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COMBINATORIAL DIFFERENTIAL GEOMETRY AND IDEAL BIANCHI–RICCI IDENTITIES II – THE TORSION CASE

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Abstract. This paper is a continuation of [2], dealing with a general, not-necessarily torsion-free, connection. It characterizes all possible systems of generators for vector-field valued operators that depend naturally on a set of vector fields and a linear connection, describes the size of the space of such operators and proves the existence of an 'ideal' basis consisting of operators with given leading terms which satisfy the (generalized) Bianchi–Ricci identities without corrections.

Methods of the paper are based on the graph complex approach developed in [8, 9]. Most of the proofs in this paper are parallel to the proofs of the analogous statements for the torsion-free case given in [2].

Plan of the paper. In Section 1 we recall the basis features of the torsion case and quote the classical reduction theorem due to Łubczonok [5]. In Section 2 we formulate the main results of the paper (Theorems A–D) and show some explicit calculations. The difference from the torsion-free case is obvious already in the formulation of Theorem A. In contrast to the corresponding [2, Theorem A], we allow the basis operators to be indexed by a two-parameter set S rather than just natural numbers $n \geq 3$ as in the torsion-free case. We had to accept this generality because the 'classical' basis consist of two families of operators – the iterated covariant derivatives of the curvature *and* the iterated covariant derivatives of the torsion, see Subsection 2.6.

All proofs are contained in Section 3. As they are parallel to the proofs in the torsion-free case of [2], we had two extremal choices – either to give no proofs at all, saying that they are 'obvious' modifications of the proofs of [2], or to modify the proofs of [2] and include them in full length. We choose a compromise and included only proofs which are 'manifestly' different from the torsion-free case, namely those dealing directly with the corresponding graph complex.

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Conventions: At several places, the abbreviation *l.o.t.* for 'lower order terms' is used. Its precise meaning will either be explained or will be clear from the context. We assume that this paper is read in conjunction with [2], so we refer to that article very often. We will however keep the formulation of the main theorems self-consistent.

Notation: We will use notation parallel to that of [2], the distinction against the torsion-free case will be marked by the tilde $\widetilde{(-)}$. For instance, while *Con* denoted in [2] the bundle of *torsion-free* connections, here *Con* denotes the bundle of *all* linear connections and $\widetilde{Con}$ the subbundle of torsion-free connections.

1. Reduction theorems for non-symmetric connections

In this paper, M will always denote a smooth manifold. The letters X, Y, Z, U, $V,\dots$, with or without indexes, will denote (smooth) vector fields on M. We also consider a linear (generally non-symmetric) connection Γ on M with Christoffel symbols $\Gamma^\lambda_{\mu\nu}$, $1 \le \lambda,\mu,\nu \le \dim(M)$, see, for example, [14, Section III.7]. The symbol ∇ will denote the covariant derivative with respect to Γ, and by $\nabla^{(r)}$ we will denote the sequence of iterated covariant derivatives up to order r, i.e. $\nabla^{(r)} = (\mathrm{id}, \nabla, \dots, \nabla^r)$. The letter R will denote the curvature $(1,3)$-tensor field and the letter T will denote the torsion $(1,2)$-tensor field of Γ. In order to get formulas compatible with the notation of our earlier paper [2], we assume $R(X,Y)(Z) = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$, i.e. our curvature tensor R differs from the curvature tensor of [14] by the sign.

For non-symmetric connections we have (see, for example, [14, Section III.5]) the first Bianchi identity

$$\underset{X,Y,Z}{\mathfrak{S}} R(X,Y)(Z) = -\underset{X,Y,Z}{\mathfrak{S}} \big[(\nabla_X T)(Y,Z) + T(T(X,Y),Z)\big]\,, \tag{1.1}$$

and the second Bianchi identity

$$\underset{U,X,Y}{\mathfrak{S}} (\nabla_U R)(X,Y)(Z) = -\underset{U,X,Y}{\mathfrak{S}} R(T(U,X),Y)(Z)\,, \tag{1.2}$$

where $\mathfrak{S}$ is the cyclic sum over the indicated vector fields. Further, if Φ is a $(1,r)$-tensor field, then we have the Ricci identity

$$\begin{aligned}(\nabla_X\nabla_Y\Phi - \nabla_Y\nabla_X\Phi)(Z_1,\dots,Z_r) &= -R(X,Y)(\Phi(Z_1,\dots,Z_r)) \\ &+ \sum_{j=1}^{r} \Phi(Z_1,\dots,R(X,Y)(Z_j),\dots,Z_r) - (\nabla_{T(X,Y)}\Phi)(Z_1,\dots,Z_r)\,.\end{aligned} \tag{1.3}$$

It is well-known, see, for example, [14, Section III.7], that Γ induces a torsion-free connection $\widetilde{\Gamma}$ whose Christoffel symbols are obtained by symmetrization of the Christoffel symbols of Γ. Then $\Gamma = \widetilde{\Gamma} + \frac{1}{2}T$ and we get

$$\begin{aligned}R(X,Y)(Z) &= \widetilde{R}(X,Y)(Z) - \tfrac{1}{2}(\widetilde{\nabla}_X T)(Y,Z) + \tfrac{1}{2}(\widetilde{\nabla}_Y T)(X,Z) \\ &\quad - \tfrac{1}{4}T(X,T(Y,Z)) + \tfrac{1}{4}T(Y,T(X,Z)) - \tfrac{1}{2}T(T(X,Y),Z)\,,\end{aligned} \tag{1.4}$$

where $\widetilde{R}$ is the curvature of $\widetilde{\Gamma}$ and $\widetilde{\nabla}$ is the covariant derivative with respect to $\widetilde{\Gamma}$. Further, $\nabla_X Y = \widetilde{\nabla}_X Y + \frac{1}{2}T(X,Y)$ which implies, for any $(1,r)$-tensor field Φ,

$$(1.5)\qquad (\nabla_X \Phi)(Y_1,\dots,Y_r) = (\widetilde{\nabla}_X \Phi)(Y_1,\dots,Y_r) + \tfrac{1}{2}T(X,\Phi(Y_1,\dots,Y_r)) - \tfrac{1}{2}\sum_{j=1}^{r}\Phi(Y_1,\dots,T(X,Y_j),\dots,Y_r)$$

If we apply covariant derivatives on the identity (1.5), we get

$$(1.6)\qquad \nabla^r\Phi = \widetilde{\nabla}^r\Phi + l.o.t.\,,$$

where $l.o.t.$ is a polynomial constructed from $\nabla^{(r-1)}\Phi$ and $\nabla^{(r-1)}T$. Especially, for the torsion tensor,

$$(1.7)\qquad \nabla^r T = \widetilde{\nabla}^r T + l.o.t.$$

Similarly, from (1.4),

$$(1.8)\qquad \nabla^r R = \widetilde{\nabla}^r \widetilde{R} + o.t.\,,$$

where $o.t.$ is a polynomial constructed from $\nabla^{(r+1)}T$ and $\nabla^{(r-1)}R$.

It is well-known, [17, p. 91] and [15, p. 162], that differential concomitants (natural polynomial tensor fields in terminology of natural bundles [3, 4, 12, 13, 16]) depending on tensor fields and a torsion-free connection can be expressed through given tensor fields, the curvature tensor of given connection and their covariant derivatives. This result is known as the first (operators on connections only) and the second reduction theorems.

Using the above splitting of connections with torsions into the symmetric connections and the torsions, we can prove the reduction theorem for connections with torsions, see Łubczonok [5]. Let us quote Łubczonok's formulation of the *reduction theorem* for connections with torsions.

Theorem 1.1. *If Ω is a differential concomitant of order r of $\{\Phi_k\}_{k=1,\dots,s}$ and of the linear connection $\Gamma^\lambda_{\mu\nu}$ with torsion, then Ω is an (ordinary) concomitant of the quantities:*

$$\{\widetilde{\nabla}_{\kappa_l,\dots,\kappa_1}\Phi_k\}\,,\qquad l=0,1,\dots,r\,,\quad k=1,\dots,s\,,$$
$$\{\widetilde{\nabla}_{\kappa_l,\dots,\kappa_1}T^\lambda_{\mu\nu}\}\,,\qquad l=0,1,\dots,r\,,$$
$$\{\widetilde{\nabla}_{\kappa_1,\dots,\kappa_l}\widetilde{R}_\rho{}^\lambda{}_{\mu\nu}\}\,,\qquad l=0,1,\dots,r-1\,,$$

where $\widetilde{R}_\rho{}^\lambda{}_{\mu\nu}$, $\widetilde{\nabla}$ denote the curvature tensor and the covariant derivative with respect to $\widetilde{\Gamma}^\lambda_{\mu\nu}$.

