GENERALIZED PLANAR CURVES AND QUATERNIONIC GEOMETRY

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Abstract. Motivated by the analogies between the projective and the almost quaternionic geometries, we first study the generalized planar curves and mappings. We follow, recover, and extend the classical approach, see e.g. [10, 11]. Then we exploit the impact of the general results in the almost quaternionic geometry. In particular we show, that the natural class of $\mathbb{H}$–planar curves coincides with the class of all geodesics of the so called Weyl connections and preserving this class turns out to be the necessary and sufficient condition on diffeomorphisms to become morphisms of almost quaternionic geometries.

Various concepts generalizing geodesics of affine connections have been studied for almost quaternionic and similar geometries. Let us point out the generalized geodesics defined via generalizations of normal coordinates, cf. [2] and [3], or more recent [4, 12]. Another class of curves was studied in [11] for the hypercomplex structures with additional linear connections. The latter authors called a curve c quaternionic planar if the parallel transport of each of its tangent vectors $\dot{c}(t_0)$ along c was quaternionic colinear with the tangent field $\dot{c}$ to the curve. Yet another natural class of curves is given by the set of all unparameterized geodesics of the so called Weyl connections, i.e. the connections compatible with the almost quaternionic structure with normalized minimal torsion. The latter connections have remarkably similar properties for all parabolic geometries, cf. [3], and so their name has been borrowed from the conformal case. In the setting of almost quaternionic structures there were studied first in [8] and they are also called Oproiu connections, see [1].

The first author showed in [6] that actualy the concept of quaternionic planar curves was well defined for the almost quaternionic geometries and their Weyl connections. Moreover, it did not depend on

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Date: November 14, 2005.

The essential part of writing of this paper has been done with support of .... during the stay of the second author at the University of Auckland. The authors announce the support of grants GACR Nr. 201/05/H005 and 201/05/2117, respectively.

the choice of a particular Weyl connection and it turned out that the quaternionic planar curves were just all unparameterized geodesics of all Weyl connections.

The aim of this paper is to find further analogies of Mikeš's classical results in the realm of the almost quaternionic geometry. On the way we simplify, recover, and extend results on generalized planar mappings, explain results from [6], and finally we show that morphisms of almost quaternionic geometries are just those diffeomorphisms which leave invariant the class of all unparameterized geodesics of Weyl connections.

1. Motivation and background on quaternionic geometry

There are many equivalent definitions of almost quaternionic geometry to be found in the literature. Let us start with the following one:

Definition 1.1. *Let M be a smooth manifold of dimension $4n$. An almost hypercomplex structure on M is a triple (I, J, K) of smooth affinors in $\Gamma(T^*M \otimes TM)$ satisfying*

$$I^2 = J^2 = -E, \quad K = I \circ J = -J \circ I$$

where $E = \mathrm{id}_{TM}$.

An almost quaternionic geometry is a rank four subbundle $\mathcal{Q} \subset T^*M \otimes TM$ locally generated by the identity E and a hypercomplex structure.

An almost complex geometry on a $2m$–dimensional manifold M is given by the choice of the affinor J satisfying $J^2 = -E$. Let us observe, that such a J is uniquely determined within the rank two subbundle $\langle E, J\rangle \subset TM$, up to its sign. (Indeed, if $\hat{J} = aE+bJ$, then the condition $\hat{J}^2 = -E$ implies $a = 0$ and $b = \pm 1$.)

Thus we may view the almost quaternionic geometry as a straightforward generalization of this case. Here, a similar simple computation reveals that the rank three subbundle $\langle I, J, K\rangle$ is invariant of the choice of the generators and this is the definition we may find in [1]. More explicitly, different choices will always satisfy $\hat{I} = aI + bJ + cK$ with $a^2 + b^2 + d^2 = 1$, and similarly for J and K. Let us also remark that the 4–dimensional almost quaternionic geometry coincides with 4-dimensional conformal Riemannian geometries.

