities rather than pretend that such software does not exist. It seems to be more acceptable by both students and teachers to use software nowadays than it used to be before the COVID-19 pandemic.

Challenge 9: *Developing questionnaires and surveys convenient for evaluation of conceptual understanding and for collecting students' feedback and suggestions.*

Achievement: The CoI prepared a number of surveys, both on paper and online, to collect students' views on teaching and learning activities, both in-class and online, on the study of mathematics for engineers in general, and on their self-evaluation. Two evaluation questionnaires were developed to examine students' conceptual understanding acquired through the inquiry-based teaching units in a long term. Data collected were both quantitative and qualitative.

Experience: Feedback from students is very important. They often have a clear idea what could be helpful for them in learning mathematics. That way the students can contribute to teachers' better understanding of student needs and optimisation of teaching and learning activities.

***Challenge 10**: Finding ways to share our experiences with colleagues who are not interested in inquiry-based mathematics education.*

Achievement: The BUT CoI has not succeeded in addressing this challenge yet. We are aware that in the busy schedule of service departments, this is a big challenge.

Experience: It is often easier to share experience with colleagues who do not teach mathematics but teach other subjects, or do not teach at all. People who are not into mathematics are often interested in approaches to learning and teaching mathematics that they are not familiar with.

17.4.2. Reflections of the CoI members about their engagement in the PLATINUM project and IBME. There were many 'lessons learned' by individual BUT CoI members. These elements contributed significantly to the development of the CoI as a whole. Some of the observations and reflections can be found in the following collection.

***Reflection 1**: Importance of a non-teacher perspective.*
"It is helpful to engage people who do not teach mathematics, and perhaps who do not teach at all, in design of learning and teaching activities. They have different experience with mathematics, which is difficult to imagine if one has never gone through it."

***Reflection 2**: Complementarity of the students' perspective.*
"I was pleased that the experience from my own education—so different from the other colleagues in the BUT CoI—can be helpful to deal/work with some of the students who might have similar experience/attitude as I had when I was a student. I remember where I made mistakes and what piece of information I skipped as not important. For example, thank to that experience we were able to design tasks for tests and exams in such way that it was more difficult to copy solution from others." (cf., Section 12.5)

***Reflection 3**: Advantage of international collaboration.* "I like the overall idea of an international project in education. Eight universities from seven countries work with the same idea and each team (local CoI) fits the idea to local conditions and investigates it in its own way. Teams and the members do not compete but collaborate. My favourite childhood TV show was 'Games Without Borders',[8] and PLATINUM is like the highest level of it where the whole international community wins."

***Reflection 4**: Observer's feedback to the teacher's activity.*
"I liked observing the teaching activity when the teacher gives a set of tasks and then walks around the classroom and supports students in their efforts. I especially appreciated that the teacher kept explaining the students that it is good for them to ask questions. What a contrast to my experience from primary and secondary school where asking questions was considered a weakness or a lack of ability!"

***Reflection 5**: Students' struggle to accept 'learning by mistakes.'*
"It was difficult to pass to the students the idea that humans learn by mistakes and that they should perceive mistakes positively to some extent. A mistake would make the students reflect on why did they made it and what to do to prevent it next time. We invited them to practice clapping hands in couples in specific patterns in a fast pace, a game we have learned at a design thinking workshop. However, the students rather slowed down the pace to avoid making a mistake."

[8] https://en.wikipedia.org/wiki/Jeux_sans_frontières

Reflection 6*: Consequences of denying 'learning by mistakes.'*
"We observed that this 'fear of making a mistake' led to a sequence of undesirable consequences:

(1) If I don't know, I don't even try.
(2) If I don't try (at home), I don't know what to do in the class.
(3) If I don't know what to do in the class, I don't ask because it might reveal that I didn't prepare at home.
(4) If I don't ask, I miss the chance to clarify and understand.
(5) If I don't understand, I am leaving it until the preparation for a test/an exam.
(6) If I have left it to the time when I prepare for the test/exam, I don't ask because it will reveal that I did not study what I should have learned.
(7) If I have never asked and understood, I don't pass the exam.

Such behaviour—postponing a problem instead of facing it—is a waste of time and harms the student."

Reflection 7*: Lack of interest of the (engineering) students in learning mathematics.*
"Part of the students who do not engage in the classroom activity occupy themselves with other subjects. Perhaps they consider other subjects more important than mathematics. However, doing this they are not taking the opportunity to learn what is going on in the classroom right now. Another part of the students is active on social networks or browses the internet thoughtlessly. That is the ultimate waste of time. These two groups might represent about 20–30% of the students."

Reflection 8*: Students' (non-)readiness to participate in tutorials.*
"Many students come unprepared to the tutorial. One of the reasons that the students are not prepared might be the size of the group. The students might have the feeling that they do not need to prepare because there are many other students to be asked to say something. Reducing the group size might help. On the other hand, students who come prepared to the tutorial are often prepared to help other students in the classroom."

Reflection 9*: First-year students' (dis)orientation in how to study.*
"The students are in their first semester at the university. They have no idea how to study. Some of them cannot 'google' the resource even if I write the query on the whiteboard. They don't know how to work with formulas that they have on their cheat sheet, or how to use one task to find solution to a similar one. I spent a lot of time to teach them how to study. I think that it might be the biggest outcome of the mathematics tutorials for some students."

Reflection 10*: Students' quick accommodation to the observer's presence.*
"It is not necessary to sit on a chair among the students. The students perceived my presence in the very beginning, but after the teacher explained the purpose of my presence, I became unnoticed, like a ghost. I moved along one side wall and the back wall to have a good view of their screens. I did not talk to them and they did not talk to me. Definitely, my presence did not scare them off from browsing the internet, pursuing other subjects or doing nothing at all."

17.4.3. Changes in the work and life experience of the CoI during the COVID-19 pandemic. About half of the PLATINUM project took place during the COVID-19 pandemic, which brought challenges to people's lives that had to be taken into account seriously. The BUT CoI members commented on issues related to keeping work/life balance, switching to digital teaching/learning environments, increasing volume of online communication and longing for on-campus activities in the following narratives. One of the big new challenges was how to keep a good balance between work and life. A CoI member reported:

> The biggest challenge is the time management: how to find a place and time for work at home, and to keep enough time for the children and my own sleep. Especially now, when the schools and free time activities are open and closed in various combinations every week, it is difficult to find time for regular work.

Due to COVID-19, new arrangements had to be made, resources prepared, and equipment purchased in order to provide students with teaching and learning activities in different learning mode. Another CoI member noticed:

> EU and other countries have spent a lot of money on digitalisation of education, supporting financially projects 24 or 36 months long. Now, most of the educational institutions will switch to digital education within one or two weeks, maybe a month, without any additional money from the EU or governments.

Then the question arises what to do with all that after the COVID-19 pandemic, as one of the CoI members noted:

> I'm very curious about the future education after the current pandemic. It has changed the relation to online communication for a lot of people. I can say from my experience that even being far away from the country where my relatives live, I helped my husband's niece with non-university level mathematics. It didn't happen in times before pandemic as I wasn't used to online learning/teaching where it is necessary to share notes/drawings. But it turned out there are a lot of already existing nice tools for that. I expect it happened to a lot of people and because of that, I'm interested in how it will affect consulting hours and (mathematics) support centres[9] after coming back to teaching in person. Whether it will be at least partially online or not as it's very easy, quick and convenient sometimes.

Another CoI member added a reflection concerning the student perception of on-campus education:

> There is a chance that after the campuses will open again and the students are back to the classrooms, they might appreciate the on-campus activities more than before the COVID-19 pandemic.

17.5. Conclusion and Future Development of the CoI

First of all, we can conclude that participation in the PLATINUM project was interesting for and contributed to professional development of all BUT CoI members. We encountered a lot of challenges and uncertainties. We were glad that we could always rely on support from more experienced colleagues from partner universities. We learned that any educational activity must take local conditions and circumstances into account. It was also inspiring and enriching to see how other partners deal with the challenges related to IBME in higher education. We noticed that it was not easy to find colleagues within our institutional environment who would be motivated to join us in the professional development. On the other hand, the form of our CoI allowed us to include members from different institutions who want to discuss and share experience with each other. Essential outcomes we see in the process of classroom observations and subsequent reflections. For most of us, it was an interesting experience to observe the classroom activity with a bird's eye view and get insight into the students' behaviour. This process always brought new questions and we learned how to use inquiry in teaching to influence students' engagement and inquiry in learning.

[9]For information about mathematics learning support see `www.sigma-network.ac.uk/about/what-is-mathematics-statistics-support/`

We acknowledge that the COVID-19 pandemic made us reflect about ideas that we would never have thought about in relation to the classroom teaching. This is especially true for real-time online teaching, asynchronous learning, and remote learning support. We learned that digitisation of non-digitised subjects is time-consuming and additional resources are required to complete the process. Last but not least, using questionnaires as an indirect form of communication with students also proved helpful for development of inquiry-based learning and teaching activities. We perceive the experience that we collected during the PLATINUM project as having significant impact on our professional lives. We will never be the same as before.

We conclude this section by an outline of the potential future development of the BUT CoI. We will keep questioning ourselves how to do better in education. We can foresee development of further IBME-based teaching and learning activities, in particular activities supporting students' teamwork and mathematical modelling competency as well as applications relevant to concrete study programmes (for inspiration, see for example Chapter 12). That should be a long-time process due to the local conditions and constraints. We plan to continue collecting feedback from students.

