Morphisms of almost product projective geometries

Jaroslav Hrdina, Jan Slovák *

Institute of Mathematics and Statistics, Faculty of Science, Masaryk University, Janackovo nam 2a, 662 95 Brno, Czech Republic

We discuss almost product projective geometry and the relations to a distinguished class of curves. Our approach is based on an observation that well known general techniques[2,5,8] apply, and our goal is to illustrate the power of the general parabolic geometry theory on a quite explicit example. Therefore, some rudiments of the general theory are mentioned on the way, too.

1. An almost product projective structure

Let M be a smooth manifold of dimension $2m$. *An almost product structure* on M is a smooth trace–free affinor J in $\Gamma(T^\star M \otimes TM)$ satisfying $J^2 = \mathrm{id}_{TM}$.

For better understanding, we describe an almost product structure at each tangent space in a fixed basis, i.e. with the help of real matrices:

$$J := \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_m \end{pmatrix}.$$

The eigenvalues of J have to be ± 1 and $T_xM = T^L_x M \oplus T^R_x M$, where the subspaces are of the form

$$T^L M := J_+ = \left\{ \begin{pmatrix} c \\ 0 \end{pmatrix} \mid c \in \mathbb{R}^m \right\},\ T^R M := J_- = \left\{ \begin{pmatrix} 0 \\ c \end{pmatrix} \mid c \in \mathbb{R}^m \right\}.$$

Thus, we can equivalently define an almost product structure J on M as a reduction of the linear frame bundle P^1M to the appropriate structure group, i.e. as a G–structure with the structure group L of all automorphisms preserving the affinor J:

$$L := \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} | A, B \in GL(m, \mathbb{R}) \right\} \cong GL(m, \mathbb{R}) \times GL(m, \mathbb{R}) \subset GL(2m, \mathbb{R}).$$

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These G–structures are of infinite type, however each choice of a linear connection ∇ compatible with the affinor J, i.e. $\nabla J = 0$, determines a finite type geometry similar to products of projective structures, which we shall study below.

The difference Υ between two projectively equivalent connections is a smooth one–form given by

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi.$$

Instead, we shall consider a class of connections parameterized also by all smooth one–forms Υ, but with the transformation rule

$$\begin{aligned}\hat{\nabla}_{\xi^L+\xi^R}(\eta^L+\eta^R) = \nabla_{\xi^L+\xi^R}(\eta^L+\eta^R) &+ \Upsilon^L(\xi^L)\eta^L + \Upsilon^L(\eta^L)\xi^L \\ &+ \Upsilon^R(\xi^R)\eta^R + \Upsilon^R(\eta^R)\xi^R,\end{aligned} \tag{1}$$

where the indices at the forms and fields indicate the components in the subbundles T^LM and T^RM, respectively. Clearly such a transformed connection will make J parallel again.

Definition 1.1. Let M be a smooth manifold of dimension $2m$. *An almost product projective structure* on M is a couple $(J, [\nabla])$, where J is an almost product structure, ∇ is a linear connection preserving J, and $[\nabla]$ is the class of connections obtained from ∇ by the transformations given by all smooth one–forms Υ as in (1).

The standard tool for the study of G–structures is the classical prolongation theory. The first step usually provides a class of distinguished connections with minimized (or preferred) torsions. In our case, the torsion

$$T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$$

of ∇ will always involve the obstructions against the integrability of T^LM and T^RM, i.e. the appropriate projections of the antisymmetric term given by the Lie bracket of vector fields.

Our next goal is to identify a class of almost product projective geometries which fit into a wider class of the so called normal parabolic geometries. We shall see, that the appropriate requirement on the torsion of ∇ will be that the integrability obstructions are the only non–zero components.

Example 1.1 (The homogeneous model). *Let us consider the homogeneous space* $M = G/P$ *given as the product of two projective spaces* $G_L/P_L \times G_R/P_R$*, i.e.* $G_L = G_R = SL(n+1, \mathbb{R})$ *while* $P = P_L \times P_R$*, where* $P_R = P_L$ *is the usual parabolic subgroup corresponding to the block upper triangular matrices of the block sizes* $(1, n)$.

Clearly, any product connection on M built of the linear connections ∇^L and ∇^R from the two projective classes on the product components provides a homogeneous example of an almost product projective structure. At the same time, the Maurer–Cartan form on $G = G_L \times G_R$ provides the homogeneous model of the $|1|$-graded parabolic geometries of type (G, P). The Lie algebra of $P \times P$ is a parabolic subalgebra of the real form $\mathfrak{sl}(n+1, \mathbb{R}) \oplus \mathfrak{sl}(n+1, \mathbb{R})$ of the complex algebra $\mathfrak{sl}(n+1, \mathbb{C}) \oplus \mathfrak{sl}(n+1, \mathbb{C})$.