1.2. **The frame bundles.** Equivalently, we can define an almost quaternionic structure $\mathcal{Q}$ on M as a reduction of the linear frame bundle P^1M to an appropriate structure group, i.e. as a G–structure with the structure group of all automorphisms preserving the subbundle $\mathcal{Q}$. We may view such frames as linear mappings $T_xM \to \mathbb{H}^n$ which carry over the multiplications by $i, j, k \in \mathbb{H}$ onto some of the possible choices for I, J, K. Thus, a further reduction to a fixed hypercomplex structure leads to the structure group $GL(n, \mathbb{H})$ of all quaternionic linear mappings on $\mathbb{H}^n$. Additionaly, we have to allow morphisms which do not leave the affinors I, J, K invariant but change them within the subbundle $\mathcal{Q}$. As well known, the resulting group is

$$G_0 = GL(n, \mathbb{H}) \times_{\mathbb{Z}_2} Sp(1)$$

where $Sp(1)$ are the unit quaternions in $GL(1, \mathbb{H})$, see e.g. [9].

We shall write $\mathcal{G}_0 \subset P^1M$ for this principal G_0–bundle defining our structure.

The simplest example of such a structure is well understood as the homogeneous space

$$\mathbb{P}_n\mathbb{H} = G/P$$

where $G_0 \subset P$ are the subgroups in $G = SL(n+1, \mathbb{H})$

$$G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix};\ A \in GL(n, \mathbb{H}),\ \mathrm{Re}(a \det A) = 1 \right\},$$

$$P = \left\{ \begin{pmatrix} a & Z \\ 0 & A \end{pmatrix};\ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in G_0,\ Z \in (\mathbb{H}^n)^* \right\}.$$

Since P is a parabolic subgroup in the semisimple Lie group G, the almost quaternionic geometry is an instance of the so called parabolic geometries. All these geometries enjoy a rich and quite uniform theory similar to the classical development of the conformal Riemannian and projective geometries, but we shall not need much of this here. We refer the reader to [3] and the references therein.

1.3. **Weyl connections.** The classical prolongation procedure for G–structures starts with finding a minimal available torsion for a connection belonging to the structure on the given manifold M. Unlike the projective and conformal Riemannian structures where torsion free connections always exist, the torsion has to be allowed for the almost quaternionic structures in general in dimensions bigger than four. The standard normalization comes from the general theory of parabolic geometries and we shall not need this in the sequel. The details may be found for example in [5], [2], another and more classical point of view can be found in [9]. The only essential point for us is that all connections compatible with the given geometry sharing the unique normalized torsion are parametrized by smooth one–forms on M. In analogy to the conformal Riemannian geometry we call them *Weyl connections* for the given almost quaternionic geometry on M.

The almost quaternionic geometries with Weyl connections without torsion are called *quaternionic geometries.*

From the point of view of prolongations of G–structures, the class of all Weyl connections defines a reduction $\mathcal{G}$ of the semiholonomic second order frame bundle over the manifold M to the structure group P and the Weyl connections ∇ are in bijective correspondence with G_0–equivariant sections $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the natural projection.

As mentioned above, the difference of two Weyl connections is a one–form and, also in full analogy to the conformal geometry, there are neat formulae for the change of the covariant derivatives of two such connections $\hat{\nabla}$ and ∇ in terms of their difference $\Upsilon = \hat{\nabla} - \nabla \in \Omega^1(M)$.

1.4. **Adjoint tractors.** In order to understand the latter formulae, we introduce the so called adjoint tractors. They are sections of the vector bundle

$$\mathcal{A} = \mathcal{G}_0 \times_{G_0} \mathfrak{g}.$$

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{H})$ carries the G_0–invariant grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix};\ A \in \mathfrak{gl}(n, \mathbb{H}),\ \operatorname{Re}(a + \operatorname{Tr} A) = 0 \right\},$$
$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix};\ Z \in (\mathbb{H}^n)^* \right\}, \quad \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix};\ X \in \mathbb{H}^n \right\}$$

Moreover, $TM = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$, $T^*M = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_1$, and we obtain on the level of vector bundles

$$\mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1 = TM \oplus \mathcal{A}_0 \oplus T^*M.$$

The key feature of $\mathcal{A}$ is that all further G_0–invariant objects on $\mathfrak{g}$ are carried over to the adjoint tractors, too. In particular, the Lie bracket on G induces an algebraic bracket $\{\ ,\ \}$ on $\mathcal{A}$.