Formally, we can write $\Omega(\partial^{(r)}\Phi_k,\partial^{(r)}\Gamma) = \widetilde{\Omega}(\widetilde{\nabla}^{(r)}\Phi_k,\widetilde{\nabla}^{(r)}T,\widetilde{\nabla}^{(r-1)}\widetilde{R})$.

Remark 1.2. The original Łubczonok's result quoted above assumes the same maximal order r of derivatives of Φ_k and Γ. But Theorem 1.1 holds if the order with respect to Γ is $(r-1)$ only, i.e. $\Omega(\partial^{(r)}\Phi_k,\partial^{(r-1)}\Gamma) = \widetilde{\Omega}(\widetilde{\nabla}^{(r)}\Phi_k,\widetilde{\nabla}^{(r-1)}T,\widetilde{\nabla}^{(r-2)}\widetilde{R})$. Theorem 1.1 is in fact valid for any order $s \geq r-1$ with respect to Γ, see, for example, [1].

Thanks to the above relations (1.6)–(1.8) between covariant derivatives with respect to Γ and $\widetilde{\Gamma}$, we can reformulate the reduction Theorem 1.1 directly for connections with torsions.

Theorem 1.3. *If Ω is a differential concomitant of order r of $\{\Phi_k\}_{k=1,\dots,s}$ and of the linear connection $\Gamma^\lambda_{\mu\nu}$ with torsion, then Ω is an ordinary concomitant of the quantities:*

$$
\begin{aligned}
&\{\nabla_{\kappa_l,\dots,\kappa_1}\Phi_k\}\,, && l=0,1,\dots,r\,,\ \ k=1,\dots,s\,,\\
&\{\nabla_{\kappa_l,\dots,\kappa_1}T^\lambda_{\mu\nu}\}\,, && l=0,1,\dots,r\,,\\
&\{\nabla_{\kappa_1,\dots,\kappa_l}R_\rho{}^\lambda{}_{\mu\nu}\}\,, && l=0,1,\dots,r-1\,,
\end{aligned}
$$

i.e.

$$\Omega(\partial^{(r)}\Phi_k,\partial^{(r)}\Gamma)=\overline{\Omega}(\nabla^{(r)}\Phi_k,\nabla^{(r)}T,\nabla^{(r-1)}R)\,.$$

Remark 1.4. We get, from Theorem 1.1 and Theorem 1.3, that $(\nabla^{(r)}\Phi_k,\nabla^{(r)}T,$ $\nabla^{(r-1)}R)$ and $(\widetilde{\nabla}^{(r)}\Phi_k,\widetilde{\nabla}^{(r)}T,\widetilde{\nabla}^{(r-1)}\widetilde{R})$ form two systems of generators of differential concomitants of order r of $\{\Phi_k\}_{k=1,\dots,s}$ and of the linear connection $\Gamma^\lambda_{\mu\nu}$ with torsion (in order r). These two systems of generators satisfy different identities. For the system $(\nabla^{(r)}\Phi_k,\nabla^{(r-1)}T,\nabla^{(r-2)}R)$ we have the Bianchi and the Ricci identities (1.1), (1.2) and (1.3) (and their covariant derivatives), while for the system of generators $(\widetilde{\nabla}^{(r)}\Phi_k,\widetilde{\nabla}^{(r-1)}T,\widetilde{\nabla}^{(r-2)}\widetilde{R})$ we have the Bianchi and the Ricci identities (and their covariant derivatives) for torsion-free connections recalled, for instance, in [2, Section 2].

It follows from the Ricci identity that we can take also the *symmetrized* covariant derivatives $(\overset{S}{\nabla}{}^{(r)}\Phi_k,\overset{S}{\nabla}{}^{(r)}T,\overset{S}{\nabla}{}^{(r-1)}R)$ and $(\overset{S}{\widetilde{\nabla}}{}^{(r)}\Phi_k,\overset{S}{\widetilde{\nabla}}{}^{(r)}T,\overset{S}{\widetilde{\nabla}}{}^{(r-1)}\widetilde{R})$ as two different bases of differential concomitants of order r. The Bianchi-Ricci identities for such symmetric bases are, however, quite involved. We will prove, in Theorem C, that there are bases whose elements satisfy the "ideal" Bianchi-Ricci identities (with vanishing right hand sides) similar to the ideal Bianchi-Ricci identities for symmetric connections, [2].

2. Main results

2.1. Operators we consider. Let *Con* be the natural bundle functor of linear, not necessarily torsion-free, connections [3, Section 17.7] and T the tangent bundle functor. We will consider natural differential operators $\mathcal{O}\colon Con\times T^{\otimes d}\to T$ acting on a linear connection and d vector fields, $d\geq 0$, which are linear in the vector field variables, and which have values in vector fields. We will denote the space of natural operators of this type by $\mathfrak{Nat}(Con\times T^{\otimes d},T)$.

To make the formulation of the main results of this paper self-consistent, we recall almost verbatim some definitions of [2]. Define the *vf-order* (vector-field order) resp. the *c-order* (connection order) of a differential operator $\mathcal{O}\colon Con\times T^{\otimes d}\to T$ as the order of $\mathcal{O}$ in the vector field variables, resp. the connection variable.

2.2. Traces. Let $\mathcal{O}$ be an operator acting on vector fields $X_1,\dots,X_d$ and a connection Γ, with values in vector fields. Suppose that $\mathcal{O}$ is a linear order 0 differential operator in X_i for some $1 \leq i \leq d$. This means that the local formula $O(\Gamma, X_1, \ldots, X_d)$ for $\mathcal{O}$ is a linear function of the coordinates of X_i and does not contain derivatives of the coordinates of X_i. In this situation we define $Tr_i(\mathcal{O}) \in \mathfrak{Nat}(Con \times T^{\otimes(d-1)}, R)$ as the operator with values in the bundle R of smooth functions given by the local formula

$$Tr_i(O)(\Gamma, X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d) := \\ \mathrm{Trace}(O(\Gamma, X_1, \ldots, X_{i-1}, -, X_{i+1}, \ldots, X_d)\colon \mathbb{R}^n \to \mathbb{R}^n)\,.$$

Whenever we write $Tr_i(\mathcal{O})$ we tacitly assume that the trace makes sense, i.e. that $\mathcal{O}$ is a linear order 0 differential operator in X_i.

2.3. Compositions. Let $\mathcal{O}'\colon Con \times T^{\otimes d'} \to T$ and $\mathcal{O}''\colon Con \times T^{\otimes d''} \to T$ be operators as in 2.1. Assume that $\mathcal{O}'$ is a linear order 0 differential operator in X_i for some $1 \leq i \leq d'$. In this situation we define the *composition* $\mathcal{O}' \circ_i \mathcal{O}''\colon Con \times T^{\otimes(d'+d''-1)} \to T$ as the operator obtained by substituting the value of the operator $\mathcal{O}''$ for the vector-field variable X_i of $\mathcal{O}'$. As in 2.2, by writing $\mathcal{O}' \circ_i \mathcal{O}''$ we signal that $\mathcal{O}'$ is of order 0 in X_i.

2.4. Iterations. By an *iteration* of differential operators we understand applying a finite number of the following 'elementary' operations:

(i) permuting the vector-fields inputs of a differential operator $\mathcal{O}$,

(ii) taking the pointwise linear combination $k' \cdot \mathcal{O}' + k'' \cdot \mathcal{O}''$, $k', k'' \in \mathbb{R}$,

(iii) performing the composition $\mathcal{O}' \circ_i \mathcal{O}''$, and

(iv) taking the pointwise product $Tr_i(\mathcal{O}') \cdot \mathcal{O}''$.

There are 'obvious' relations between the above operations. The operations $\circ_i$ in (iii) satisfy the 'operadic' associativity and compatibility with permutations in (i), see properties (1.9) and (1.10) in [10, Definition II.1.6]. Other 'obvious' relations are the commutativity of the trace, $Tr_j(\mathcal{O}' \circ_i \mathcal{O}'') = Tr_i(\mathcal{O}'' \circ_j \mathcal{O}')$ and its 'obvious' compatibility with permutations of (i).