In matrix form, we can illustrate the grading from our example as:

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{g}_0^L & \mathfrak{g}_1^L & 0 & 0 \\ \mathfrak{g}_{-1}^L & \mathfrak{g}_0^L & 0 & 0 \\ 0 & 0 & \mathfrak{g}_0^R & \mathfrak{g}_1^R \\ 0 & 0 & \mathfrak{g}_{-1}^R & \mathfrak{g}_0^R \end{pmatrix}$$

2. Parabolic geometries and Weyl connections

The classical prolongation procedure for G–structures starts with finding a minimal available torsion for connections belonging to the structure on the given manifold M. Our normalization will come from the general theory of parabolic geometries.

As a rule, the $|1|$–graded parabolic geometries are completely given by certain classical G–structures on the underlying manifolds.[1,2] In our case, however, both components of the semisimple Lie algebra belong to the series of exceptions and only the choice of an appropriate class of connections defines the Cartan geometry completely.[1,2]

The normalization of the Cartan geometries is based on cohomological interpretation of the curvature. More explicitly, the normal geometries enjoy co–closed curvature.[1,2]

In our case, the appropriate cohomology is easily computed by the Künneth formula from the classical Kostant's formulae and the computation[10] provides all six irreducible components of the curvature, only two of which are of torsion type. Of course, the integrability obstructions of the bundles $T^L M$ and $T^R M$ are just those two. Therefore, a *normal almost product projective structure* $(M, J, [\nabla])$ has this minimal torsion.

Now, the general theory provides for each normal almost product projective structure $(M, J, [\nabla])$ the construction of the unique principal bundle $\mathcal{G} \to M$ with structure group P, equipped by the normal Cartan connection. Furthermore, there is the distinguished class of the so called *Weyl connections* corresponding to all choices of reductions of the parabolic structure group to its reductive subgroup G_0. All Weyl connections are parametrized just by smooth one–forms and they all share the torsion of the Cartan connection.

The transformation formulae for the Weyl connections are generally given by the Lie bracket in the algebra in question. Of course, this is just the formula (1) we used for the definition of the almost product projective structures. Let us express (1) as:[2]

$$\hat{\nabla}_X Y = \nabla_X Y + [[X, \Upsilon], Y] \tag{2}$$

where we use the frame forms $X, Y : \mathcal{G} \to \mathfrak{g}_{-1}$ of vector fields, and similarly for $\Upsilon : \mathcal{G} \to \mathfrak{g}_1$. Consequently, $[\Upsilon, X]$ is a frame form of an affinor valued in $\mathfrak{g}_0$ and the bracket with Y expresses the action of such an affinor on the differentiated field. According to the general theory, this transformation rule works for all covariant derivatives ∇ with respect to Weyl connections.[2]

The general theory of the $|1|$–graded geometries also provides a formula for the unique normal Cartan connection ω in terms of any chosen Weyl connection and its curvature. Technically, this formula computes the difference between the two connections as the so called Schouten Rho tensor $\mathrm{P} = -\Box^{-1}\partial^* R$, where R is the curvature of the Weyl connection, ∂^* is the Kostant's codifferential, and $\Box^{-1}$ is the inverse of the Kostant's Laplacian.[2]

This observation shows that our normal almost product projective geometries form a category equivalent to normal Cartan connections of type (G, P) with homogeneous model discussed in Example 1.1.

3. *J*–planar curves

Let us remind the notion of planarity with respect to affinors:[8]

Definition 3.1. Let (M, J) be an almost product structure. A smooth curve $c : \mathbb{R} \to M$ is called J–planar with respect to a linear connection ∇ if $\nabla_{\dot{c}}\dot{c} \in \langle \dot{c}, J(\dot{c})\rangle$, where $\dot{c}$ means the tangent velocity field along the curve c and the brackets indicate the linear hulls of the two vectors in the individual tangent spaces.

Next, we observe that there is a nice link between J–planar curves and connections from the class defining an almost product projective structure:

Theorem 3.1. *Let $(M, J, [\nabla])$ be a smooth almost product projective structure on a manifold M. A curve c is J–planar with respect to at least one Weyl connection $\bar{\nabla}$ on M if and only if there is a parametrization of c which is a geodesic trajectory of some Weyl connection ∇. Moreover, this happens if and only if c is J–planar with respect to all Weyl connections.*

Proof. First, let us compute the bracket $[[\dot{c},\Upsilon],\dot{c}]$ appearing in (2). We write $\dot{c} = c^L + c^R$ and similarly for Υ:

$$\left[\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ c^L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^R & 0 \end{pmatrix}, \begin{pmatrix} 0 & \Upsilon^L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Upsilon^R \\ 0 & 0 & 0 & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & 0 & 0 & 0 \\ c^L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^R & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c^L\Upsilon^L(c^L) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Upsilon^R(c^R)c^R & 0 \end{pmatrix}.$$

We may write shortly $[[\dot{c},\Upsilon],\dot{c}] = c^L\Upsilon^L(c^L) + \Upsilon^R(c^R)c^R$.