However, what we perceive as most important for the future is keeping in touch and sharing experience within the CoI and beyond, with colleagues who are interested in professional development in mathematics teaching and learning. It is essential for us to be members of the professional community, the CoI. We were lucky to become a part of the PLATINUM CoI. We can foresee that relationships and collaboration across universities as well as with the MU CoI would make our BUT CoI more sustainable. We expect participation of the BUT CoI in more educational projects like PLATINUM because we like the taste of inquiry and we want to push our professional development further.

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CHAPTER 18

Experience in Implementing IBME at the Borys Grinchenko Kyiv University

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18.1. Development of an IBME Community at BGKU

In recent years, Borys Grinchenko Kyiv University has faced the problem of low motivation of future students in choosing mathematics programmes, and in teaching—with the problem of involving students in active learning through the use of new methods of teaching mathematics. One of the ways to tackle these problems is the use of innovative pedagogy and educational technology by mathematics teachers, which stimulate students' motivation to study mathematics and their involvement in the learning process. This includes inquiry-based mathematics education (IBME). With this in mind, an educational community of mathematics teachers was created at the university to acquaint teachers with IBME and the peculiarities of its use. The purpose of creating such a community was: to acquaint teachers with the concept, types, and cyclic structure of inquiry; examples of use; discussion of the use of inquiry in the teaching and learning of mathematics at the university. Initially, theoretical problems were analysed, the experience of creating educational communities was studied as well as forms of their activities to attract university teachers to participate in the community.

Quarantine measures in response to the COVID-19 pandemic led to significant changes in the lives of everyone in all countries of the world during 2020, including the organisation of the educational process in both secondary education and universities. The special conditions imposed on the work of educators have made apparent that one of the factors in the progress of educational reforms depends on the individual and collective ability of teachers to contribute to the transformation of the educational process. One of the ways to improve the quality of the educational process is the constant exchange of experience between teachers, discussion of existing pedagogical problems, analysis of best innovative educational practices, their implementation, and further discussion in the community of educators who are experts in a particular field. Therefore, the introduction and dissemination of such professional communities are relevant and important. Educational communities create favourable conditions and motivation for constant professional development of academic staff in higher education institutions.

Research confirms that the activities of educational communities have a positive impact on the results of the educational process for both teachers and students (Hattie, 2012; Hord, 1997; Jaworski, 2005; Marzano, 2003; Solomatin, 2015; Brodie & Chimhande, 2020). When teachers are part of the professional educational community, it reduces their isolation (certainly during quarantine periods), increases their commitment to the mission and understanding of the goals of the institution, creates conditions to support joint responsibility for the formation and development of professional competencies of students, and supports positive motivation to improve skills. This allows us to share the best teaching practices and expands the understanding of the content of educational material and the new role of the teacher in the digital transformation of education.

The term "educational community" has been introduced in Ukrainian pedagogy only recently. It is often associated with cyberspace (Maluhin & Aristova, 2020; Levchenko, 2020), while foreign studies do not narrow this concept to the web interface (Tönnies, 2002) and speak of 'community of practice' (Wenger, 1998) and 'community of inquiry' (Jaworski, 2005). Educational communities allow all participants to develop both personally and professionally. Training depends on the educational sector and tasks of the community. Participation in a community means, first and foremost, access to its resources, which can be both tangible and intangible (Tönnies, 2002). Some educators view the educational community as a dissemination of classroom practice in a community, using the resources of that community, both material and human. Others understand the educational community as involving specialists in educational institutions to improve the curriculum and learning objectives for students' educational activities (Hersi et al., 2016). For some educators, community activities involve the mutual learning of students, academic staff, and administrators, through the use of various organisational forms, pedagogical, and digital technologies.

Identification of these four important components was the first step in the formation of the community (Figure 18.1). These components were declared "important" a priori and served as a (normative) guideline/aim for the process of forming a community. The development of the community at BGKU confirmed the importance of each of these components.

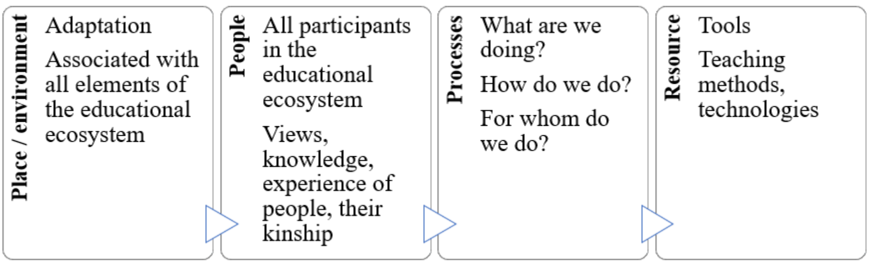


FIGURE 18.1. Components of communities.

Initially, the community consisted only of BGKU faculty who are members of the PLATINUM project. One of the main conditions for the creation and functioning of the educational community is the common goal of its members. Because the established community is engaged in the study of IBME, the name of the community was narrowed to IBME-community, a group of people that explore and disseminate inquiry-based mathematics education. The main target of the IBME-community in BGKU was implementing inquiry-based learning in high school mathematics education. Over time, the community was joined by mathematics teachers from other universities, employees of the Institute of Mathematics (National Academy of Sciences).

The main stages of forming such a community of teachers in BGKU were:

- holding an organisational meeting with teachers;
- developing a community promotion strategy;

- creating a questionnaire for the IBME survey;
- conducting a survey;
- conducting seminars on IBME;
- defining the main features of the professional community;
- creating a site for the community of mathematicians and a page on the Wiki portal;
- creating a Facebook page for the community;
- facebook page support, site creation, and support;
- activity planning and community development.

The initial survey of community members involved determining the respondents' experience in teaching, the list of subjects they teach, the educational institution where they work, their educational needs, and problems in teaching mathematics in university.

Community activities include discussion of open lectures, brainstorming in solving didactic problems, analysis of scenarios for involving students in active learning in mathematics, pedagogical technologies for teaching students inquiry, reflection on the introduction of inquiry technologies in various types of classes—lectures, practical training, discussions, surveys of teachers and students, group work, workshops, discussion of ways to use digital technologies in teaching mathematics.

At the stage of integrating the academic staff into the community, practical seminars, and workshops were held, during which the following issues were discussed:

- The concept of "educational community." Features of community activities and their functions.
- STEM-education and innovative methods—problem-based learning, project-based learning, inquiry-based learning. Commonalities and differences between these approaches.
- Inquiry questions. Criteria for inquiry questions.
- The three-layer model of inquiry adopted in PLATINUM (Chapter 2). Inquiry in mathematics in the classroom using the 5E-model of instruction to develop students' research skills (Bybee et al., 2006)
- Examples of mathematical research environments in the Go-Lab online laboratory,[1] which is a tool for learning and using Inquiry-Based Science Education (IBSE) in practice.

During the study of practical aspects of creating and organising the work of the IBME-community of teachers at BGKU, an empirical method was used (initial and repeated questionnaires of teachers of higher education institutions), as well as analysis of the results. The questionnaire in electronic form was created and sent to all members of the BGKU community. The initial survey was conducted at the stage of community formation, and the second after six months of cooperation of teachers in the community. A total of 72 respondents took part in the survey. The purpose of the survey was: to determine the impact of teacher activities in the community to change its methods of teaching and learning mathematics associated with the use of inquiry in teaching mathematics. For example, the objective of one question was to find out "What teaching methods do you use most often in your pedagogical activity?" before participating in the community and during joint activities in the community. Therefore, whether community participation influenced the use of innovative pedagogical methods of teaching mathematics. The results of the survey showed the percentage of teachers who began to use project-based learning in teaching mathematics (at

[1] `www.golabz.eu`

the beginning of the community 30.6%, at the stage of community activity 50%), research (33.3% and 66.7% respectively), and inquiry-based learning approach (41.7% and 83.3%, respectively) (Figure 18.2).

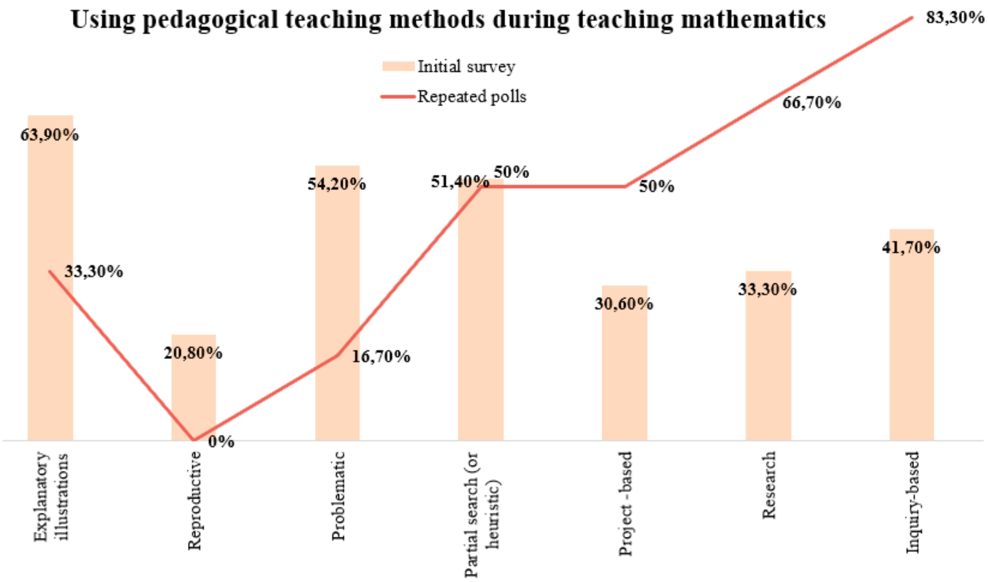


FIGURE 18.2. Teaching approaches reported within the community at the start of the project (the bar chart) and half a year later (the line graph).