2.5. We denote, for each $n \geq 2$, by $E^0(n)$ the induced representation

$$E^0(n) := \mathrm{Ind}\,^{\Sigma_n}_{\Sigma_{n-2}\times\Sigma_2}(\mathbf{1}_{\Sigma_{n-2}} \otimes \mathbb{R}[\Sigma_2])\,,$$

where $\mathbb{R}[\Sigma_2]$ is the regular representation of Σ_2 and $\mathbf{1}_{\Sigma_{n-2}}$ the trivial representations of the symmetric group Σ_{n-2}. The space $E^0(n)$ expresses the symmetries of the derivative

$$\frac{\partial^{n-2}\Gamma^{\omega}_{\rho_{n-1}\rho_n}}{\partial x^{\rho_1}\cdots\partial x^{\rho_{n-2}}},\ n \geq 2\,, \tag{2.9}$$

of the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$, which is totally symmetric in the first $(n-2)$ indexes but, unlike the torsion-free case, *not* in the last two ones. Elements of $E^0(n)$ are linear combinations

$$\sum_{\sigma\in\Sigma'_n} \alpha_\sigma \cdot (1_{n-2} \otimes \mathrm{id}_2)\sigma\,, \tag{2.10}$$

where $1_{n-2} \otimes \mathrm{id}_2 \in \mathbf{1}_{n-2} \otimes \mathbb{R}[\Sigma_2]$ is the generator, $\alpha_\sigma \in \mathbb{R}$, and σ runs over the set Σ'_n of all permutations $\sigma \in \Sigma_n$ such that $\sigma(1) < \cdots < \sigma(n-2)$. We also denote $E^1(n)$ be the trivial Σ_n-module $\mathbf{1}_n$ and by

$$\vartheta_E \colon E^0(n) \to E^1(n)$$

the equivariant map that sends the generator $1_{n-2} \otimes \mathrm{id}_2 \in \mathbf{1}_{n-2} \otimes \mathbb{R}[\Sigma_2]$ to $-1_n \in \mathbf{1}_n$. Analogously to the torsion-free case discussed in [2], the leading terms of the basis tensors are parametrized by a choice of generators for the kernel $\mathcal{K}(n) \subset E^0(n)$ of the map ϑ_E.

The first main theorem of the paper reads:

Theorem A. *Let $D^i_n(\Gamma, X_1, \ldots, X_n)$, $(n,i) \in S := \{n \geq 2,\ 1 \leq i \leq k_n\}$, be differential operators in $\mathfrak{Nat}(Con \times T^{\otimes n}, T)$ whose local expressions are*

$$D^{i,\omega}_n\big(\Gamma^\lambda_{\mu\nu}, X^{\delta_1}_1, \ldots, X^{\delta_n}_n\big) = \sum_{\sigma \in \Sigma'_n} \alpha^i_{n,\sigma} \cdot X^{\rho_1}_{\sigma(1)} \cdots X^{\rho_n}_{\sigma(n)} \frac{\partial^{n-2}\Gamma^\omega_{\rho_{n-1}\rho_n}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-2}}} + l.o.t. \tag{2.11}$$

where l.o.t. is an expression of differential order $< n-2$, and $\{\alpha^i_{n,\sigma}\}_{\sigma \in \Sigma'_n}$ are real constants such that the elements

$$\sum_{\sigma \in \Sigma'_n} \alpha^i_{n,\sigma} \cdot (1_{n-2} \otimes \mathrm{id}_2)\sigma\,, \quad 1 \leq i \leq k_n\,,$$

generate the Σ_n-module $\mathcal{K}(n)$ for each $n \geq 2$.

Let moreover $V_n(\Gamma, X_1, \ldots, X_n)$, $n \geq 1$, be differential operators in $\mathfrak{Nat}(Con \times T^{\otimes n}, T)$ of the form

$$V^\omega_n\big(\Gamma^\lambda_{\mu\nu}, X^{\delta_1}_1, \ldots, X^{\delta_n}_n\big) = X^{\rho_1}_1 \cdots X^{\rho_{n-1}}_{n-1} \frac{\partial^{n-1} X^{\omega_n}_n}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-1}}} + l.o.t.\,,$$

where l.o.t. is an expression of differential order $< n-1$.

Suppose that the operators $D^i_n(\Gamma, X_1, \ldots, X_n)$ are of vf-order 0 and $V_n(\Gamma, X_1, \ldots, X_n)$ of order 0 in $X_1, \ldots, X_{n-1}$. Then each differential operator $\mathcal{O} \colon Con \times T^{\otimes d} \to T$ is an iteration, in the sense of 2.4, of some of the operators $\{D^i_n\}_{(n,i) \in S}$ and $\{V_n\}_{n \geq 1}$.

On manifolds of dimension ≥ 3, each sequence of operators that generates all operators in $\mathfrak{Nat}(Con \times T^{\otimes n}, T)$ is of the form required by Theorem A. We leave the precise formulation of this modification of [2, Theorem B] to the reader. Let us spell out two preferred choices of the leading term of the operators $D^i_n(\Gamma, X_1, \ldots, X_n)$ in Theorem A.

2.6. The classical choice. In this case $k_2 := 1$ and $k_n := 2$ for $n \geq 3$. We put, for $n \geq 3$,

$$r^\omega_n\big(\Gamma^\lambda_{\mu\nu}, X^{\delta_1}_1, \ldots, X^{\delta_n}_n\big) := X^{\rho_1}_1 \cdots X^{\rho_n}_n \frac{\partial^{n-3}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-3}}} \Big(\frac{\partial \Gamma^\omega_{\rho_{n-2}\rho_n}}{\partial x^{\rho_{n-1}}} - \frac{\partial \Gamma^\omega_{\rho_{n-1}\rho_n}}{\partial x^{\rho_{n-2}}} \Big) \tag{2.12}$$

and, for $n \geq 2$,

$$(2.13) \quad t_n^\omega\big(\Gamma_{\mu\nu}^\lambda, X_1^{\delta_1}, \ldots, X_n^{\delta_n}\big) := X_1^{\rho_1} \cdots X_n^{\rho_n} \frac{\partial^{n-2}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-2}}} \big(\Gamma^\omega_{\rho_{n-1}\rho_n} - \Gamma^\omega_{\rho_n\rho_{n-1}}\big)\,.$$

Then t_2 (resp. r_n and t_n if $n \geq 3$) generate, in the sense required by Theorem A, the kernel $\mathcal{K}(2)$ (resp. $\mathcal{K}(n)$). So any system of operators D^1_n with the leading term t_n, $n \geq 2$, and operators D^2_n with the leading term r_n, $n \geq 3$, satisfy the requirements of Theorem A.

The reader certainly noticed that r_n's (resp. t_n's) are the leading terms of the iterated covariant derivatives of the curvature (resp. the torsion), see also Example 2.10. This explains why we called this choice *classical*. The term r_n has the following symmetries:

- (s1) antisymmetry in X_{n-2} and X_{n-1},
- (s3) for $n \geq 4$, cyclic symmetry in X_{n-3}, X_{n-2}, X_{n-1}, and
- (s4) for $n \geq 4$, total symmetry in $X_1, \ldots, X_{n-3}$,

so there is no symmetry (s2) of [2] typical for the torsion-free case. The term t_n is

- (t1) antisymmetric in X_{n-1} and X_n, and
- (t2) for $n \geq 3$, totally symmetric in $X_1, \ldots, X_{n-2}$.

The terms r_n and t_n are not independent but tied, for $n \geq 3$, by the vanishing of the sum

$$(2.14) \quad \sum \big(r_n(\Gamma, X_1, \ldots, X_{n-3}, X_a, X_b, X_c) + t_n(\Gamma, X_1, \ldots, X_{n-3}, X_a, X_b, X_c)\big) = 0\,,$$

running over all cyclic permutations $\{a, b, c\}$ of the set $\{n-2, n-1, n\}$.