Now, suppose $c : \mathbb{R} \to M$ is a geodetics with respect to a connection ∇ and compute:

$$\begin{aligned}\hat{\nabla}_{\dot{c}}\dot{c} = \nabla_{\dot{c}}\dot{c} + [[\dot{c},\Upsilon],\dot{c}] = [[\dot{c},\Upsilon],\dot{c}] = c^L\Upsilon^L(c^L) + \Upsilon^R(c^R)c^R = \\ = (\Upsilon^L(c^L) + \Upsilon^R(c^R))\dot{c} + (\Upsilon^L(c^L) - \Upsilon^R(c^R))J(\dot{c}) \in \langle \dot{c}, J(\dot{c})\rangle.\end{aligned}$$

Thus, the geodetics c is J–planar with respect to connection $\hat{\nabla}$, i.e. with respect to all Weyl connections.

On the other hand, let us suppose that $c : \mathbb{R} \to M$ is J–planar with respect to $\bar{\nabla}$, i.e. $\bar{\nabla}_{\dot{c}}\dot{c} = a(\dot{c})\dot{c} + b(\dot{c})J(\dot{c})$ for some functions $a(\dot{c})$ and $b(\dot{c})$ along the curve. We have to find a one form $\Upsilon = \Upsilon^L + \Upsilon^R$ such that the formula for the transformed connection kills all the necessary terms along the curve c. Since there are many such forms Υ, there is a Weyl connection ∇ such that $\nabla_{\dot{c}}\dot{c} = 0$. □

4. Generalized planar curves and mappings

Definition 4.1.[5] Let M be a smooth manifold of dimension n. Let A be a smooth l–rank ($l < n$) vector subbundle in $T^*M \otimes TM$, such that the identity affinor $E = id_{TM}$ restricted to T_xM belongs to $A_x \subset T^*M \otimes TM$ at each point $x \in M$. We say that M is equipped by ℓ–dimensional A–*structure.*

An almost product projective structure $(M, J, [\nabla])$ carries the A–structure with $A = \langle E, J\rangle$. Let us remind, that the A–planarity does not depend on the choice of the Weyl connection ∇ in the class in view of Theorem 3.1.

For any tangent vector $X \in T_xM$, we shall write $A(X)$ for the vector subspace

$$A(X) = \{F(X) |\, F \in A_xM\} \subset T_xM,$$

and we call $A(X)$ the A*–hull of the vector* X. In order to work out relations between morphisms of our geometries and planarity, we shall follow our earlier work.[5] We start by quoting a few definitions and results:

Definition 4.2. Let (M, A) be a smooth manifold M equipped with an ℓ–dimensional A–structure. We say that A–structure has

- *generic rank* ℓ if for each $x \in M$ the subset of vectors $(X, Y) \in T_xM \oplus T_xM$, such that the A–hulls $A(X)$ and $A(Y)$ generate a vector subspace $A(X) \oplus A(Y)$ of dimension 2ℓ is open and dense.
- *weak generic rank* ℓ if for each $x \in M$ the subset of vectors
$$\mathcal{V} := \{X \in T_xM |\, \dim A(X) = \ell\}$$
is open and dense in T_xM.

Lemma 4.1. *Every almost product structure (M, J) on a manifold M, $\dim M \geq 2$, has weak generic rank 2.*

Proof. Let as consider that $X \notin \mathcal{V}$, this fact implies that $\exists F \in A : F(X) = 0$, i.e. the vector X has to be an eigenvector of J, i.e. X has to belong to m–dimensional subspace T^LM or T^RM of TM. Finally, the complement $\mathcal{V}$ is open and dense. □

Theorem 4.1.[4] *Let (M, A) be a smooth manifold of dimension n with ℓ–dimensional A–structure, such that $\ell \geqq 2 \dim M$.*

- *If A_x is an algebra (i.e. for all $f, g \in A_x$, $fg := f \circ g \in A_x$) for all $x \in M$ and A has weak generic rank ℓ, then the structure has generic rank ℓ.*
- *If $A_x \subset T^\star_xM \otimes T_xM$ is an algebra with inversion then A has generic rank ℓ.*

Each almost product structure on a smooth manifold M has generic rank 2 because of lemma and theorem above.