Now we may write down easily the transformation formula. Let $\hat{\nabla}$ and ∇ be two Weyl connections, $\hat{\nabla} - \nabla = \Upsilon \in \Gamma(\mathcal{A}_1)$. Then for all tangent vector fields $X, Y \in \Gamma(\mathcal{A}_{-1})$,

$$\hat{\nabla}_X Y = \nabla_X Y + \{\{X, \Upsilon\}, Y\}, \tag{1}$$

see [3] or [2, 5] for the proof. Notice that the internal bracket results in an endomorphism on TM, while the external bracket is just the evaluation of this endomorphism on Y (all this is read of the brackets in the Lie algebra easily).

2. Generalized planar curves and mappings

Various geometric structures on manifolds are defined as smooth distributions in the vector bundle $T^*M \otimes TM$ of all endomorphisms of the tangent bundle. We have seen the two examples of almost complex and almost quaternionic structures above. Let us extract some formal properties from these examples.

Definition 2.1. *Let A be a smooth ℓ–dimensional vector subbundle in $T^*M \otimes TM$, such that the identity affinor $E = \mathrm{id}_{TM}$ restricted to T_xM belongs to $A_xM \subset T^*_xM \otimes T_xM$ at each point $x \in M$. We say that M is equipped by an A–structure.*

For any tangent vector $X \in T_xM$ we shall write $A(X)$ for the vector subspace

$$A(X) = \{F(X);\ F \in A_xM\} \subset T_xM$$

and we call $A(X)$ the *A–hull of the vector X*. Similarly, the *A–hull of a vector field* will be the subbundle in TM obtained pointwise. Notice that the dimension of such a subbundle in TM may vary pointwise.

We say that the A–structure has *generic rank ℓ* if the dimension of A is ℓ, and for each $x \in M$, the subset of vectors $(X, Y) \in T_xM \times T_xM$, such that the A–hulls $A(X)$ and $A(Y)$ generate a vector subspace $A(X) \oplus A(Y)$ of dimension 2ℓ, is open and dense.

Let us point out some examples:

- The $\langle E \rangle$–structure is of generic rank one on all manifolds of dimensions at least 2.
- Any almost complex structure or almost product structure $\langle E, J \rangle$ is of generic rank two on all manifolds of dimensions at least 4.
- Any almost quaternionic structure is of generic rank four on all manifolds of dimensions at least 8.

Definition 2.2. *Let M be a smooth manifold with a given A–structure and a linear connections ∇. A smooth curve $c : \mathbb{R} \to M$ is told to be A–planar if*

$$\nabla_{\dot c}\dot c \in A(\dot c).$$

Clearly, A planarity means that the parallel transport of any tangent vector to c has to stay within the A–hull $A(\dot c)$ of the tangent vector field $\dot c$ along the curve. Moreover, this concept does not depend on the parametrization of the curve c.

Definition 2.3. *Let M be a manifold with a linear connection ∇ and an A–structure, while N be another manifold with a linear connection*

$\hat{\nabla}$ and a B–structure. A diffeomorphism $f : M \to N$ is called (A, B)–planar if each A–planar curve c on M is mapped onto the B–planar curve $F \circ c$ on N.