Interesting are the changes in the forms of work that began to be used by members of the community with students in teaching mathematics. At the beginning of joint activities, teachers most often used the group form of work (70.8%) and individual (70.8%). After the exchange of experience and participation in workshops, the group form of student work became a priority (83.3%). At the same time, the percentage of using the individual form of work decreased by 20.8%.

One of the didactic techniques discussed during the community meetings was flipped classroom. The result of the exchange of experience and identification of the peculiarities of the organisation of this innovative pedagogical technology in the study of mathematics was an increase in the percentage of teachers (at the beginning of the community 50%, at the stage of forming community 66.7%), who began to use the flipped classroom in their professional activities, using digital tools. Members of the BGKU IBME-community defined their role in the use of IBME. The primary and secondary surveys showed a difference in priorities. Participants ranked the teacher's role from 1 (not important) to 7 (extremely important). As the result shows, teachers are interested in the stage of engagement, motivating students to inquiry activities, while planning itself has become less important, which demonstrates the willingness of teachers and students to use the method of "open inquiry" in this approach (Figure 18.3).

Analysing the role of the community in implementing IBME for teaching mathematics, all faculty members answered the question "Did you learn anything new about the inquiry approach after sharing experiences in the community?" with "Yes, a lot of interesting cases and tricks." To the question "Did the experience of working in the community allow you to share your work, research, observations?" two-thirds answered "Yes," and one-third "Partially."

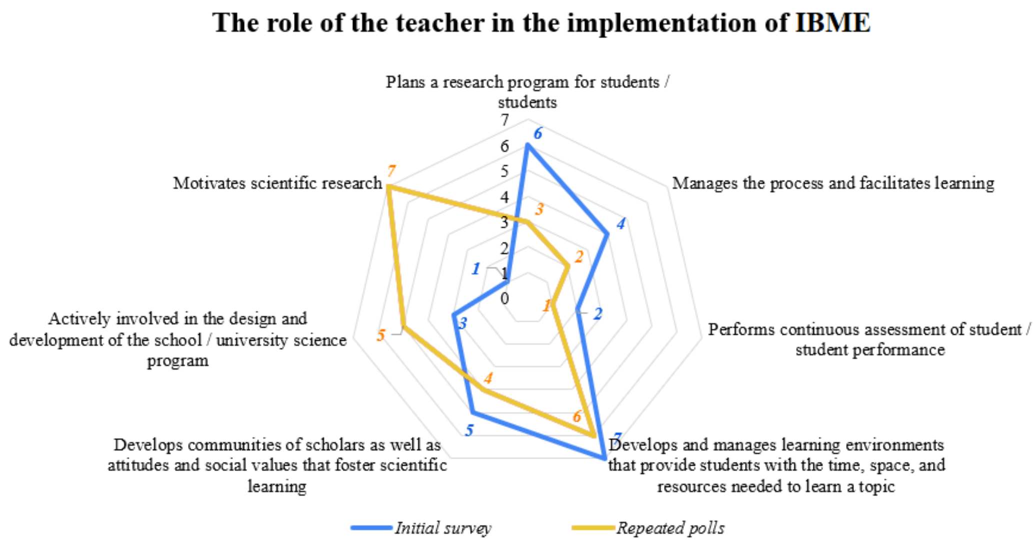


FIGURE 18.3. The role of the teacher in the implementation of IBME.

One of the stages of activity in the IBME-community consists of the creation and discussion of case studies, exchange of experience in implementing IBME, conducting open classes, development of templates for research tasks, creation of a database of modelling tasks, discussion of students' academic achievements and their reflection on changes in educational activities during the implementation of IBME. Currently, community teachers are actively working at this stage. This allows exploring the impact of community functioning on the professional activities of teachers and the process of implementing IBME in teaching mathematics, on positive changes in motivation and interest of students in mathematics, the results of their academic achievements in mathematics. In the next section we describe one of the stages of activity in the IBME community, namely, a case on the use of IBME in the study of mathematical analysis (lecture description, description of the organisation of students' homework) to improve the conceptual understanding of mathematics and the formation of conceptual knowledge.

18.2. IBME for the Formation of Conceptual Knowledge During Teaching of Mathematical Analysis

18.2.1. Conceptual and Procedural Knowledge. A deep understanding of mathematics and the ability to use it in further professional activities require two types of knowledge: conceptual and procedural. First of all, let's find out what conceptual knowledge is. There are different interpretations of the term "conceptual knowledge." Most of them, despite some differences, agree that conceptual knowledge involves not only knowledge of individual concepts, facts, methods, but also an understanding of the relationships between them, seeing how some facts follow from others, the ability to see the key idea of one or another method, to feel in what contexts it can be useful, to apply it in problem-solving, etc. (Hiebert & Lefevre, 1986; Cobb, 1988; Byrnes & Wasik, 1991; Haapasalo & Kadijevich, 2000; Star, 2005). All these characteristics of conceptual mathematical knowledge are quite accurately conveyed by Hiebert and LeFevre's definition: "Conceptual knowledge is characterised most clearly as the knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information." In addition, this definition leads to an important conclusion for teaching mathematics, which Star (2005) emphasises: "The term conceptual knowledge has come to encompass not only what is known (knowledge of concepts) but also one way that concepts can be known (e.g., deeply and with rich connections)" (p. 408). That is, the learning process aimed at achieving conceptual understanding and the formation of conceptual knowledge requires a significant reorganisation of existing knowledge, not just its accumulation. From the point of view of Hiebert and Lefevre (1986), procedural knowledge—rules or algorithms—are represented mainly in symbolic form. Haapasalo and Kadijevich (2000, p. 141) see procedural knowledge as "dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation form(s)."

The question of procedural knowledge vs. conceptual knowledge has been the focus of many researchers (e.g., Lauritzen, 2012). There are four main models of causal relationships between conceptual (C) and procedural (P) knowledge:

(1) *Genetic*: the presence of C automatically ensures obtaining P, but the formation of P does not ensure the formation of C.
(2) *Dynamic interaction*: the presence of P and problem-solving forms C, but the formation of C does ensure getting P.
(3) *Simultaneous activation*: the presence of P and problem-solving forms C, and the formed C, in turn, helps to obtain P.
(4) *Inactivation*: P and C are not related

Our teaching practice over the years shows that the third model improves our personal experiences. It follows that conceptual knowledge can both precede or influence procedural knowledge, and procedural knowledge can precede conceptual knowledge. However, conceptual knowledge without procedural knowledge is ineffective, and procedural knowledge without conceptual knowledge is superficial, and can lead to serious errors in the use of mathematics later, and can hardly be applied in unfamiliar contexts. Conceptual knowledge improves the student's ability to detect misuse of a method (procedure) or inconsistency of the method in a given situation, and to analyse and evaluate an answer (Lauritzen, 2012). Moreover, as shown in (Chappell & Killpatrick, 2003), students for whom the concept-based learning environment was created showed already during training much better results than students in the procedure-based learning environment. Both conceptual understanding of mathematics and procedural skills were assessed in this reference.

Thus, conceptual knowledge is a necessary component of teaching Mathematics and considerable attention must be given to its formation. But, unlike procedural knowledge, "conceptual knowledge is the most difficult to acquire. It's difficult because knowledge is never acquired *de novo*; a teacher cannot pour concepts directly into students' heads." (Willingham, 2009, p. 18). When teaching higher mathematics in the context of concept-based learning, we must provide conceptual understanding so that students

- understand which mathematical ideas are key and why;
- are aware of the systemic nature of mathematics and see the relationship of its areas;
- understand what ideas can be applied in a particular context, and understand the basic methods of mathematical proofs and the scope of their application; and
- can adapt prior experience to new problems, especially to nonstandard ones.

Let us note that it is much more difficult to assess the level of formation of conceptual knowledge and identify their gaps rather than procedural ones. To do this, it is necessary to select appropriate problems that require a systematic approach, application of previous experience, and analytical thinking of the student. During the teaching of mathematics it is necessary to evaluate the following learning outcomes:

- knowledge of mathematical concepts, statements, theorems, properties, features, methods, and ideas; the ability to apply the acquired knowledge and skills to solve educational and practical problems, when the method of such a solution must be found by herself/himself (*conceptual knowledge*);
- knowledge of the methods of activity that can be presented in the form of a system of actions (rules, algorithms); ability to perform already known actions following the learned rules, algorithms (*procedural knowledge*).

In Figure 18.4 are shown examples of problems that we proposed when studying Rolle's theorem (in the context of Mathematical Analysis).

Problem A (*procedural-oriented*):
Can Rolle's Theorem be applied to the function $f(x) = 1 - \sqrt[3]{x^2}$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$?

Problem B (*conceptual-oriented*):
Can the equation $e^x = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$, have four roots?

FIGURE 18.4. Two problems related to Rolle's Theorem.

Although the formulation of both problems looks problematic ("can"), problem A is purely procedural. To obtain the answer it is sufficient to check the fulfilment of the conditions of Rolle's Theorem and to be able to find the derivative of the function by known rules. To solve problem B, you need (1) to find out whether the equation can have roots at all; (2) to form a hypothesis ("cannot"), and for this to resort to graphical interpretation, conditional experiment; and (3) to prove it (feel that the proof should be carried out by the method proof by contradiction, and use Rolle's Theorem). However, Task A could be made conceptual-oriented with the help of inquiry questions that encourage students to continue their research. For example: "If your answer to question A is negative, does this mean that there is no point at this interval where the derivative of the function is zero?" (Answer: no); "Is there an interval at which Rolle's theorem can be applied to this function?" (Answer: no).