2.7. The canonical choice. Now $k_n := 1$ for all $n \geq 2$. Let $l_2^\omega(\Gamma) := \Gamma^\omega_{\rho_1\rho_2} - \Gamma^\omega_{\rho_2\rho_1}$ and l_n be, for $n \geq 3$, given by the local formula

$$l_n^\omega\big(\Gamma_{\mu\nu}^\lambda, X_1^{\delta_1}, \ldots, X_n^{\delta_n}\big) := X_1^{\rho_1} \ldots X_n^{\rho_n} \frac{\partial^{n-3}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-3}}} \Big(6 \frac{\partial \Gamma^\omega_{\rho_{n-1}\rho_n}}{\partial x^{\rho_{n-2}}} - \sum_{a,b,c} \frac{\partial \Gamma^\omega_{\rho_a\rho_b}}{\partial x^{\rho_c}}\Big)$$

where $\{a, b, c\}$ runs over all permutations of $\{\rho_{n-2}, \rho_{n-1}, \rho_n\}$. We call the choice *canonical* because it is given by the canonical Σ_n-equivariant projection of $E^0(n) = \mathcal{K}(n) \oplus \mathbf{1}_n$ onto $\mathcal{K}(n)$. The system $\{l_n\}_{n\geq 2}$ enjoys the following symmetries:

- (l1) $l_2(\Gamma, X_1, X_2)$ is antisymmetric in X_1, X_2 and, for $n \geq 3$,

$$\sum_\omega l_n(\Gamma, X_1, \ldots, X_{n-3}, X_{\omega(n-2)}, X_{\omega(n-1)}, X_{\omega(n)}) = 0\,,$$

  with the sum over all permutations ω of $\{n-2, n-1, n\}$,
- (l2) for $n \geq 3$, total symmetry in $X_1, \ldots, X_{n-3}$,

(l3) for $n \geq 4$,

$$\sum_{\omega} (-1)^{\mathrm{sgn}(\omega)} \cdot l_n(\Gamma, X_1, \dots, X_{n-4}, X_{\omega(n-3)}, X_{\omega(n-2)}, X_{\omega(n-1)}, X_n) = 0\,,$$

where ω runs over all permutations of $\{n-3, n-2, n-1\}$, and

(l4) for $n \geq 4$,

$$\sum_{\tau,\lambda} (-1)^{\mathrm{sgn}(\tau)+\mathrm{sgn}(\lambda)} \cdot l_n(\Gamma, X_1, \dots, X_{n-4}, X_{\tau(n-3)}, X_{\tau(n-2)}, X_{\lambda(n-1)}, X_{\lambda(n)}) = 0\,,$$

with the sum over all permutations τ (resp. λ) of $\{n-3, n-2\}$ (resp. of $\{n-1, n\}$).

The following theorem specifies more precisely which of the basis operators may appear in the iterative representation of operators $Con \times T^{\otimes d} \to T$.

Theorem B. *Assume that* $\dim(M) \geq 2d-1$ *and that* $\{D^i_n\}_{(n,i)\in S}$, $\{V_n\}_{n\geq 1}$ *be as in Theorem A. Let* $\mathcal{O}\colon Con \times T^{\otimes d} \to T$ *be a differential operator of the vf-order* $a \geq 0$*. Then it has an iterative representation with the following property. Suppose that an additive factor of this iterative representation of* $\mathcal{O}$ *via* $\{D^i_n\}_{(n,i)\in S}$ *and* $\{V_n\}_{n\geq 2}$ *contains* $V_{q_1}, \dots, V_{q_t}$*, for some* $q_1, \dots, q_t \geq 2$, $t \geq 0$*. Then*

$$q_1 + \cdots + q_t \leq a + t\,.$$

In particular, if $\mathcal{O}$ *is of vf-order* 0*, it has an iterative representation that uses only* $\{D_n\}_{(n,i)\in S}$*.*

Theorem B implies the following two 'reduction' theorems. The first one uses the 'classical' choice of the generators of the kernels $\mathcal{K}(n)$, $n \geq 2$.

Theorem 2.8. *Let* R_n, $n \geq 3$*, be operators of the form*

$$R^\omega_n\big(\Gamma^\lambda_{\mu\nu}, X_1^{\delta_1}, \dots, X_n^{\delta_n}\big) = X_1^{\rho_1} \dots X_n^{\rho_n} \frac{\partial^{n-3}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-3}}} \Big(\frac{\partial \Gamma^\omega_{\rho_{n-2}\rho_n}}{\partial x^{\rho_{n-1}}} - \frac{\partial \Gamma^\omega_{\rho_{n-1}\rho_n}}{\partial x^{\rho_{n-2}}}\Big) + l.o.t.$$

and T_n, $n \geq 2$*, operators of the form*

$$T^\omega_n\big(\Gamma^\lambda_{\mu\nu}, X_1^{\delta_1}, \dots, X_n^{\delta_n}\big) = X_1^{\rho_1} \dots X_n^{\rho_n} \frac{\partial^{n-2}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-2}}} \big(\Gamma^\omega_{\rho_{n-1}\rho_n} - \Gamma^\omega_{\rho_n\rho_{n-1}}\big) + l.o.t.$$

If $\dim(M) \geq 2d-1$*, the all differential concomitants* $\mathcal{O}\colon Con \times T^{\otimes d} \to T$ *of the connection* $\Gamma^\kappa_{\mu\nu}$ *(i.e. operators of the vf-order* 0*) are ordinary concomitants of* $\{R_n\}_{n\geq 3}$ *and* $\{T_n\}_{n\geq 2}$*.*

The 'canonical' choice of the generators of the kernels $\mathcal{K}(n)$ leads to

Theorem 2.9. *Let* $L_2(X_1, X_2) := T(X_1, X_2)$ *be the torsion and* L_n*, for* $n \geq 3$*, be operators of the form*

$$L^\omega_n\big(\Gamma^\lambda_{\mu\nu}, X_1^{\delta_1}, \dots, X_n^{\delta_n}\big) = X_1^{\rho_1} \dots X_n^{\rho_n} \frac{\partial^{n-3}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-3}}} \Big(6\frac{\partial \Gamma^\omega_{\rho_{n-1}\rho_n}}{\partial x^{\rho_{n-2}}} - \sum_{a,b,c} \frac{\partial \Gamma^\omega_{\rho_a\rho_b}}{\partial x^{\rho_c}}\Big) + l.o.t.\,,$$

where the sum runs over all permutations $\{a,b,c\}$ *of* $\{n-2,n-1,n\}$*. If* $\dim(M) \geq 2d-1$*, then all differential concomitants* $\mathcal{O}\colon Con \times T^{\otimes d} \to T$ *of the connection* $\Gamma^{\kappa}_{\mu\nu}$ *are ordinary concomitants of the tensors* $\{L_n\}_{n\geq 2}$*.*

Example 2.10. Tensors required by the above theorems (and therefore also by Theorem A) exist. One may, for instance, take

$$\begin{aligned} R_n(\Gamma, X_1, \ldots, X_n) &:= (\nabla^{n-3}R)(X_1, \ldots, X_{n-3})(X_{n-2}, X_{n-1})(X_n)\,,\ n \geq 3\,, \\ T_n(\Gamma, X_1, \ldots, X_n) &:= (\nabla^{n-2}T)(X_1, \ldots, X_{n-2})(X_{n-1}, X_n)\,,\ n \geq 2\,, \end{aligned} \tag{2.15}$$

where R and T are the curvature and torsion tensors, respectively. For the operators L_n, $n \geq 3$, in Theorem 2.9, one can take

$$\begin{aligned} L_n(\Gamma, X_1, \ldots, X_n) :=& -3R_n(\Gamma, X_1, \ldots, X_n) \\ & - R_n(\Gamma, X_1, \ldots, X_{n-3}, X_{n-1}, X_n, X_{n-2}) \\ & + R_n(\Gamma, X_1, \ldots, X_{n-3}, X_n, X_{n-2}, X_{n-1}) \\ & + 2T_n(\Gamma, X_1, \ldots, X_n) \\ & - 2T_n(\Gamma, X_1, \ldots, X_{n-3}, X_{n-1}, X_n, X_{n-2}). \end{aligned} \tag{2.16}$$

where T_n and R_n are as in (2.15).

Observe that, while the choice (2.15) in Theorem 2.8 represents operators via the iterated covariant derivatives of both the curvature *and* the torsion, the choice (2.16) in Theorem 2.9 packs both series into one. Recall the following important definition of [2].

Definition 2.11. We say that $\mathfrak{S} \in \mathbb{R}[\Sigma_n]$ is a *quasi-symmetry* of an operator D^i_n in (2.11) if

$$\Big(\sum_{\sigma\in\Sigma_n} \alpha^i_{n,\sigma}\sigma\Big)\mathfrak{S} = 0$$

in the group ring $\mathbb{R}[\Sigma_n]$. We say that $\mathfrak{S}$ is a *symmetry* of D^i_n if $D^i_n\mathfrak{S} = 0$.

A quasi-symmetry $\mathfrak{S}$ of D^i_n, by definition, annihilates its leading term, therefore $D^i_n\mathfrak{S}$ is an operator of c-order $\leq (n-3)$ that does not use the derivatives of the vector field variables. We can express this fact by writing

$$D^i_n\mathfrak{S}(\Gamma, X_1, \ldots, X_n) = \mathfrak{D}^{i,\mathfrak{S}}_n(\Gamma, X_1, \ldots, X_n)\,, \tag{2.17}$$

where $\mathfrak{D}^{i,\mathfrak{S}}_n \in \mathfrak{Nat}(Con \times T^{\otimes n}, T)$ ($\mathfrak{D}$ abbreviating "deviation") is a degree $\leq n-3$ operator which is, by Theorem B, an iteration of the operators D^i_u with $2 \leq u \leq n-1$ (no V_n's). By definition, $\mathfrak{S}$ is a symmetry of D^i_n if and only if $\mathfrak{D}^{i,\mathfrak{S}}_n = 0$. We explained in [2] that (2.17) offers a conceptual explanation of the Bianchi and Ricci identities. As in the torsion-free case, one can prove that the iterative presentation of Theorem A is unique up to the quasi-symmetries and the 'obvious' relations, see [2, Theorem D] for a precise formulation. The following theorem guarantees the existence of "ideal" tensors.