Example 2.4. *The 1–dimensional A structures are all trivial since the distribution $A = \langle E \rangle$ must be given just as the linear hull of the identity affinor E, by the definition. Obviously, the $\langle E \rangle$–planar curves on a manifold M with a linear connection ∇ are exactly the unparametrized geodesics. Moreover, two connections ∇ and $\bar{\nabla}$ without torsion are projectively equivalent (i.e. they share the same unparametrized geodesics) if and only if their difference satisfies $\bar{\nabla}_X Y - \nabla_X Y = \alpha(X)Y + \alpha(Y)X$ for some one–form α on M. The latter condition can be rewritten as*

$$\bar{\nabla} - \nabla \in \Gamma(T^*M \odot \langle E \rangle) \subset \Gamma(S^2T^*M \otimes TM). \tag{2}$$

The latter condition on projective structures may be also rephrased in the terms of morphisms: A diffeomorphism $f : M \to M$ is geodetical (an automorphism of the projective structure) if $f \circ c$ is an unparametrized geodesic for each geodesic c and this happens if and only if the symmetrization of the difference $f^*\nabla - \nabla$ is a section of $T^*M \odot \langle E \rangle$. We are going to generalize the above example in the rest of this section.

In the case $A = \langle E \rangle$, the $(\langle E \rangle, B)$–planar mappings are called simply *B–planar*. They map each geodesic curve on (M, ∇) onto a B–planar curve on $(N, \hat{\nabla}, B)$.

Each ℓ dimensional A structure $A \subset T^*M \otimes TM$ determines the distribution $A^{(1)}$ in $S^2T^*M \otimes TM$, given at any point $x \in M$ by

$$A^{(1)}_x M = \{\alpha_1 \odot F_1 + \cdots + \alpha_\ell \odot F_\ell;\ \alpha_i \in T^*_x M, F_i \in A_x M\}.$$

Theorem 2.5. *Let M be a manifold with a linear connection ∇, let N be a manifold of the same dimension with a linear connection $\hat{\nabla}$ and an A–structure of generic rank ℓ, and suppose* $\dim M \geq 2\ell$. *Then a diffeomorphism $f : M \to N$ is A–planar if and only if*

$$\mathrm{Sym}(f^*\hat{\nabla} - \nabla) \in f^*(A^{(1)}) \tag{3}$$

where Sym *denotes the symmetrization of the difference of the two connections.*

Proof. Let us first observe that the entire claim of the theorem is of local character. Thus, identifying the objects on N with their pullbacks on M, we may assume that $M = N$ and $f = \mathrm{id}_M$.

Next, let us observe that the A–planarity of $f : M \to N$ does not at all depend on the possible torsions of the connection. Indeed, we always test expressions od the type $\nabla_{\dot{c}}\dot{c}$ for a curve c and thus deforming ∇ into $\nabla' = \nabla + T$ by adding some torsion will not effect the results.

Thus, without any loss of generality, we may assume that the connections ∇ and $\hat{\nabla}$ share the same torsion, and then we may ommit the symmetrization from equation (3).

Finally, we may fix some (local) basis $E = F_0, F_i$, $i = 1, \dots \ell - 1$, of A, i.e. $A = \langle F_0, \dots, F_{\ell-1}\rangle$. Then the condition in equation (3) says

$$\hat{\nabla} = \nabla + \sum_{i=0}^{\ell-1} \alpha_i \odot F_i \tag{4}$$

for some suitable one–forms α_i on M. Of course, the existence of such forms does not depend on our choice of the basis of A.

The quite simplified statement we now have to prove is:

Claim 1. *Let M be a manifold of dimension at least 2ℓ, ∇ and $\hat{\nabla}$ two connections on M with the same torsion, and consider an A–structure of generic rank ℓ on M. Then each geodesic curve with respect to ∇ is A–planar with respect to $\hat{\nabla}$ if and only if there are one–forms α_i satisfying equation (4).*

Assume first we have such forms α_i, and let c be a geodesic for ∇. Then equation (4) implies $\hat{\nabla}_{\dot{c}}\dot{c} \in A(\dot{c})$ so that c is A–planar, by definition.