The key to students' high academic achievements is *active learning*, which means the use of such methods and techniques that require students' conscious educational activities, involve them in the process of constructing new knowledge, research skills, and their use. Active student participation is a key factor that influences the success of the entire educational process in higher education. This is found in many studies and not exclusively in mathematics education. Active learning has been found to improve conceptual understanding (Laws et al., 1999), improving performance better than increasing study time (Redish et al., 1997). For example, an analysis of 225 studies comparing students' performance in active and traditional (passive-informative) learning in Science, Technology, Engineering, and Mathematics confirmed the effectiveness of active learning in STEM education. (Freeman et al., 2014).

Information can be obtained passively, but not understanding, because it requires the connection between prior and new knowledge. And this is possible only as a result of active mental actions. Learning based on memorising and using algorithms saves time but does not contribute to the formation of conceptual knowledge and the development of critical thinking. Below are three of the students' most typical opinions about this, expressed after six months of studying mathematical analysis.

> Now I see that nothing needs to be crammed in mathematics. Finally (!) I understood where many of the formulas we studied at school came from. At school, they just aimed at memorising and that's all.
>
> At the beginning of my studies, I tried to memorise definitions, theorems, formulas to reproduce when asked. But they did not keep in mind, because they were very unusual and incomprehensible. Fortunately, I soon concluded that the main thing is to understand concepts, facts, imagine them, look for and find convincing arguments. Living and studying have become much easier and more interesting.
>
> I really like the way we study. We reflect on new concepts and facts. But at the same time we can always count on the friendly help of the teacher. And it's very inspiring.

Thus, active learning strategies can act as a mechanism for the development of conceptual understanding of mathematical structures, creative thinking, research competencies, and meta-skills. One of such strategies is IBME.

18.2.2. Characteristics of the Discipline and the Cohort of Students. An example is a case of IBME implementation to form conceptual knowledge for the teaching of mathematical analysis to first-year students majoring in Mathematics, by a lecturer—Associate Professor of the Department of Computer Science and Mathematics, Ph.D. (Physical and Mathematical Sciences) Mariia Astafieva.

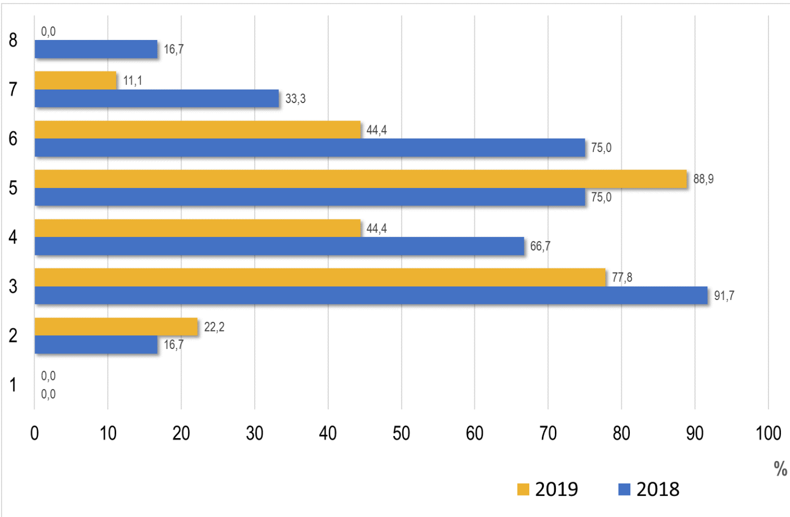
Mathematical Analysis is a compulsory subject of the BGKU Bachelor's programme in Mathematics. The course aims to provide first-year students with systematic knowledge of the basics of classical analysis of univariate real functions. Educational activities (teaching and organisation of students' self-study) are aimed at students to master the classical methods of mathematical analysis, theoretical principles, and basic applications of mathematical analysis in various problems of mathematics, mechanics, other subject areas, their use in further courses in mathematics and mechanics. It is also necessary to promote the development of critical and logical thinking of students, research skills, and instrumental competencies.

In this example, we describe the learning of mathematics students only. Their groups in BGKU are small: each study year there are 8 to 12 students. These are secondary school graduates who enter the university based on the results of an external independent evaluation.[2]

The entrance assessment of knowledge and skills of freshmen, which we conduct annually on the first days of their studies at the university, traditionally (unfortunately) reveals significant gaps in most students' basic mathematical preparation—contradictory or misinterpreted concepts, fragmented and useless knowledge. A month and a half after the start of classes, we survey first-year students to identify the difficulties they encounter in studying Mathematical Analysis. We offer students a questionnaire in which they choose one or more of the proposed reasons for difficulties. Figure 18.5 shows the histogram of the distribution of responses of students in 2018 (12 students) and 2019 (9 students). The analysis of the survey results showed that the main obstacles to the successful learning of Mathematical Analysis according to students are gaps in school basic mathematical training, in particular: inability to prove

[2]External Independent Assessment in Mathematics (EIA) is an all-Ukrainian measurement of learning outcomes in mathematics for students aged 16-17 who are completing general secondary education and plan to continue their education in higher education institutions (HEIs). The tasks for EIA are developed annually by the Ukrainian Centre for Educational Quality Assessment, kept secret until the day of the event, the same for all students, and presented in the form of a test. The results of EIA of graduates of the system of complete general secondary education are used for admission to HEIs. Only those entrants who have passed the "threshold score" can enter HEI. All entrants, whose results are above the "threshold score," will receive a score on a scale of 100-200 points and will have the right to participate in the competitive selection for admission.

theorems, inability to substantiate the use of formulas; inability to independently process the material in the textbook. In 2018, students most frequently indicated as a cause of difficulties "gaps in school mathematical preparation" (statement 3), students in 2019 predominantly chose "it is difficult for me to prove theorems and formulas" (statement 5). This result, in our opinion, shows the lack of students' understanding of the essence of mathematics as a process of proof and the relationship between different concepts.



(1) No difficulties arise
(2) There is something I don't understand due to the high level of abstraction
(3) I find gaps in school mathematical preparation
(4) I understand the teacher's explanation, but very rarely can I form a hypothesis on my own, identify the essential features of a new concept, give examples and counter-examples, guess the idea of solving or proving
(5) It is difficult for me to prove theorems and formulas since it was never done at school, we only were asked to learn the formulations
(6) It is difficult to independently (without additional explanation) process the educational material in the textbook
(7) I find it difficult to remember the definitions and formulations of the theorems
(8) Not enough patience / no habit to do homework

FIGURE 18.5. Survey results of first-year students majoring in Mathematics.

Also, in addition to closing gaps in school mathematics knowledge, our goal was to give students self-confidence, direct experience of mathematical discovery, and the joy and satisfaction of their mathematical research. To achieve this goal and implement the objectives of the course, the entire educational process in the Mathematical Analysis course was based on research-oriented approaches to learning (i.e., on IBME). Because the course is taught to first-year students who do not yet have sufficient experience in guiding the trajectory of their learning, structured and guided inquiry prevail. The sequence of involving students in active learning can be described by a chain: motivation (raising interest) $\rightarrow$ active action under the guidance of a teacher (constructing new knowledge) $\rightarrow$ own initiative (independently stating problems, proposing alternative solutions, and so on).

The student's learning and cognitive motivation depend on whether the learning goals become a motivated need and personal value and interest for them, and to what extent the educational material meets these needs, values, and interests. Learning methodology largely takes into account the students' attitude to their educational activities. To form a positive permanent motivation for learning, it is important that each student feels like a subject of the educational process in which s/he plays an active role, consciously striving for self-improvement.

The specific motivational background is created by the mathematical content itself, which has several features. Such features include, in particular: a high level of abstraction; complex logical structure of many definitions and theorems; the orientation of the content is not so much on the assimilation of specific information, as on the mastery of a certain mode of action; dialectical interaction of strict proofs and heuristic considerations; the key role of tasks that motivate research activities; significant internal connections between different topics; wide possibilities of applications in various fields; as well as maximum accuracy and persuasiveness, creative inexhaustibility, beauty, and aesthetic perfection. We try to use the motivational potential generated by these features not only to stimulate the situational activity of students but also to form in them a deep inner interest in mathematics. To this end, real-life problems, mathematical problems that challenge thinking, are proposed. For example, at the beginning of the study of the topic "Definite Integration" the teacher offers students several practical problems that lead to the integral (see Section 8.4. Furthermore, acquaintance with numerical series begins with the search for "Achilles' heel" in Zeno's paradox about Achilles and the tortoise. This introduction aims to attract students, arouse interest and enthusiasm, which will give enough impetus, help to further master complex, abstract, and even boring, but necessary things and see in them a kind of beauty and harmony, as well as enjoy the mathematical activities.

Familiarity with the concept of the limit, the operation of boundary-crossing causes considerable difficulties for first-year students, including psychological, because it is something fundamentally different from what they learned in secondary school. Therefore, special attention is paid to the formation of mathematical concepts based on conceptual understanding.

To stimulate students' active mental activity, a teacher encourages students to use earlier learned material in their considerations to explain a new idea, provoking the formulation of inquiry questions that help students draw conclusions, encourage and support discussion, reflection, mutual assistance, and mutual learning. For example, to bring first-year students to the concept of continuity at a point x_0, the teacher shows on the screen 5 to 7 graphs of functions, of which only one is continuous at x_0, and invites students to find the extra one. Students do it easily. But it is difficult for them to explain their choice in mathematical terms. Here, as a rule, discussions start, which ultimately lead to the idea of using the concept of a limit (studied earlier) to define continuity. Continuing this 'game', students come to the concept of discontinuity points and their classification.