Theorem C. *For each choice of the leading terms*

$$\sum_{\sigma\in\Sigma'_n} \alpha^i_{n,\sigma}\cdot X^{\rho_1}_{\sigma(1)}\dots X^{\rho_n}_{\sigma(n)} \frac{\partial^{n-2}\Gamma^{\omega}_{\rho_{n-1}\rho_n}}{\partial x^{\rho_1}\cdots\partial x^{\rho_{n-2}}}\,,\quad (n,i)\in S\,, \tag{2.18}$$

where S is of the same form as in Theorem A, such that

$$\sum_{\sigma\in\Sigma'_n} \alpha^i_{n,\sigma} = 0 \tag{2.19}$$

for each $(n,i)\in S$, there exist 'ideal' operators $\{J^i_n\}_{(n,i)\in S}$ as in (2.11), for which all the "generalized" Bianchi-Ricci identities (2.17) are satisfied without the right hand sides. In other words, all quasi-symmetries, in the sense of Definition 2.11, are actual symmetries of the operators $\{J^i_n\}_{(n,i)\in S}$.

Observe that (2.19) means that $\sum_{\sigma\in\Sigma'_n}\alpha^i_{n,\sigma}\cdot(1_{n-2}\otimes \mathrm{id}_2)\sigma$ belongs to the kernel $\mathcal{K}(n)$, but, in contrast to Theorem A, we do not assume that the elements corresponding to (2.18) generate the kernel.

Ideal tensors. Theorem C implies the existence of streamlined versions of the tensors $\{R_n\}_{n\geq 3}$, $\{T_n\}_{n\geq 2}$ and $\{L_n\}_{n\geq 2}$ for which the quasi-symmetries induced by the symmetries (s1), (s3), (s4), (t1), (t2), (l1), (l2), (l3), (l4) and equation (2.14) given on pages 66–68 are actual symmetries. So one has tensors $\overline{R}_n$, $n\geq 3$, $\overline{T}_n$, $n\geq 2$ and $\overline{L}_n$, $n\geq 2$, such that

$$\overline{R}_n(\Gamma, X_1,\dots,X_{n-2},X_{n-1},X_n) + \overline{R}_n(\Gamma, X_1,\dots,X_{n-1},X_{n-2},X_n) = 0\,, \tag{2.20}$$

$$\mathfrak{S}_\sigma\overline{R}_n\big(\Gamma, X_1,\dots,X_{n-4},X_{\sigma(n-3)},X_{\sigma(n-2)},X_{\sigma(n-1)},X_n\big) = 0\,,\quad n\geq 4\,, \tag{2.21}$$

where $\mathfrak{S}$ is the cyclic sum over the indicated indexes, and

$$\overline{R}_n\big(\Gamma, X_{\omega(1)},\dots,X_{\omega(n-3)},X_{n-2},X_{n-1},X_n\big) = \overline{R}_n(\Gamma, X_1,\dots,X_n)\,, \tag{2.22}$$

for each $n\geq 4$ and a permutation $\omega\in\Sigma_{n-3}$. The tensors $\overline{T}_n$ satisfy

$$\overline{T}_n(\Gamma, X_1,\dots,X_{n-2},X_{n-1},X_n) + \overline{T}_n(\Gamma, X_1,\dots,X_{n-2},X_n,X_{n-1}) = 0\,, \tag{2.23}$$

and, for $n\geq 3$, also

$$\overline{T}_n\big(\Gamma, X_{\omega(1)},\dots,X_{\omega(n-2)},X_{n-1},X_n\big) = \overline{T}_n(\Gamma, X_1,\dots,X_n)\,, \tag{2.24}$$

for each permutation $\omega\in\Sigma_{n-2}$. Moreover,

$$\mathfrak{S}_\sigma\overline{R}_n\big(\Gamma, X_1,\dots,X_{n-3},X_{\sigma(n-2)},X_{\sigma(n-1)},X_{\sigma(n)}\big) = -\mathfrak{S}_\sigma\overline{T}_n\big(\Gamma, X_1,\dots,X_{n-3},X_{\sigma(n-2)},X_{\sigma(n-1)},X_{\sigma(n)}\big)\,, \tag{2.25}$$

with the sums running over cyclic permutations σ of $\{n-2,n-1,n\}$.

The tensor $\overline{L}_2$ is antisymmetric. The tensors $\overline{L}_n$ satisfy, for $n\geq 3$,

$$\sum_{\omega}\overline{L}_n\big(\Gamma, X_1,\dots,X_{n-3},X_{\omega(n-2)},X_{\omega(n-1)},X_{\omega(n)}\big) = 0\,, \tag{2.26}$$

where ω runs over all permutations of $\{n-2, n-1, n\}$. For $n \geq 4$ they also satisfy

$$\sum_{\omega} (-1)^{\mathrm{sgn}(\omega)} \cdot \overline{L}_n(\Gamma, X_1, \dots, X_{n-4}, X_{\omega(n-3)}, X_{\omega(n-2)}, X_{\omega(n-1)}, X_n) = 0\,, \tag{2.27}$$

where ω runs over all permutations of $\{n-3, n-2, n-1\}$,

$$\overline{L}_n\left(\Gamma, X_{\omega(1)}, \dots, X_{\omega(n-3)}, X_{n-2}, X_{n-1}, X_n\right) = \overline{L}_n\left(\Gamma, X_1, \dots, X_n\right), \tag{2.28}$$

for each permutation $\omega \in \Sigma_{n-3}$, and

$$\sum_{\tau,\lambda} (-1)^{\mathrm{sgn}(\tau)+\mathrm{sgn}(\lambda)} \times \overline{L}_n(\Gamma, X_1, \dots, X_{n-4}, X_{\tau(n-3)}, X_{\tau(n-2)}, X_{\lambda(n-1)}, X_{\lambda(n)}) = 0\,, \tag{2.29}$$

with the sum over all permutations τ (resp. λ) of $\{n-3, n-2\}$ (resp. of $\{n-1, n\}$).

In Examples 2.13–2.15 below we explicitly calculate the ideal tensors $\overline{R}_n$, $\overline{T}_n$ and $\overline{L}_n$ for $n \leq 4$. Our calculation is facilitated by the following lemma whose straightforward though technically involved proof we omit.

Lemma 2.12. *Let $n \geq 3$ and $\mathfrak{X}$, V be vector spaces over a field of characteristic zero. Denote by $\mathcal{F}_L$ the space of all linear maps $L\colon \mathfrak{X}^{\otimes n} \to V$ with symmetry* (2.26) *and, if $n \geq 4$, also* (2.27)–(2.29)*. Denote further by $\mathcal{F}_{(R,T)}$ the space of all pairs (R, T) of linear maps $R, T\colon \mathfrak{X}^{\otimes n} \to V$ satisfying* (2.20), (2.23)–(2.25) *and, if $n \geq 4$, also* (2.21) *and* (2.22)*. Define finally the map $\Phi = (\Phi_R, \Phi_T)\colon \mathcal{F}_L \to \mathcal{F}_{(R,T)}$ by*

$$\Phi_R(X_1, \dots, X_n) := \frac{1}{6}\big[L(X_1, \dots, X_{n-3}, X_{n-1}, X_{n-2}, X_n) - L(X_1, \dots, X_n)\big], \text{ and}$$
$$\Phi_T(X_1, \dots, X_n) := \frac{1}{6}\big[L(X_1, \dots, X_n) - L(X_1, \dots, X_{n-2}, X_n, X_{n-1})\big]\,,$$

and the map $\Psi\colon \mathcal{F}_{(R,T)} \to \mathcal{F}_L$ by

$$\begin{aligned}\Psi(X_1, \dots, X_n) := &- 3R(X_1, \dots, X_n) - R(X_1, \dots, X_{n-3}, X_{n-1}, X_n, X_{n-2}) \\ &+ R(X_1, \dots, X_{n-3}, X_n, X_{n-2}, X_{n-1}) + 2T(X_1, \dots, X_n) \\ &- 2T(X_1, \dots, X_{n-3}, X_{n-1}, X_n, X_{n-2})\,.\end{aligned}$$

Then Φ and Ψ are well-defined mutual inverses, $\Phi\colon \mathcal{F}_L \cong \mathcal{F}_{(R,T)} \colon \Psi$.