The other implication is the more difficult one. Assume each (unparametrized) geodesic c is A–planar. This implies that the symmetric difference tensor $P = \hat{\nabla} - \nabla \in \Gamma(S^2T^*M \otimes TM)$ satisfies

$$P(\dot{c}, \dot{c}) = \hat{\nabla}_{\dot{c}}\dot{c} \in \langle \dot{c}, F_1(\dot{c}), \dots, F_{\ell-1}(\dot{c})\rangle.$$

Let $\mathcal{V} \subset TM$ be the open and dense subspace of all vectors $X \in TM$ for which $\{X, F_1(X), \dots, F_{\ell-1}(X)\}$ are linearly independent. Now, for each $X \in \mathcal{V}$ there are the unique coefficients $\alpha_i(X) \in \mathbb{R}$ such that

$$P(X, X) = \sum_{i=0}^{\ell-1} \alpha_i(X) F_i(X). \tag{5}$$

The essential technical step in the proof of our Claim 1 is to show that all functions α_i are in fact restrictions of smoth one–forms on M. Let us notice, that P is a symmetric bilinear tensor and thus it is determined by the restriction of $P(X, X)$ to arbitrarily small open non–empty subset of the arguments X in each fiber.

Claim 2. *If a smooth symmetric tensor P is determined over the above defined subspace $\mathcal{V}$ by (5), then the functions $\alpha_i : \mathcal{V} \to \mathbb{R}$ are smooth and their restrictions to the individual rays (half–lines) generated by vectors in $\mathcal{V}$ are linear.*

Let us fix a local smooth basis $e_i \in TM$, the dual basis e^i, and consider the induced dual bases e_I and e^I on the multivectors and exterior forms. Let us consider the smooth mapping

$$\chi : \Lambda^\ell TM \setminus \{0\} \to \Lambda^\ell T^*M, \quad \chi(\textstyle\sum a_I e^I) = \displaystyle\sum \frac{a_I}{\sum a_I^2} e_I.$$

Now, for all non–zero tensors $\Xi = \sum a_I e^I$, the evaluation $\langle \Xi, \chi(\Xi) \rangle$ is the constant function 1, while $\chi(k \cdot \Xi) = k^{-1}\chi(\Xi)$.

Next, we define for each $X \in \mathcal{V}$

$$\tau(X) = \chi\big(X \wedge F_1(X) \wedge \cdots \wedge F_{\ell-1}(X)\big)$$

and we may compute the unique coefficients α_i from (5):

$$\begin{aligned}
\alpha_0(X) &= \langle P(X,X) \wedge F_1(X) \wedge F_2(X) \wedge \cdots \wedge F_{\ell-1}(X), \tau(X) \rangle \\
\alpha_1(X) &= \langle X \wedge P(X,X) \wedge F_2(X) \wedge \cdots \wedge F_{\ell-1}(X), \tau(X) \rangle \\
&\ \vdots \\
\alpha_{\ell-1}(X) &= \langle X \wedge F_1(X) \wedge F_2(X) \wedge \cdots \wedge P(X,X), \tau(X) \rangle.
\end{aligned}$$

In particular, this proves the first part our Claim 2.

Let us now consider a fixed vector $X \in \mathcal{V}$. The defining formula (5) for α_i implies $\alpha_i(kX) = k\alpha_i(X)$, for each real number $k \neq 0$. Passing to zero with positive k shows that α does have the limit 0 in the origin and so we may extend the definition of α_i's (and validity of formula (5)) to the entire cone $\mathcal{V} \cup \{0\}$ by setting $\alpha_i(0) = 0$ for all i.

Finally, along the ray $\{tX; t > 0\} \subset \mathcal{V}$ the derivative $\alpha'(tX) = \frac{d}{dt}\alpha(tX)$ satisfies $\alpha'(kX) = \alpha'(X)$ for all $X \in \mathcal{V}$, and thus $\alpha'(tX)$ is constant along the ray. This proves the rest of Claim 2.

Now, in order to complete the proof of Theorem 2.5, we have to prove the following assertion.