Learning is a social activity, it is what we do together in interaction with each other. Therefore, we were interested in how to organise this joint activity in and out of class, in what is the role of the teacher in this activity, and in how to ensure effective pedagogical mediation. The main goal is to achieve an improved conceptual understanding of mathematics by students and to achieve their cognitive development in general through the joint construction of knowledge in the so-called "Zone of Proximal Development" (Vygotsky, 1978, 1987). In the next section, examples from our practice illustrate attempts to achieve this goal, some positive results, and also some problems.

18.2.3. Description of the Lesson. The lesson duration was 80 minutes; the class was attended by 8 students; the topic was: absolute and conditional convergence of a numerical series. The purpose is to acquaint students with the concept of absolute (conditional) convergence of a series and the possibilities of applying the convergence of non-positive series to research, to form a conceptual understanding of these concepts, and to improve research and procedural skills.

Expected learning outcomes are: *Knowledge*—the concept of absolutely and conditionally convergent series, a sufficient condition for the convergence of a non-positive series; *Skills*—the study of absolute (conditional) convergence of series; and *Research and procedural skills*—the ability to make empirical reasoning, make assumptions, and understand the essence of mathematical proof; the ability to present one's judgements. Requested preliminary knowledge consists of the concept of a numerical series, its convergences/divergence, its sum; properties of convergent series; a necessary condition for the convergence of the series; Cauchy convergence criterion; convergence tests of positive series; Leibniz criterion of convergence of alternating series; understanding what a sufficient and necessary condition is; understanding in which cases the use of the necessary condition of convergence of a series can be effective and the ability to use it; skills of investigating of convergence of positive and alternating series. The lecture was conducted in the form of a video conference on the Zoom platform (due to the COVID-19 pandemic) with PLATINUM participants from different partner universities present.

The lecturer brought students to the definition of absolute and conditionally convergent series gradually. First, she posed the following problem:

Investigate the convergence of the non-positive series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$.

This created the conditions in which the student must recognise (see or feel) the need for new knowledge. Note that by choosing this series for research, the teacher anticipated and even deliberately provoked students to a misconception, which in turn (after being rejected) intensified the intrigue and desire to solve the problem. Below is shown how a way to solve the problem was found (excerpt of the discussion).

Excerpt

S1: (*immediately*) The series is convergent based on the comparison test $\frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so this series is convergent.

(Pause)

Lecturer (L): Does everyone agree with *S1*?

S2: No.

L: Why?

S2: Because the comparison test is for positive series, and our series contains both positive and negative terms.

L: Maybe based on the Leibniz criterion?

S3: No, it is not possible, because the signs of the terms of the series do not alternate: the first three terms are positive, then a few negative, then again positive, and so on.

(Pause)

L: Okay. Let's form from this series an alternating series:

$$\left(\frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2}\right) + \left(\frac{\sin 4}{4^2} + \frac{\sin 5}{5^2} + \frac{\sin 6}{6^2}\right) + \left(\frac{\sin 7}{7^2} + \frac{\sin 8}{8^2} + \frac{\sin 9}{9^2}\right) + \cdots (*)$$

In it, the first term is positive, the second is negative, the third is positive again, etc. We will separately collect successive positive and negative terms in groups.

S4: It will not help. Because according to the Leibniz criterion, the absolute values of the terms of a series must decrease and tend to zero, and in the series, you have formed, it is unknown whether this is the case.

L: At least, it's not obvious. Well, if we somehow proved that the absolute values of the terms of the formed series decrease and tend to zero, then could it be concluded that the series under study is convergent?

S5: Yes.

S3: No. Because, even if we established that this series is convergent, it will still not follow the convergence of the initial series. The series (*) is formed by grouping the terms of the given in the problem series (combining in parentheses), without changing the order of their sequence. We know that a convergent series has the associative property. That is, if we have a *convergent* series and group its terms, we also get a convergent series. Not the other way around. The other way around is even wrong, which can be easily illustrated with an example.

L: Convincing. And what other tools are there?

S6: A necessary condition for convergence. But it also does not help to solve the problem, because $\frac{\sin n}{n^2} \to 0$ if $n \to \infty$.

S7: And there is the Cauchy criterion and the definition of the convergence of the series. But they are inconvenient for practical use.

L: So what do we do?

S1: (*emotionally*) But our series is still convergent! Well, look: if you take a positive series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$, it is convergent, based on comparison test. And in our series, some terms are just negative. Well, for example, take the sum $1+2+3+4+5 = 15$. And now we will change some terms for the opposite, the sum will decrease. For example, $1-2+3+4+5 = 11 < 15$. It will be the same in the case of a series.

L: But we know that it is risky to automatically transfer facts that are valid in finite sets to infinite sets.

S1: Well, that's just a hypothesis

L: Then formulate it.

S1: (*formulates a hypothesis*) If the series $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

The given excerpt is an example of using the inquiry approach in the organisation of research activity of students. With a series of purposeful inquiry questions, the teacher directed the students' progress to their independent formulation of the hypothesis. Despite the lack of direct contact among the audience, an atmosphere of cooperation was created (although not without certain losses), and the teacher periodically moved from the role of facilitator to the role of the team member, offering options for solving the problem. Changing this role had a purpose—not to direct students to the shortcut "a straight line"—but to lead them through a 'maze' with access to all sorts of 'dead ends.' This technique contributed to the conceptual understanding of the research problem and helped to develop systemic and flexible thinking.

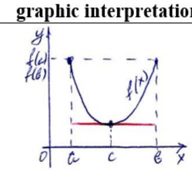
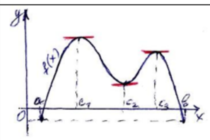
One of the indicators of conceptual knowledge is the ability to apply it in practice. In addition to conceptual understanding, procedural knowledge is very important here. Its formation, as well as the deepening of conceptual knowledge 'in action,' occurs mainly in practical classes and in the process of solving independent practice problems. Our practice has shown that a significant problem for students is the ability to recognise a particular mathematical theory in a practical problem, the content of which does not directly indicate this theory. Thus, in one of the practical classes on "Numerical Series," students were given the task to find the area and perimeter of the Koch Snowflake. But they could not independently 'see' a numerical series in this problem. Instead, the same students, in the same class, brilliantly coped with a rather difficult (conceptual) task to study the convergence of a series, and they solved it in different ways.

18.2.4. Organisation of Extracurricular Collective Work of Students. In Section 18.2.3 we described how it is convenient to organise the collective work of students in the classroom, real or virtually (although less successfully). And how to organise the interaction of students when doing homework. We tried to find models for organising such cooperation outside of classes and tested three forms: the so-called 'conceptual tables,' and the FORUM and WIKI tools in the LMS MOODLE in an e-learning course (ELC), which were developed by lecturer Maria Astafieva and used in the educational process.

A conceptual table is the summarised, organised, and structured information about the content on a particular topic. The table is filled by a group of students (2-4 people) in class, and more often in extracurricular time. They formulate questions themselves and answer them. At the same time, students demonstrate an understanding of the essence of concepts, facts of this topic, their connection with the previously studied material, the ability to correlate different forms of presentation of a mathematical topic (verbal, symbolic, graphic). The organisation of the next activity with the filled conceptual tables depends on what purpose the teacher pursues: to continue training or to check and estimate knowledge. Depending on this, group discussions can be employed, mutual reviews or a check of the table by the teacher. An example of a conceptual table on Rolle's Theorem is given below (Table 18.1).

Conceptual table

Course: Mathematical analysis. Topic: Rolle's theorem
Students: S1; S2; S3. Date: December 19, 2019

Question	Answer		
	verbal	symbolic	graphic interpretation
How is Rolle's theorem formulated?	If a real-valued function is continuous on a proper closed interval and differentiable at each of its interior points, and at the ends of the interval it acquires equal values, then inside the interval there is a point at which the derivative of the function is equal to zero.	$f(x)\in C[a;b]\wedge$ $\wedge\exists f'(x)\forall x\in(a;b)\wedge$ $\wedge f(a)=f(b)\Rightarrow\exists c\in(a;b)$ $f'(c)=0$	
Is point c unique?	No. For example, the function $\sin x$ on the proper closed interval $[0; 2\pi]$ satisfies the conditions of Rolle's theorem and inside this interval there are two points $\pi/2$ and $3\pi/2$, at which the derivative of the sinus $(\cos x)$ is equal to zero.	$\exists c_1, c_2, c_3\in(a;b):$ $f'(c_1)=f'(c_2)=f'(c_3)=0$	
Is it possible to find a point (points) c by using the theorem?	The theorem does not answer the question of where exactly on the interval is the point c; it only guarantees the existence of such a point.	-	-
What conditions for the existence of a zero derivative are given by the theorem: sufficient; necessary; necessary and sufficient?	The theorem sets sufficient conditions	If $A\Rightarrow B$, then A is sufficient for B	-

Example of a conceptual table (*continued on next page*).