Definition 4.3.[5,8] Let M be a smooth manifold equipped with an A–structure and a linear connection ∇.

- A smooth curve C is told to be *A–planar* if there is trajectory $c : \mathbb{R} \to M$ such that $\nabla_{\dot{c}}\dot{c} \in A(\dot{c})$.
- Let $\bar{M}$ be another manifold with a linear connection $\bar{\nabla}$ and B–structure. A diffeomorphism $f : M \to \bar{M}$ is called *(A, B)–planar* if each A–planar curve C on M is mapped onto the B–planar curve $f_\star C$ on M. In the special case, where A is the trivial structure given by $\langle E\rangle$, we talk about B–planar maps.

Theorem 4.2.[5] *Let M be a manifold with a linear connection ∇, let N be a manifold of the same dimension with a linear connection $\bar{\nabla}$ and with A-structure of generic rank ℓ, and suppose $dimM \geqq 2\ell$. Then a diffeomorphism $f : M \to N$ is a A-planar if and only if*

$$\mathrm{Sym}(f^*\bar{\nabla} - \nabla) \in f^*(A^{(1)}) \tag{3}$$

where Sym *denotes the symmetrization of the difference of the two connection.*

Theorem 4.3.[5] *Let M be a manifold with linear connection ∇ and an A-structure, N be a manifold of the same dimension with a linear connection $\bar{\nabla}$ and B-structure with generic rank ℓ. Then a diffeomorphism $f : M \to N$ is (A, B)-planar if and only if f is B-planar and $A(X) \subset (f^\star(B))(X)$ for all $X \in TM$.*

Theorem 4.4.[4,5] *Let (M, A), (M', A') be smooth manifolds of dimension m equipped with A–structure and A'–structure of the same generic rank $\ell \leq 2m$ and assume that the A–structure satisfies the property*

$$\forall X \in T_xM,\, \forall F \in A, \exists c_X \mid \dot{c}_X = X,\, \nabla_{\dot{c}_X}\dot{c}_X = \beta(X)F(X), \tag{4}$$

where $\beta(X) \neq 0$. If $f : M \to M'$ is an (A, A')–planar mapping, then f is a morphism of the A–structures, i.e $f^*A' = A$.

Finally, we can apply the above concepts and theorems to our situation:

Theorem 4.5. *Let $f : M \to M'$ be a diffeomorphism between two almost product projective manifolds of dimension at least four. Then f is a morphism of the almost product structures if and only if it preserves the class of unparameterized geodesics of all Weyl connections on M and M'.*

Proof. In view of the series of theorems above and the fact that $f^*A = A$ implies $f^*J = \pm J$ (i.e. f^*J preserves the eigenspaces of J), we only have to prove that an almost product structure (M, J) has the property (4). Consider $F = aE + bJ \in \langle E, F\rangle$ and $X \in TM$. First we may solve the system of equations:

$$a(X) + b(X) = 2\Upsilon^L(X^L)$$
$$a(X) - b(X) = 2\Upsilon^R(X^R)$$

with respect to Υ. Second, we may to define a new connection $\hat{\nabla}$, where:

$$\hat{\nabla}_Y Z = \nabla_Y Z - [[Y, \Upsilon], Z]$$

and we may find a geodetics c of $\hat{\nabla}$, such that $\dot{c} = X$. Finally, we recognize that c is the requested curve from (4) because:

$$\nabla_X X = \bar{\nabla}_X X - [[X, \Upsilon], X] = [[X, \Upsilon], X] =$$

$$= (\Upsilon^L(X^L) + \Upsilon^R(X^R))X = a(X)X + b(X)J(X) = F(X). \qquad \square$$

5. Almost complex projective structure

Another possibility of a Lie algebra whose complexification is $\mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})$ is the complex algebra $\mathfrak{sl}(n, \mathbb{C})$ viewed as a real algebra. The corresponding geometry analogous to the almost product projective geometry is the almost complex projective geometry.

Definition 5.1. Let M be a smooth manifold of dimension $2m$. *An almost complex projective structure* on M is defined by a smooth affinor I on M satisfying $I^2 = -\operatorname{id}_{TM}$ and by a choice of a linear connection ∇ with $\nabla I = 0$.

In this case, the minimal torsion equals the Nijenhuis tensor obstructing the integrability of I and we may use the same technique as above to verify that this is the only component of the torsion of the normal almost complex projective geometry.

There has been a lot of interest on such geometries recently. All the above approach works equally well and we shall come to further discussion on these questions in the context of existing literature elsewhere.

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