Claim 3. *If a smooth symmetric tensor P is determined over the above defined subspace $\mathcal{V} \cup \{0\}$ by (5), then the coefficients α_i are smooth one–forms on M and the tensor P is given by*

$$P(X,Y) = \frac{1}{2}\sum_{i=0}^{\ell-1}\big(\alpha_i(Y)F_i(X) + \alpha_i(X)F_i(Y)\big).$$

The entire tensor P is obtained through polarization from its evaluations $P(X,X)$, $X \in TM$,

$$P(X,Y) = \tfrac{1}{2}\big(P(X+Y, X+Y) - P(X,X) - P(Y,Y)\big), \tag{6}$$

and again, the entire tensor is determined by its values on arbitrarily small non–empty open subset of X and Y in each fiber.

The summands on the right hand side have values in the following subspaces:

$$\begin{aligned}P(X+Y,X+Y) &\in \langle X+Y, F_1(X+Y),\dots,F_{\ell-1}(X+Y)\rangle \subset \\ &\quad \langle X, F_1(X),\dots F_{\ell-1}(X), Y, F_1(Y),\dots,F_{\ell-1}(Y)\rangle,\\ P(X,X) &\in \langle X, F_1(X),\dots F_{\ell-1}(X)\rangle,\\ P(Y,Y) &\in \langle Y, F_1(Y),\dots F_{\ell-1}(Y)\rangle.\end{aligned}$$

Since we have assumed that A has generic rank ℓ, the subspace $\mathcal{W} \in TM \times_M TM$ of vectors (X,Y) such that all the values

$$\{X, F_1(X),\dots,F_{\ell-1}(X), Y, F_1(Y),\dots,F_{\ell-1}(Y)\}$$

are linearly independent is open and dense. Clearly $\mathcal{W} \subset \mathcal{V}\times_M\mathcal{V}$. Moreover, if $(X,Y)\in\mathcal{W}$ than $F_0(X+Y),\dots,F_{\ell-1}(X+Y)$ are independent, i.e. $X+Y\in\mathcal{V}$. Inserting (5) into (6), we obtain

$$P(X,Y) = \sum_{i=0}^{\ell-1}\big(d_i(X,Y)F_i(X) + e_i(X,Y)F_i(Y)\big).$$

For all $(X,Y)\in\mathcal{W}$, the coefficients $d_i(X,Y) = \frac{1}{2}(\alpha_i(X+Y)-\alpha_i(X))$ at $F_i(X)$, and $e_i(X,Y) = \frac{1}{2}(\alpha_i(X+Y)-\alpha_i(Y))$ at $F_i(Y)$ in the latter expression are uniquely determined. The symmetry of P implies $d_i(X,Y) = e_i(Y,X)$. If $(X,Y)\in\mathcal{W}$ then also $(sX,tY)\in\mathcal{W}$ for all non-zero reals s,t and the linearity of P in the individual arguments yields for all real parameters s,t

$$std_i(X,Y) = sd_i(sX,tY).$$

Thus the functions α_i satisfy

$$\alpha_i(sX+tY)-\alpha_i(sX) = t(\alpha_i(X+Y)-\alpha_i(X)).$$

Since $\alpha_i(tX) = t\alpha_i(X)$, in the limit $s\to 0$ this means

$$\alpha_i(Y) = \alpha_i(X+Y)-\alpha_i(X).$$

Thus α_i are additive over the open and dense set $(X,Y)\in\mathcal{W}$. Choosing a basis of T_xM such that each couple of basis elements is in $\mathcal{W}$, this shows that α_i are restrictions of linear forms, as required. □

Theorem 2.6. *Let M be a manifold with a linear connection ∇ and an A–structure, N be a manifold of the same dimension with a linear connection $\hat\nabla$, and suppose that B has generic rank ℓ. Then a diffeomorphism $f: M\to N$ is (A,B)–planar if and only if f is B–planar and $A(X)\subset (f^*(B))(X)$ for all $X\in TM$.*

Proof. As in the proof of Theorem 2.5, we may restrict ourselves to some open submanifolds, fix generators F_i for B, assume that $f = \mathrm{id}_M$ and both connections ∇ and $\hat{\nabla}$ share the same torsion, and prove the equivalent local assertion to our theorem:

Claim. *Each A–planar curve c with respect to $\hat{\nabla}$ is B–planar with respect to ∇, if and only if the symmetric difference tensor $P = \hat{\nabla} - \nabla$ is of the form (5) with smooth one–forms α_i, $i = 0, \dots, \ell - 1$ and $A(X) \subset B(X)$ for each $X \in TM$.*

Obviously, the condition in this statement is sufficient. So let us deal with its neccessity.