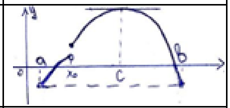
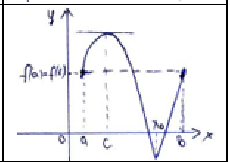
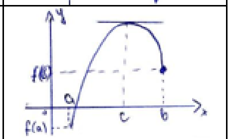
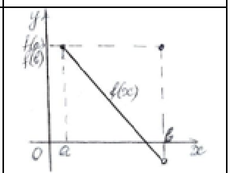
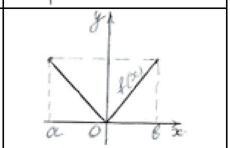
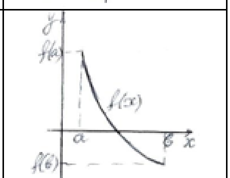
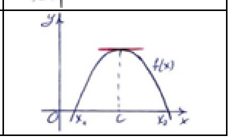
Are the conditions of the theorem necessary?	No. None of the three conditions is necessary. That is, the condition is not fulfilled, and the zero derivative exists. This is well illustrated by the appropriate graphics for the cases:	-	-
	a) the function is not continuous on the proper closed interval, and the other two conditions are satisfied	$\exists c \in (a;b): f'(c)=0$, although x_0 – the breakpoint of the function	
	b) inside the proper closed interval there are points at which the function is not differentiable, and the other two conditions are fulfilled	$x_0 \in (a;b) \wedge \nexists f'(x_0)$, but $\exists c \in (a;b): f'(c)=0$.	
	c) at the ends of the proper closed interval the values of the function are not the same, and the other two conditions are satisfied	$f(a) \neq f(b)$, but $\exists c \in (a;b): f'(c)=0$.	
Are the conditions of the theorem significant?	Yes. Each condition is significant and if it is not fulfilled, there may be no zero derivative (see relevant graphics):	-	-
	a) the condition of continuity of the function is not fulfilled	b – the breakpoint of the function and $f'(x) \neq 0\ \forall x \in (a;b)$	
	b) the condition of differentiation is not fulfilled	$\nexists f'(0), 0 \in (a;b)$ i $\nexists c \in (a;b): f'(c)=0$.	
	c) at the ends of the interval the values of the function are not the same	$f(a) \neq f(b)$ i $\nexists c \in (a;b): f'(c)=0$.	
What is the consequence of Rolle's theorem?	For example: "Between two zeros of a differentiable function is a zero of its derivative"	$f(x_1)=f(x_2)=0 \wedge \exists f'(x) \forall x \in (x_1;x_2) \Rightarrow \exists c \in (x_1;x_2): f'(c)=0$	

TABLE 18.1. Example of a conceptual table on Rolle's Theorem.

Another activity that we used in Mathematical Analysis ELC in the LMS MOODLE for students to perform tasks together is the WIKI activity. This tool allows participants to add and edit web pages. The history of all changes is preserved and this allows the teacher to follow the trajectory of each student and respond on time and evaluate the educational process, using the technique of formative assessment. The Wiki collections created by students are a virtual analogue of the conceptual table (Astafieva et al., 2019).

The initiative is an important component of active learning, and at the same time, evidence of a high level of student motivation, and active involvement in the research process is expressed in the ability to inquire, to put forward new ideas or proposals for research, to offer different solutions, and to formulate new tasks.

An indicator of the attainment of a high level of conceptual knowledge by some students is the discussion of the topic "Numerical Series" at the Forum of the electronic training course "Mathematical Analysis." At the end of the study of the topic (in May, 2020), the teacher invited freshmen to ask mathematical questions about this topic on the forum and give answers to them. Five students asked conceptual questions and participated in the discussion. Here is an example of one of these questions and the discussion of students caused by it.

Excerpt

S1: The Leibniz criterion of convergence of an alternating series requires that the sequence of its terms decreases and tends to zero. How important is the condition of decreasing?

S2: The condition of decreasing the sequence of terms of the series is significant. If this sequence converges to zero but does not decrease, then the series may be divergent. An example of such a series is

$$\frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{3^2} + \frac{1}{4} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{5^2} + \cdots \qquad (**)$$

The terms of this series tend to zero, but non-monotonically: $a_1 > a_2$, $a_2 < a_3$, $a_3 > a_4$, $a_4 < a_5$, etc.

Let us show that this series is divergent. To do this, let us group its terms as follows:

$$\left(\frac{1}{2} - \frac{1}{2^2}\right) + \left(\frac{1}{3} - \frac{1}{3^2}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n^2}\right) + \cdots = \sum_{n=2}^{\infty}\left(\frac{1}{n} - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty}\frac{n-1}{n^2}$$

We obtained the divergent series, it is compared with a harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$.

Thus, the series (**) is also divergent, because if it was convergent, then the series formed from it by grouping the terms would be convergent. The significance of the condition of decreasing the sequence of terms of a series in the Leibniz criterion is proved.

S3: The condition of decreasing the sequence of terms of the series is important. The following example proves this. Let's take the series $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \cdots$. The terms of this series tend to zero, but they decrease, then increase. Let's form the series

$$\left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{3} - \frac{1}{4^2}\right) + \left(\frac{1}{5} - \frac{1}{6^2}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{(n+1)^2}\right) + \cdots$$

The n-th term of this series is $a_n = \frac{1}{n} - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - n}{n(n+1)^2} = \frac{n^2+n+1}{n(n+1)^2}$.

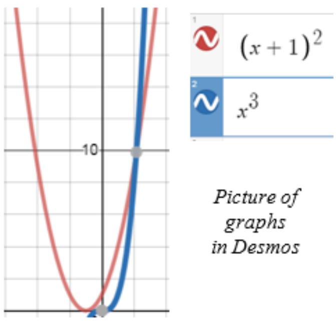
The series is divergent, because we have $\lim_{n\to\infty}\left(\frac{n^2+n+1}{n(n+1)^2} : \frac{1}{n}\right) = 1$, based on the comparison test, and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. Therefore, the initial series is divergent, because if it was convergent, then after combining the terms in parentheses, it would remain convergent.

S4: Although the condition of decreasing the sequence an is essential in the Leibniz criterion, it is not necessary. Take, for example, the following series:

$$\frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{4^2} - \frac{1}{4^3} + \frac{1}{5^2} - \frac{1}{5^3} + \cdots$$

The sequence of its terms $\{a_n\}$: $\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{5^2}, \frac{1}{5^3}, \cdots$ converges to zero. But this is not a decreasing sequence, because in it the first four terms are decreasing, and the fifth is greater than the fourth; the sixth member is less than the fifth, and the seventh is greater than the sixth; etc. Nevertheless, the series is convergent, because it can be interpreted as the sum of convergent series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ and $\sum_{n=2}^{\infty} \frac{-1}{n^3}$.

S5: A good example was provided by S4. But I don't understand how it can be seen that the sequence $\{a_n\}$ is not decreasing. The phrase "etc." did not convince me. There was no certainty that starting with the fifth term, as *S4* claimed, each term with an odd number is greater than the previous one, i.e., $\frac{1}{(n+1)^2} > \frac{1}{n^3}$. But it turns out that this is indeed the case. The graphic image in Desmos[3] helped me to see it.



Picture of graphs in Desmos

$$(n+1)^2 < n^3 \qquad \forall n > 2.$$

Then:

$$\frac{1}{(n+1)^2} > \frac{1}{n^3} \qquad \forall n > 2$$

Thus, we have:

$$\frac{1}{n^3} < \frac{1}{n^2} \qquad \forall n > 1$$

It is obvious. But already $\frac{1}{4^2} > \frac{1}{3^3}$, $\frac{1}{5^2} > \frac{1}{4^3}$, $\frac{1}{6^2} > \frac{1}{5^3}$, etc.

That is the sequence $\{a_n\}$ is not monotonic.

The fragment above shows that the students who participated in the discussion had the ability to:

- independently formulate a research problem and willingness to work on its solution;
- find the right idea to solve the problem;
- understand what is sufficient, necessary, and essential condition;
- understand the essence and methods of mathematical proof;
- feel the internal need for full evidence;
- make strict logical reasoning;
- choose convincing arguments for argumentation and critically evaluate provided arguments;
- apply previous experience and knowledge to solve a new problem; and to
- establish a connection between different interpretations of mathematical concepts and facts, in particular, to use a graphic image for illustration and argumentation.

18.3. Evaluating Effectiveness of IBME to Achieve Educational Goals

Evaluation of the effectiveness of IBME to achieve educational goals was carried out according to a scheme developed by the BGKU team based on a template created by project participants from the Complutense University of Madrid (see Chapter 9),

[3] www.desmos.com

and taking into account student feedback, teacher self-analysis and collective discussions of project participants in academia. The evaluation consists of five blocks.

The First Block is the general information about the lesson: date, course, speciality, course, number of students, topic and purpose of the lesson, type of lesson (lecture, practical), duration of the lesson, equipment, software used during the lesson, expected learning outcomes, prior knowledge that students should have.

The Second Block is a description of educational activities during the lesson: the actions of teachers and students.

The Third Block is the characteristics of the student group (formulated by the teacher), which allows determining the level of internal motivation to study mathematics and what it is caused by; whether students have experience in managing their learning trajectory; the initiative of students in self-study; the ability to persistently, purposefully overcome the difficulties and obstacles that arise in the process of solving a problem.

The Fourth Block is a description of students' activities during the lesson: the level of involvement in the educational process; how actively students participate in the study (discussion of problematic issues); ability to formulate different types of questions: clarifying, research, hypothetical; research and procedural skills demonstrated by students during the class; ability to put forward their hypotheses; ability to self-reflection (what I learned, how my knowledge, skills, abilities have changed); how the students saw (felt) the relationship with the previously studied material.

The Fifth Block of assessment of a lesson provides an assessment (low, average, high) of achievement of the purposes set by the teacher and its substantiation.

According to this scheme, we present the analysis and evaluation of the lesson "Absolute and conditional convergence of a numerical series," described in Section 18.2.3.