The maps Φ and Ψ of Lemma 2.12 produce from ideal tensors $\overline{R}_n, \overline{T}_n$ the ideal tensor $\overline{L}_n$ and vice versa. Since the ideal tensors $\overline{R}_n, \overline{T}_n$ can be constructed as modification of the covariant derivatives of the classical curvature and torsion tensors, we start in examples below with them and obtain $\overline{L}_n$ as $\Psi(\overline{R}_n, \overline{L}_n)$.

Example 2.13. If $n = 2$, the tensor $\overline{T}_2 = T_2 = T$ satisfies the antisymmetry (2.23), so $\overline{L}_2 = \overline{T}_2 = T$. There is, of course, no $\overline{R}_2$.

To make formulas shorter, in the following two examples we drop the implicit Γ from the notation.

Example 2.14. If $n = 3$, then the tensor $R_3 = R$ satisfies (2.20), the tensor $T_3 = \nabla T$ satisfies (2.23) and, trivially, also (2.24), but the couple $(R_3, T_3) = (R, \nabla T)$ *does not* satisfy (2.25). If one takes, instead of $T_3 = \nabla T$, a streamlined version

$$\overline{T}_3(X,Y,Z) := (\nabla_X T)(Y,Z) - T(X, T(Y,Z))\,,$$

then $\overline{T}_3$ satisfies (2.23), (2.24), and the couple $(\overline{R}_3 = R_3, \overline{T}_3)$ satisfies (2.25) which is in this case precisely the first Bianchi identity (1.1) for the curvature of a connection with nontrivial torsion.

It follows from Lemma 2.12 that the tensor $\overline{L}_3$ defined by

$$\begin{aligned} \overline{L}_3(X,Y,Z) :=& -3\,\overline{R}_3(X,Y,Z) - \overline{R}_3(Y,Z,X) + \overline{R}_3(Z,X,Y) \\ &+ 2\,\overline{T}_3(X,Y,Z) - 2\,\overline{T}(Y,Z,X) \end{aligned} \tag{2.30}$$

satisfies (2.26), so it is the ‘ideal’ $\overline{L}_3$. On the other hand, by the same lemma, given $\overline{L}_3$ satisfying (2.26), we have

$$\overline{R}_3(X,Y,Z) = -\frac{1}{6}\big[\overline{L}_3(X,Y,Z) - \overline{L}_3(Y,X,Z)\big] \tag{2.31}$$

satisfying (2.20). Further

$$\overline{T}_3(X,Y,Z) = \frac{1}{6}\big[\overline{L}_3(X,Y,Z) - \overline{L}_3(X,Z,Y)\big] \tag{2.32}$$

satisfies (2.23) and, trivially, (2.24). Moreover, the pair $(\overline{R}_3, \overline{T}_3)$ satisfies (2.25). If we put $\overline{R}_3$ and $\overline{T}_3$ calculated from (2.31) and (2.32) into (2.30), we recover $\overline{L}_3$. Likewise, if we substitute $\overline{L}_3$ calculated from (2.30) into (2.31) and (2.32), we get $\overline{R}_3$ and $\overline{T}_3$, because the transformations (2.30) and (2.31)–(2.32) are, by Lemma 2.12, mutually inverse.

Example 2.15. If $n = 4$, the tensor $R_4 = \nabla R$ satisfies (2.20) and, trivially, also (2.22) but *does not* satisfy (2.21) because of the non vanishing right hand side of the 2nd Bianchi identity (1.2). We found the following explicit formula for a streamlined couple $(\overline{R}_4, \overline{T}_4)$ in which $\overline{R}_4$ is given by

$$\begin{aligned} \overline{R}_4(X_1,\dots,X_4) &= (\nabla_{X_1}R)(X_2,X_3)(X_4) \\ &+ \frac{1}{2}\big[R(T(X_1,X_2),X_3)(X_4) + R(X_2,T(X_1,X_3))(X_4)\big] \\ &- \frac{1}{2}\big[T(R(X_2,X_3)(X_1),X_4) + T((\nabla_{X_1}T)(X_2,X_3),X_4) \\ &\qquad + T(T(T(X_2,X_3),X_1),X_4)\big] \\ &+ \frac{1}{4}\big[-2\,(\nabla_{X_1}T)(T(X_2,X_3),X_4) - (\nabla_{X_2}T)(T(X_1,X_3),X_4) \\ &\qquad + (\nabla_{X_3}T)(T(X_1,X_2),X_4)\big] \\ &+ \frac{1}{8}\big[T(T(X_3,X_4),T(X_1,X_2)) - T(T(X_2,X_4),T(X_1,X_3)) \\ &\qquad + 2\,T(T(X_2,X_3),T(X_1,X_4))\big]\,. \end{aligned}$$

It satisfies identities (2.20), (2.21) and, trivially, also (2.22). For $\overline{T}_4$ we found

$$\begin{aligned}\overline{T}_4(X_1,X_2,X_3,X_4) = {} & \frac{1}{2}\big[(\nabla_{X_1}\nabla_{X_2}T)(X_3,X_4) + (\nabla_{X_2}\nabla_{X_1}T)(X_3,X_4)\big] \\ & - \frac{1}{4}\big[R(X_1,X_3)(T(X_4,X_2)) + R(X_2,X_3)(T(X_4,X_1)) \\ & \quad - R(X_1,X_4)(T(X_3,X_2)) - R(X_2,X_4)(T(X_3,X_1))\big] \\ & + \frac{3}{4}\big[(\nabla_{X_1}T)(T(X_2,X_3),X_4) + (\nabla_{X_2}T)(T(X_1,X_3),X_4) \\ & \quad - (\nabla_{X_1}T)(T(X_2,X_4),X_3) - (\nabla_{X_2}T)(T(X_1,X_4),X_3)\big] \\ & + \frac{1}{2}\big[T((\nabla_{X_1}T)(X_2,X_3),X_4) + T((\nabla_{X_2}T)(X_1,X_3),X_4) \\ & \quad - T((\nabla_{X_1}T)(X_2,X_4),X_3) - T((\nabla_{X_2}T)(X_1,X_4),X_3)\big]\,.\end{aligned}$$

It is easy to see that $\overline{T}_4$ satisfies identities (2.23) and (2.24), and the pair $(\overline{R}_4\,,\,\overline{T}_4)$ satisfies (2.25). By Lemma 2.12, we may put

$$\begin{aligned}&\overline{L}_4\,(X_1,X_2,X_3,X_4) = -3\,\overline{R}_4(X_1,X_2,X_3,X_4) - \overline{R}_4(X_1,X_3,X_4,X_2)\\ &\overline{T}_4(X_1,X_4,X_2,X_3) + 2\,\overline{T}_4(X_1,X_2,X_3,X_4) - 2\,\overline{T}_4(X_1,X_3,X_4,X_2)\,.\end{aligned}$$

On the other hand, given an 'ideal' $\overline{L}_4$ satisfying (2.26)–(2.29), the equations

$$\begin{aligned}\overline{R}_4(X_1,X_2,X_3,X_4) &:= \frac{1}{6}\big[\overline{L}_4(X_1,X_3,X_2,X_4) - \overline{L}_4(X_1,X_2,X_3,X_4)\big] \quad \text{and}\\ \overline{T}_4(X_1,X_2,X_3,X_4) &:= \frac{1}{6}\big[\overline{L}_4(X_1,X_2,X_3,X_4) - \overline{L}_4(X_1,X_2,X_4,X_3)\big]\end{aligned}$$

determine 'ideal' $\overline{R}_4$ and $\overline{L}_4$.

We saw above that calculating the ideal tensors $\overline{R}_n, \overline{T}_n$ and $\overline{L}_n$ is difficult already for $n = 4$. To find explicit formulas for arbitrary $n \geq 3$ is, as in the torsion-free case [2], a challenging task.

Let $\mathcal{K}$ be the collection of the kernels (3.7) and $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$ the space spanned by graphs with d black vertices (3.1), one vertex $\overset{\blacksquare}{\uparrow}$ and a finite number of vertices decorated by elements of $\mathcal{K}$, see pages 74–76 of Section 3 for a precise definition. The size of the space of natural operators $Con \times T^{\otimes d} \to T$ is described in:

Theorem D. *On manifolds of dimension $\geq 2d-1$, the vector space $\mathfrak{Nat}(Con \times T^{\otimes d}, T)$ is isomorphic to the graph space $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$.*

Example 2.16. As in the torsion-free case, the calculation of the dimension of $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$ is a purely combinatorial problem. For $d = 1$ we get $\dim(\mathcal{G}\mathrm{r}[\mathcal{K}](d)) = 1$, with the corresponding natural operator the identity $X \mapsto X$.