Since every (A, B)–planar mapping is also B–planar, Theorem 2.5 (or the equivalent Claim 1 in its proof) says that

$$P(X, X) = \sum_{j=0}^{\ell} \alpha_i(X) F_i(X)$$

for uniquely given smooth one–forms α_i.

Now, consider a fixed $F \in A$ and suppose $F(X) \notin B(X)$. Since we assume that all $\langle E, F\rangle$–planar curves c in M are B–planar, we may proceed exactly as in the beginning of the proof of Theorem 2.5 to deduce that

$$P(X, X) = \sum_{j=0}^{\ell} \alpha_i(X) F_i(X) + \beta(X) F(X)$$

on a neighborhood of X, with some unique functions α_i and β.

The comparison of the latter two unique expressions for $P(X, X)$ shows that $\beta(X)$ vanishes. But since $F(X) \neq X$, there definitly are curves which are $\langle E, F\rangle$–planar and tangent to X, but not $\langle E\rangle$–planar. Thus, the assumption in the theorem would lead to $\beta(X) \neq 0$. Consequently, our choice $F(X) \notin B(X)$ cannot be achieved and we have proved $A(X) \subset B(X)$ for all $X \in TM$. $\square$

3. Results on quaternionic geometries

The main result of this section is:

Theorem 3.1. *Let $f : M \to M'$ be a diffeomorphism between two almost quaternionic manifolds of dimension at least eight. Then f is a morphism of the geometries if and only if it preserves the class of unparametrized geodesics of all Weyl connections on M and M'.*

This theorem will follow easily from the results of Section 2 and its proof requires only a few quite simple formal steps. Let Q be the subbundle in $T^*M \otimes TM$ defining the almost quaternionic geometry on M as in 1.1.

Lemma 3.2. *A curve $c : \mathbb{R} \to M$ is Q–planar with respect to at least one Weyl connection ∇ on M if and only if c is Q–planar with respect to all Weyl connections on M.*

Proof. For a Weyl connection ∇ and a curve $c : \mathbb{R} \to M$, the defining equation for Q–planarity reads $\nabla_{\dot c}\dot c \in Q(\dot c)$. If we choose some hypercomplex structure within Q, we may rephraze this condition as: $\nabla_{\dot c}\dot c = \dot c \cdot q$ for a quaternion q. Now the formula (1) for the deformation of the Weyl connections implies

$$\hat\nabla_{\dot c}\dot c = \nabla_{\dot c}\dot c + \{\{\dot c, \Upsilon\}, \dot c\} = 2\dot c \cdot \Upsilon(\dot c).$$

Indeed, this is the consequence of the computation of the Lie bracket in $\mathfrak{g}$ of the corresponding elements $\dot c \in \mathfrak{g}_{-1}$, $\Upsilon \in \mathfrak{g}_1$:

$$\begin{aligned} [[\dot c, \Upsilon], \dot c] &\simeq \left[\left[\begin{pmatrix} 0 & 0 \\ \dot c & 0 \end{pmatrix}, \begin{pmatrix} 0 & \Upsilon \\ \dot 0 & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & 0 \\ \dot c & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} 0 & 0 \\ 2\dot c \cdot \Upsilon(\dot c) & 0 \end{pmatrix} \simeq 2\dot c \cdot \Upsilon(\dot c). \end{aligned}$$

Thus we see that if there is such a quaternion q for one Weyl connection, then it exists also for all of them. □