The First Block. General information about the lesson (already presented in Section 18.2.3)

The Second Block. Description of educational activities during the lesson.

The teacher implemented the 5E model of instruction (Bybee et al., 2006) during the lecture.

In the *Engage* phase, students were asked to investigate the convergence of a series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$. Because the day before, when studying positive series, a seemingly similar positive series $\sum_{n=1}^{\infty} \frac{\sin(\pi/n)}{n^2}$ was investigated for convergence, the teacher thus 'programmed' an error—due to external similarity to consider the proposed series as positive. This plan worked and some students decided that the series is positive and one of the signs of convergence of the positive series can be used directly to investigate it. However, the series is not positive. Students noticed this upon closer analysis and were therefore convinced that the comparison test did not work. There was a need to look for a new one.

During the *Explore* phase, students tried to find a way to solve the given problem. At this stage, the teacher played the role of facilitator or member of the team of 'researchers' when discussing the problem and ways to solve it: the main tasks are then to help, guide, provoke, and ask questions that push to formulate a hypothesis. During the discussion, students analysed, reflected, asked questions that help to advance in the study, expressed ideas and opposed; formulated a hypothesis and looked for a way to prove it.

In the *Explain* phase, students together with the teacher implemented the idea of proof found during the previous phase. The teacher wrote down the proof of the theorem, involving students in commenting. Students together with the teacher also wrote down the proof. The teacher formulated the definition of absolutely and conditionally convergent series.

In the *Elaborate* phase, the teacher sought to expand the conceptual understanding of the proven sign of the convergence of the series. To do this, he enquired: "We have just found that a series is convergent if the series formed from the absolute values of its terms is convergent. And what can be said about the convergence of this series, if the series of absolute values of its terms is divergent?" Based on the discussion, students concluded that a proven condition is not necessary. They gave a suitable example.

Several specially selected exercises were solved next that helped to notice one important detail: if the divergence of a series $\sum_{n=1}^{\infty} |a_n|$ is established based on D'Alembert or Cauchy criterion, then the series $\sum_{n=1}^{\infty} a_n$ is divergent. The teacher again encouraged the students to study with the question: "Do you think this is a coincidence?" In the process of a short discussion led by a teacher, a reasonable answer to the question was given. The teacher together with the students concluded how the established fact can help in practice.

In the *Evaluate* phase students summed up, upon the request of the teacher, what they have learned and what is the practical value of the knowledge gained.

Block 3. Characteristics of the student group.

Students demonstrated intrinsic motivation to study mathematics, had little experience in managing the trajectory of their learning; a large part of the group could persistently, purposefully overcome the difficulties and obstacles that arose in the process of solving the problem (for example, if the next step is not obvious or it is necessary to restore some previous knowledge for further progress).

Block 4. Characteristics of students activities during the lesson.

During the lesson, students were involved in the learning process. The teacher created conditions in which students had to recognise the need for new knowledge, because they found themselves in a situation where the knowledge they already was not enough to solve the problem.

In the process of research, most students actively participated in the discussion, formulated questions independently. Some students showed intrinsic motivation to solve the problem without the support of the teacher. At the end of the lesson, students assessed their progress and expressed their impressions of the lecture in a chat.

Block 5. Evaluation of the achievement of goals.

The purpose of the lecture was achieved. In particular, students under the guidance of a teacher concluded that the absolute convergence of a series was a sufficient condition for its convergence, and proved it. The examples demonstrated the ability to apply the proven criterion to the investigation of the convergence of non-positive series. In addition, during the class, students demonstrated an understanding of the relationships with previously studied material. Such a result indicates the effectiveness of IBME: selected educational content, the organisation of active research activities of students, learning through scaffolding gave a positive result.

18.4. Discussion of the Case in the Community of Inquiry

Throughout the semester, Mathematical Analysis classes were attended by members of the academic community – PLATINUM project participants and other interested teachers. The described case has been repeatedly discussed in the community. The ways and methods of application of IBME used by the teacher, their expediency, and efficiency were discussed. The problems faced by teachers and students (what worked and what didn't? why?) were considered, especially during the implementation of IBME in distance and blended learning (through 2020 during quarantine), and recommendations for their solution were developed.

The judgements of the students, which they expressed about their academic achievements, attitudes to mathematics, and the teaching methods used during Mathematical Analysis were also taken into account. Students noted the positive dynamics of the achieved results in terms of subject knowledge and skills. In addition, they indicated a significant improvement in understanding mathematical facts, the acquisition of certain research skills (ability to observe, analyse, doubt, the ability to ask right questions, reason logically, express hypothesis, test it, prove facts, properly express opinions, draw conclusions and generalisations, etc.). They also noted the development of imagination, increased interest, and motivation, the ability to learn independently. Students responded positively to the teaching methods used (comfortable, friendly atmosphere of discussion of problems, ideas, motivation to search and research, learning to ask the right questions, help, etc.). And the fact that the discussion (questions, answers, discussions, reflections) on the forum of the distance e-course "Mathematical Analysis" continued even after the students passed the exam, is evidence of their persistent interest, intrinsic motivation, and comfort in learning. There are, of course, some unresolved issues, in particular, there are difficulties with the processing of book texts. Below are some excerpts from students' considerations.

> My understanding of mathematics has greatly improved. Now I not only understand the proof but also draw the right logical conclusions, ask the right questions to move forward. And it's very interesting. I liked mathematics back in school, but now I felt what a beautiful and interesting science it is, I loved it.
>
> It has become much easier to study Mathematical Analysis than it was at the beginning. Although even now there are difficulties – I do not always understand everything from the first time. But I consider it great progress that I already know how to ask competently, to explain what I do not understand. I think this is the main thing I learned in half a year.
>
> During the six months that we have been studying Mathematical Analysis, it is thanks to the teaching methods that my perception of mathematics has changed a lot. I began to see and understand the connections between different mathematical concepts and facts, even from different mathematical courses.
>
> I learned to use mathematical symbols. Thanks to geometric interpretations I intuitively feel some mathematical facts, ideas of their proof. But there are still problems: it is difficult to study the material in the textbook, I do not always understand the evidence written there.
>
> I became more confident. I'm not afraid to express my opinion, to suggest the idea of proving a theorem or solving a problem.
>
> The teacher always helps when needed, but never gives a ready-made solution or answer, we always come to them ourselves. The big problem was the inability to read mathematical literature on my own, even a textbook. It is getting much better now.
>
> Discussing tasks in small groups helps a lot to understand the learning material. After all, each of us is faced with a problematic issue, to solve which everyone needs to express

their opinion. But the discussion is much more effective when the teacher participates in it. He corrects the course of our discussion, helps to resolve disputes, gives some clarifications.

I'm used to proving every theorem now, we didn't practice that at school. I distinguish between necessary and sufficient conditions. I learned to understand what it is about, you need to ask the right question, it helps in understanding the material. There is no longer any fear of making assumptions or hypotheses. It remained difficult for me: not to confuse something in the definition of the limit in the "$\epsilon - \delta$" language, but I almost overcame it.

At the end of the academic year, I started to have my hypotheses to solve the given problem. The teacher always encourages me to ask the right questions, give examples and counterexamples. Now I'm set up to prove the problem myself, not just rewrite and learn by heart like it was done at school.

The best results of the exam in the Mathematical Analysis course of entrants 2019 (the teacher modified the course based on IBME), compared to 2018 also confirm the effectiveness of the used IBME strategy and justify the pedagogical expectations (see Figure 18.6).

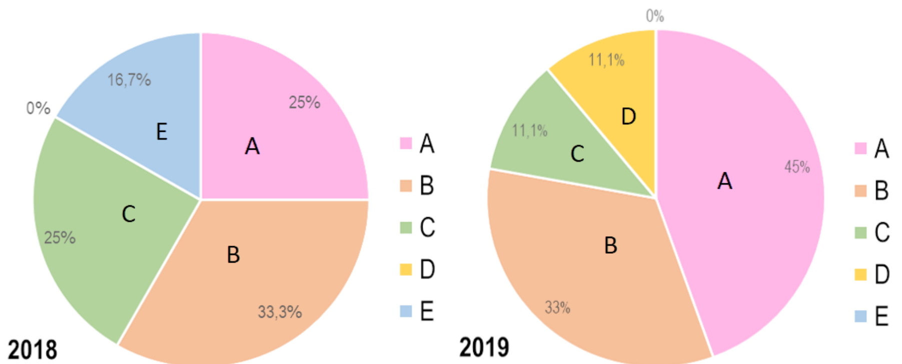


FIGURE 18.6. Mathematical analysis of exam results (12 students in 2018, 9 students in 2019) (coloured version in the ebook).

As it was mentioned above, the need for transition to blended and distance learning, which is urgent today, creates certain problems in the organisation of the educational process. The main problems of distance learning, which were noted by students, are low quality of communication during video conferencing, lack of constant communication between students and the teacher, which makes it harder to understand the learning material, and learning becomes slower. Students unanimously preferred in-person learning because of the possibility of direct communication and cooperation. Among the advantages of online classes, students included only the fact that after the class you can watch a video at a convenient time if necessary and that you do not need to spend time on the way to university. However, the students appreciated the conducted classes in an online format, noting that the teacher put maximum effort to make the lectures and practical classes full-fledged.

The teacher also noted the worsened conditions for communication in online learning through video conferencing. In particular, the possibility of organising productive work in small groups is practically lost, because the teacher cannot hear the discussion in all groups at the same time, as it happens in in-person education. Therefore, the teacher cannot react on time, join the discussion, leaving the process of forming students' research skills to themselves.