One sees that, on manifolds of dimension ≥ 3, $\dim(\mathfrak{Nat}(Con \times T^{\otimes 2}, T)) = 7$. The corresponding operators are

$$\nabla_X Y,\ \nabla_Y X,\ Tr(\nabla_- Y)\cdot X \text{ and } Tr(\nabla_- X)\cdot Y$$

as in the torsion-free case (see [2, Example 3.18]), plus three operators

$$T(X,Y),\ Tr(T(-,Y))\cdot X \text{ and } Tr(T(-,X))\cdot Y$$

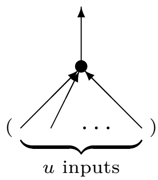
involving the torsion.

3. Proofs

As everywhere in this paper, we use the notation parallel to that of [2], but the reader shall keep in mind that we dropped the torsion-free assumption. As expected, the proofs will be based on a suitable graph complex describing operators of a given type which was, in fact, already been described in Section 4 of [2], see 4.7 of that section in particular. We only briefly recall its definition, leaving the details and motivations to [2] and [9].

We consider the graded graph complex $\mathcal{G}\mathrm{r}^*(d)$ whose degree m part $\mathcal{G}\mathrm{r}^m(d)$ is spanned by oriented graphs with precisely d 'black' vertices

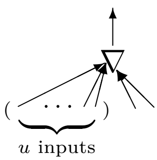
(3.1) $b_u :=$



, $u \geq 0$,

labelled $1, \ldots, d$, some number of '∇-vertices'

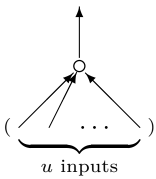
(3.2)



, $u \geq 0$.

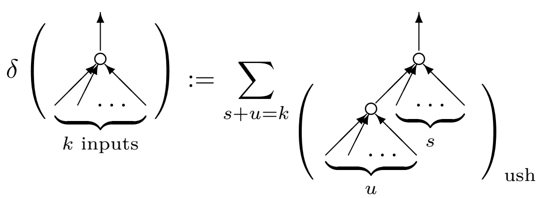
precisely m 'white' vertices

(3.3)



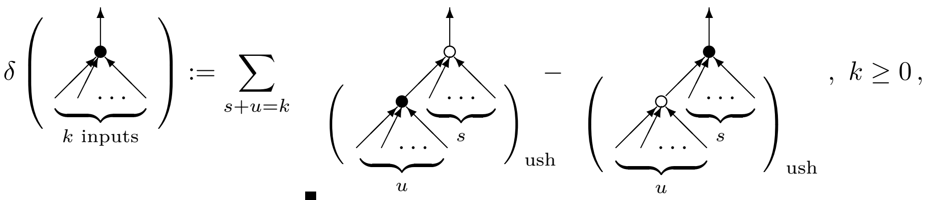
, $u \geq 2$,

and one vertex  (the anchor). We will usually omit the parentheses ( ) indicating that the inputs they encompass are fully symmetric. In contrast to the torsion-free case, the ∇-vertex (3.2) is <u>not</u> symmetric in the rightmost two inputs. The interpretation of the vertices is explained in [2, Section 4]. The differential is given by the replacement rules that are 'informally' the same as these in [2, Section 4] (but formally not, since the symmetries of the ∇-vertex are different), i.e.

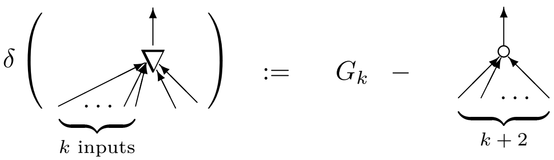


, $k \geq 2$,

for white vertices,



for black vertices and $\delta(\,\blacksquare\!\!\uparrow) = 0$ for the anchor. The braces $(\quad)_{\rm ush}$ in the right hand sides indicate the summations over all $(u, s-1)$-unshuffles. The replacement rule for the ∇-vertices is of the form

(3.4)


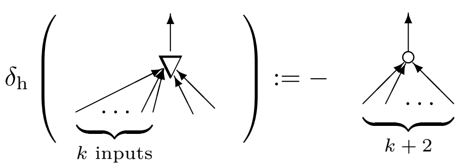
where G_k is a linear combination of 2-vertex trees with one ∇-vertex (3.2) with $u < k$, and one white vertex (3.3) with $u < k + 2$. The concrete form of G_k is not relevant for our paper, the interested reader may find some examples in [2, Section 4]. The central statement is

Theorem 3.1 ([9])**.** *Each element in* $H^0(\mathcal{G}{\rm r}^*(d), \delta) = Ker\big(\delta : \mathcal{G}{\rm r}^0(d) \to \mathcal{G}{\rm r}^1(d)\big)$ *represents a natural operator* $Con \times T^{\otimes d} \to T$*. On manifolds of dimension* $\geq 2d - 1$ *this correspondence is an isomorphism* $H^0(\mathcal{G}{\rm r}^*(d), \delta) \cong \mathfrak{Nat}(Con \times T^{\otimes d}, T)$*.*

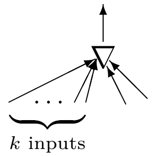
Proof of Theorem D consists of calculation of the homology $H^0(\mathcal{G}{\rm r}^*(d), \delta)$ which, of course, differs from the torsion-free case. The first step is to observe that $(\mathcal{G}{\rm r}^*(d), \delta)$ is the total complex of the following bicomplex (see [6, §XI.6] for the terminology). For $p, q \in \mathbb{Z}$, let

$$\mathcal{G}{\rm r}^{p,q}(d) := Span\big\{\text{graphs } \Lambda \in \mathcal{G}{\rm r}^{p+q}(d); \quad \text{the number of } \nabla\text{-vertices } = -p\big\}. \tag{3.5}$$

Define the horizontal differential $\delta_{\rm h} \colon \mathcal{G}{\rm r}^{p,q}(d) \to \mathcal{G}{\rm r}^{p+1,q}(d)$ by
(3.6)



while $\delta_{\rm h}$ is trivial on remaining vertices. The vertical differential $\delta_{\rm v} \colon \mathcal{G}{\rm r}^{p,q}(d) \to \mathcal{G}{\rm r}^{p,q+1}(d)$ is defined by requiring that $\delta_{\rm v} := \delta$ on black vertices (3.1), white vertices (3.3) and the anchor , while

δ_{v} () $:= G_k,$

where G_k is the same as in (3.4). We prove

Lemma 3.2. *The bicomplex* $\mathcal{G}\mathrm{r}^{*,*}(d) = (\mathcal{G}\mathrm{r}^{*,*}(d), \delta_{\mathrm{h}} + \delta_{\mathrm{v}})$ *has the following properties.*

(i) $\mathcal{G}\mathrm{r}^{*,*}(d)$ *is concentrated in the sector* $0 \le -p \le q$,

(ii) $\mathcal{G}\mathrm{r}^{p,*} = 0$ *for* $p \ll 0$, *and*

(iii) *the horizontal cohomology of* $\mathcal{G}\mathrm{r}^{*,*}(d)$ *is concentrated on the diagonal* $p+q = 0$, *i.e.*

$$H^p(\mathcal{G}\mathrm{r}^{*,q}, \delta_{\mathrm{h}}) = 0 \text{ for } p+q \neq 0 \text{ or, equivalently, } H^m(\mathcal{G}\mathrm{r}^{*}, \delta_{\mathrm{h}}) = 0 \text{ for } m \neq 0\,.$$

Proof. As in [2], properties (i)–(ii) follow from simple graph combinatorics. To verify (iii), we follow [9] and observe that $(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{v}})$ is a particular case of the following construction. For each collection $(E^*, \vartheta) = \{(E^*(s), \vartheta)\}_{s \ge 2}$ of right dg-Σ_s-modules $(E^*(s), \vartheta)$, one considers the complex $\mathcal{G}\mathrm{r}^*[E^*](d) = (\mathcal{G}\mathrm{r}^*[E^*](d), \vartheta)$ spanned by graphs with d black vertices (3.1), one vertex  and a finite number of vertices decorated by elements of E. The grading of $\mathcal{G}\mathrm{r}^*[E^*](d)$ is induced by the grading of E^* and the differential ϑ replaces E-decorated vertices, one at a time, by their ϑ-images, leaving other vertices unchanged. Since the assignment $(E^*, \vartheta) \mapsto (\mathcal{G}\mathrm{r}^*[E^*](d), \vartheta)$ is an exact functor ([11], see also [7, Theorem 21]),

$$H^*(\mathcal{G}\mathrm{r}^*[E^*](d), \vartheta) \cong \mathcal{G}\mathrm{r}^*[H^*(E, \vartheta)](d)\,.$$