Definition 3.3. *A curve $c : \mathbb{R} \to M$ is called $\mathbb{H}$–planar if it is Q–planar with respect to each Weyl connection ∇ on M.*

Theorem 3.4. *Let M be a manifold with an almost quaternionic geometry. Then a curve $c : \mathbb{R} \to M$ is $\mathbb{H}$–planar if and only if c a geodesic of some Weyl connection, up to parametrization.*

Proof. Let us remark that c is a geodesic for ∇ if and only if $\nabla_{\dot c}\dot c = 0$. Thus, the statement follows immediately from the computation in the proof of Lemma 3.2. Indeed, if c is $\mathbb{H}$–planar, then choose any Weyl connection ∇ and pick up Υ so that $\hat\nabla_{\dot c}\dot c$ vanishes. □

3.5. **Proof of Theorem 3.1.** Every morphism of almost quaternionic geometries preserves the class of Weyl connections and thus also the class of their geodesics.

We have to prove the opposit implication. This means, we have two manifolds with almost quaternionic structures (M, Q), (N, Q') and a diffeomorphism $f : M \to N$ which is (Q, Q')–planar. Then Theorem 2.6 implies that $Q(X) = (f^*Q')(X)$ for each $X \in TM$ (since they both have the same dimension). But this is equivalent to the statement $Q \subset f^*Q'$, i.e. we have proved that f preserves the defining subbundles and thus is a morphism of the almost quaternionic structures.

This still has to be checked! Probably nearly the same computation as showing the relations of hypercomplex structures within Q, but we have to show that the coefficients do not depend on X!!

3.6. **Final remarks.** All curves in the four–dimensional quaternionic geometries are $\mathbb{H}$–planar by the definition. Thus this concept starts to be interesting in higher dimensions only, and all of them are covered by Theorem 3.1.

The class of the unparametrized geodesics of Weyl connections is well defined for all parabolic geometries. Our result for the quaternionic geometries suggests the question, whether a similar statement holds for other geometries as well.

?????Some more????

References

[1] Alekseevsky, D.V.; Marchiafava, S.; Pontecorvo, M., Compatible Complex Structures on Almost Quaternionic Manifolds, Trans. Amer. Math. Soc 351 (1999), no. 3, 997–1014.

[2] Bailey, T.N.; Eastwood, M.G., Complex paraconformal manifolds – their differential geometry and twistor theory, Forum Math., 3 (1991), 61-103.

[3] Čap, A.; Slovák, J., Weyl Structures For Parabolic geometries, Math. Scand. 93 (2003), 53-90.

[4] Čap, A.; Slovák, J.; Žádník, V.; On distinguished curves in parabolic geometries, Transform. Groups, Vol. 9, No. 2, ()2004), 143–166.

[5] Gover, A.R.; Slovák, J., Invariant local twistor calculus for quaternionic structures and related geometries, J. Geom. Physics, 32 (1999), no. 1, 14–56.

[6] Hrdina, J., $\mathbb{H}$–planar curves, Proceedings Int. Conf. Differential Geometry and Applications, Prague, August 2004, to appear.

[7] Joyce, D.D., Compact Manifolds with Special Holonomy, Oxford Mathematical Monographs, 2000.

[8] Oproiu, V., Integrability of almost quaternal structures, An. st. Univ. ”Al. I. Cuza” Iasi 30 (1984), 75-84.

[9] Salamon, S.M.; Differential Geometry of Quaternionic Manifolds, Ann. Scient. ENS 19 (1987), 31–55.

[10] Mikes J., Sinyukov N.S. On quasiplanar mappings of spaces of affine connection, Sov. Math. 27, No.1 (1983), 63–70.

[11] Mikeš, J.; Němčíková, J.; Pokorná, O., On The Theory Of The 4-Quasiplanar Mappings Of Almost Quaternionic Spaces, Rediconti del circolo matematico di Palermo, Serie II, Suppl. 54 (1998), 75–81.

[12] Žádník, V., Generalized Geodesics, Ph.D. thesis, Masaryk university in Brno, 2003.

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