In addition, our 2018 survey of students on their perception of ELC teaching materials in mathematical courses, posted in the LMS Moodle, shows that most ELCs did not provide interactive learning and did not create positive intrinsic motivation in students, i.e., did not promote active, research-oriented learning in partnership. We offered some didactic and methodological approaches to the preparation of content and organisation of activities in ELC in Mathematics during the implementation of blended learning based via LMS Moodle to improve their quality and efficiency (Astafieva et al., 2019).

It is important to note that the practice of using IBME is of interest to teachers of mathematics courses, who are not participants of the project but are part of the academic community. Dissemination of community outcomes occurs through the exchange of experiences with colleagues in seminars and the involvement of colleagues in research, which is reflected in joint publications, the project website, and social media pages.

We are aware that our practice is neither the only correct one nor the only possible to achieve high results in learning mathematics. The described case only confirms that inquiry-based approaches can be effective, that our proposed approach to learning with its help can be useful, and that some of the ideas about IBME can be implemented in all practices of mathematics teachers.

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Part 4

Lessons Learned

Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project
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CHAPTER 19

Epilogue

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The production of this book is the last stage in a journey which 31 authors, belonging to the 8 university institutions, partners of the PLATINUM Project, have shared focusing on Inquiry-Based Mathematics Education (IBME). The journey has been a recurring metaphor in an attempt to illustrate the meaning of life. We find the term in the displacement from one territory to another, the walk that separates a departure from an arrival, as a space to give meaning to what has been lived. The metaphor of the journey thus becomes a commonplace worth analysing and so we wish to use it to synthesise the lessons learned from the production of this book. In the following paragraphs we will take a brief tour, like someone moving her finger to find herself on a map, to analyse this commonplace that we recognise as valuable for the university community in mathematics.

19.1. Conceptualisation of Inquiry at University Level

It would be pertinent to look back on the content page in Inquiry in University Mathematics Teaching and Learning. In Part 1 about conceptual foundations of the PLATINUM Project, taking an inquiry perspective on mathematics education we have tried to specify what exactly we mean by inquiry. In our definition of inquiry, we used the idea of emulating mathematicians: seeking to know, exploring, investigating, finding out, asking questions, solving problems, looking critically, developing Inquiry "as a way of being." In Part 2, we explained and provided examples of how we addressed the processes and principles with which we engaged in our PLATINUM activity. With all chapters in Part 3, all cases of the partners' development of inquiry-based teaching, we set out to explain our approach and how to help others achieve it. What we have done is to set out on a journey in which the reader has been encouraged to participate, to consider the way in which Inquiry processes can take place in University Mathematics Teaching and Learning. The reader can find examples in the fields of analysis, algebra, statistics, etc. to understand what makes some experiences successful and help others to act as mathematical thinkers on their learning journey.

For us, as editors, it has been an interesting study to see the development of various dimensions of an approach to inquiry in mathematics. One that has reinforced engagement and enhanced learning is the concept of inquiry community. Inquiry communities permeated PLATINUM activity at all levels from small groups of colleagues working together to develop teaching, to the whole PLATINUM community working together to produce this book. We recognise this throughout the chapters. Although we all agreed on the support of these dimensions, the underlying theoretical frameworks in mathematics education offered different perspectives and nuances. This is unsurprising since we come from different cultures and different traditions. Although it has led to differences along the way, requiring a rapprochement of positions and a deeper grounding of our original proposal, we have recognised it as valuable. It has brought richness and depth to the project and enhanced our personal conceptions of learning and teaching. (Chapters 3, 7 and 9 provide examples of this).

19.2. Pathways in the Design of Materials in University Mathematics

Chapter 4 of Part 1 and the chapters in Part 2 provide insights into the learning avenues experienced in the design of materials for teaching and learning under the inquiry approach. From our perspective, what counts are the actions developed in teaching: This means teaching in ways that result in students' developing deep understandings of mathematics as a subject matter.

First: We have, individually and collectively, gained ideas about productive teaching practices in inquiry-based learning.

Second: Since a great deal of current classroom practice does not match our ideas about productive teaching practices at university level, we would like to see changes, first of all in our own professional practice. This book represents progress in our understandings of the complex construct of inquiry approach at university level which we hope readers will find valuable for their own teaching.

Third: The commitment to social integration in the educational institutions of the partners has encouraged breaking down barriers and ensuring an inclusive learning environment so that students with identified needs are able to participate "independently and equally." Although all the partners have experience of teaching students with identified (special) needs, the specialised focus, presented in Chapter 4 considering the diversity of students, has influenced units and tasks in the courses that were designed.

In the creating teaching units for student inquiry, different authors pointed out several emphases that have become in the context of IBME:

(1) the 'authenticity' of inquiry questions, the connection of students' activities with their real life;
(2) the epistemological relevance of inquiry questions from a mathematical perspective in the statements and in the guides formulated by the teacher;
(3) the experimental and applied dimensions of mathematics and the interdisciplinary knowledge that this demands;
(4) the collaborative dimension of the inquiry process;
(5) student diversity as a cross-cutting issue to be taken into account alongside developing understanding of addressing identified needs.

We learn from the exploration of these ways of teaching that promote students' inquiry-based learning of mathematics. We look at where we were now in terms of lectures, didactic and pedagogic processes. Based on these experiences, we are more aware of how we can use inquiry-based processes to help students engage with mathematics more conceptually. We become familiar with exploring and developing our own practices, which is itself an inquiry process. So, in the formal presentation of material to students in university mathematics in this volume—including mathematics majors and mathematics as a service subject—we recognise conceptual obstacles that make the pathway very difficult for students to travel successfully. There are still avenues that will require further deepening in the future: inquiry with technology, inquiry and modelling, inquiry and algorithmic processes, inquiry and interdisciplinary projects.

19.3. Methods and Materials for Professional Development of Lecturers

As stated in Chapters 7 and 16, the PLATINUM team's idea of 'professional development' is based on an epistemology of professional knowledge which takes into account the contextualised nature of the lecturer's experience (experience knowledge) and the personalised knowledge of the practice.

In the development of this project and in the preparation of this book there has been a continuous dialogue between local and universal perspectives. The countries that take part have different cultural and social contexts even though we are all in the university environment, the different mathematics and mathematics education traditions. Community of Inquiry and the associated concept of Critical Alignment have been central to both the theoretical and practical aspects of PLATINUM. Community of Inquiry (CoI) can be seen to derive from CoP (community of practice) where the 'alignment' requirement of CoP is developed to become 'Critical Alignment.' This is mentioned (briefly) in Chapter 2. In the PLATINUM project, in relation to the professional development of lecturers, we took an expansive view of the notion, where there are no geographical boundaries to such communities.

Members of these communities have developed their teaching in certain ways, both personal and institutional. In the experience of local and international workshops for professional development lecturers highlighted that teaching decisions and actions are not 'just' actions, they manifest a wealth of knowledge, goals, and orientations. Lecturers, being members of the large community of educative practice, have been enculturated into a set of pedagogical and didactical assumptions (that is, beliefs and orientations) that shape their practice in mathematics education and which have varied in accord with the different national cultures.

A challenge in PLATINUM was the creation of nested communities of inquiry even if they came from different countries with respect to several issues essential to the project's aims at university level (conceptualisation of inquiry, teacher professional development and assessment). The achievement of joint work in these communities can be in the production of the different intellectual outputs whose work is reflected in different chapters. For example, in Part 2 of the book: in Chapter 7 the joint work on teacher education of Norway, Germany, Netherlands and Spain, in Chapter 8 the joint work on mathematical modelling and inquiry-based learning of Norway, Czech Republic and Ukraine, or in Chapter 9 the joint work of Czech Republic, Netherlands, Spain and UK on methods and instruments in assessment from the perspective of inquiry-based mathematics education. This is important if we want to think about lecturer/teacher change, in that it helps to identify the orientations of lecturers which are valuable in this perspective or which should be changed.

We would like to point out that, in PLATINUM, inquiry-based practice is itself an important source of professional development as can be seen in the chapters in Part 3. We worked together during more than three years, we are all more knowledgeable in what inquiry means for us, in what we can do to engage with inquiry, and the differing ways in which we can engage. If readers of this book or new colleagues join us, we can draw them into our communities and they can learn through working alongside others with critical alignment.

19.4. And to Conclude

Our project aims to be ambitious in promoting a classroom culture change and lecturer identity from the IBME approach to teaching and learning. Much of the literature in IBL, IBME etc. focuses on inquiry in mathematics. In PLATINUM, this is largely the central layer of the model (Chapters 2 and 3). We extend it into the middle layer as lecturers think about and design mathematical tasks for students' inquiry-based activity. The PLATINUM model extends much further to the process of inquiring into learning and teaching and the associated development that comes with this. We learn from the experience, to ask questions, to share with colleagues, to develop our own practice through this process, and to research the outcomes. In our journey, in order to systematise the experience and to become more systematic and rigorous, through critical alignment, we have been engaged in what we call 'Developmental Research.' Developmental research is research that both studies the developmental process and also contributes to the development itself. It develops new knowledge in both theory and practice, encapsulated within the three-layer model. This is the uniqueness of the PLATINUM model.

We point out this dimension of novelty and uniqueness of the PLATINUM model within scientific production because a successful change in university teaching requires both a set of new teaching techniques and a constellation of orientations (about mathematics, about concepts, about what students can do, and about classroom practices) in order to take hold. Change in improving teaching can be long and slow journey (it has taken us more than three years), but it is worth it. We hope that the contributions in this volume have offered progress towards this goal.