Let now $(E^*, \vartheta) = \{(E^*(s), \vartheta)\}_{s \ge 2}$ be such that $E^0(s)$ is spanned by the symbols (3.2) with $u + 2 = s$, $E^1(s)$ by the symbols (3.3) with $u = s$, and $E^m(s) = 0$ for $m \ge 2$. The differential ϑ is defined by the replacement rule (3.6). An equivalent description of $\vartheta \colon E^0(s) \to E^1(s)$ is given on page 66. It is clear that

$$(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{h}}) \cong (\mathcal{G}\mathrm{r}^*[E^*](d), \vartheta).$$

Since $\vartheta \colon E^0(s) \to E^1(s)$ is onto, $H^*(E, \vartheta) = \{H^*(E(s), \vartheta)\}_{s \ge 2}$ is concentrated in degree 0, with $H^0(E(s), \vartheta)$ the kernel

$$\mathcal{K}(s) := Ker\left(\vartheta : E^0(s) \to E^1(s)\right)\,. \tag{3.7}$$

Denoting by $\mathcal{K}$ the collection $\mathcal{K} := \{\mathcal{K}(s)\}_{s \ge 2}$ we conclude that

$$H^*(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{h}}) \cong H^0(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{h}}) \cong \mathcal{G}\mathrm{r}[\mathcal{K}](d). \tag{3.8}$$

In particular, $H^m(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{h}}) = 0$ for $m \neq 0$ which establishes (iii). □

Properties (i)–(iii) of Lemma 3.2 imply, by a standard spectral sequence argument and the description (3.8) of the horizontal cohomology, that

$$H^0(\mathcal{G}\mathrm{r}^*(d), \delta) \cong H^0(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{h}}) \cong \mathcal{G}\mathrm{r}[\mathcal{K}](d).$$

This, along with Theorem 3.1, implies Theorem D.

Proof of Theorem C. Consider a bicomplex $\mathbb{B} = (B^{*,*}, \delta = \delta_{\rm h} + \delta_{\rm v})$ fulfilling (i)–(iii) of Lemma 3.2 and denote by $Z_{\rm h} := \bigoplus_{r\geq 0} Z^r_{\rm h}$ the (finite, by (ii)) sum of the subspaces

$$Z^r_{\rm h} := Ker(\delta_{\rm h} : B^{-r,r} \to B^{-r+1,r}) .$$

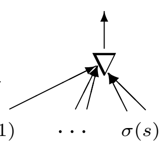
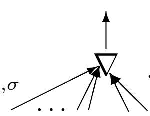
The proposition below is a combination of Proposition 5.1 and Corollary 5.4 of [2].

Proposition 3.3. *Let G be a group and assume that the bicomplex $\mathbb{B}$ consists of reductive G-modules and the differentials $\delta_{\rm h}$ and $\delta_{\rm v}$ are G-equivariant. Then there exists a G-equivariant map $\beta\colon Z_{\rm h} = \bigoplus_{r\geq 0} Z^r_{\rm h} \to \bigoplus_{r\geq 0} B^{-r,r}$ such that, for each $r \geq 0$ and $z \in Z^r_{\rm h}$, $\beta(z)$ is a cocycle in the total complex $\mathrm{Tot}(\mathbb{B})$ of the form $\beta(z) = z + l.o.t.$, with some $l.o.t. \in \bigoplus_{p>r} B^{-p,p}$.*

By a simple spectral sequence argument, any β as in Proposition 3.3 induces an isomorphism (denoted β again)

$$\beta\colon Z_{\rm h} \xrightarrow{\cong} H^0(\mathrm{Tot}(\mathbb{B})) .$$

Let $\alpha^i_{n,\sigma}$, $\sigma \in \Sigma'_n$, be coefficients as in Theorem A. If we take the symbol (3.2), with the inputs numbered consecutively from left to right by $\{1, \ldots, s\}$, as the generator of $E^0(s)$, then $\mathcal{K}(s)$ is, as a Σ_s-module, generated by the linear combinations

(3.9) $\xi^i_s := \sum_{\sigma\in\Sigma'_s} \alpha^i_{s,\sigma}$  $= \sum_{\sigma\in\Sigma'_s} \alpha^i_{s,\sigma}$  $\cdot\,\sigma\ ,\ 1 \leq i \leq k_s .$

For an arbitrary k, $0 \leq k \leq d$, denote by $\mathcal{G}\mathrm{r}^*(d)_k$ the subspace of $\mathcal{G}\mathrm{r}^*(d)$ spanned by graphs with a distinguished labelled subset $\{\bullet_1, \ldots, \bullet_k\}$ of the set of black vertices (3.1) with $u = 0$. There is a right Σ_k-action on the space $\mathcal{G}\mathrm{r}^*(d)_k$ that permutes the labels of the distinguished vertices. Let $\mathcal{G}\mathrm{r}^*_k := \bigoplus_{d\geq k} \mathcal{G}\mathrm{r}^*(d)_k$. We wish to have, for each $(n, i) \in S$, cochains $\varsigma^i_n \in \mathcal{G}\mathrm{r}^0(n)_n$ of the form

$$\varsigma^i_n = \xi^i_n + l.o.t. \tag{3.10}$$

where ξ^i_n is as in (3.9) and l.o.t. a linear combination of graphs with at least two ∇-vertices. We also wish to have, for each $n \geq 0$, cochains $\nu_n \in \mathcal{G}\mathrm{r}^0(n+1)_n$ of the form

$$\nu_n = b_n + l.o.t. \tag{3.11}$$

where b_n denotes the black vertex (3.1) with $u = n$. The abbreviation l.o.t. means here a linear combination of graphs in $\mathcal{G}\mathrm{r}^0(n+1)_n$ that has at least one ∇-vertex.

Proposition 3.4. *There are 'equivariant' cocycles $\{\nu_n\}_{n\geq 2}$ and $\{\varsigma^i_n\}_{(n,i)\in S}$ that enjoy the same symmetries as the elements $\{b_n\}_{n\geq 2}$ and $\{\xi^i_n\}_{(n,i)\in S}$.*

Proof. The obvious modification of the bigrading (3.5) turns $\mathcal{G}r^*(n+1)_n$ into a bicomplex satisfying conditions (i)–(iii) of Lemma 3.2. The group Σ_n permutes the distinguished vertices. This action satisfies the requirements of Proposition 3.3 which therefore gives a Σ_n-equivariant β. The element $\nu_n := \beta(b_n)$ is then the required 'equivariant' cocycle. An 'equivariant' ς_n^i can be constructed in the same fashion, taking $\mathcal{G}r^*(n)_n$ instead of $\mathcal{G}r^*(n+1)_n$. □

The 'ideal' tensors in Theorem C are the natural operators related, in the correspondence of Theorem 3.1, to the cocycles $\{\varsigma_n^i\}_{(n,i)\in S}$ and $\{\nu_n\}_{n\geq 2}$ constructed in Proposition 3.4.

Proof of Theorem A. Each iteration as in 2.4 is clearly a linear combination of terms given by contracting 'free' indexes of the local coordinate expressions of the operators $\{D_n^i\}_{(n,i)\in S}$ and $\{V_n\}_{n\geq 2}$. Each such a contraction is determined by a 'contraction scheme,' which is a graph with vertices of the following two types:

- vertices d_n^i, $(n,i)\in S$, with n linearly ordered input edges and one output, and
- vertices v_n, $n\geq 0$, labeled $1,\dots,d$, with n linearly ordered edges and one output.

Denote by $\mathcal{C}\mathrm{ont}(d)$ the space spanned by the above contraction schemes. One has the diagram

$$\mathcal{G}r[\mathcal{K}](d) \overset{\pi}{\twoheadleftarrow} \mathcal{C}\mathrm{ont}(d) \overset{\Psi}{\to} \mathcal{G}r^0(d) \tag{3.12}$$

in which the map π replaces each vertex d_n^i of a contraction scheme $K\in\mathcal{C}\mathrm{ont}(d)$ by ξ_n^i defined in (3.9) and each vertex v_n by b_n defined in (3.1). The map Ψ is the cocycle representing the iteration determined by K.

The fact that π is an epimorphism can be established, as in [2], by constructing a right inverse $s\colon \mathcal{G}r[\mathcal{K}](d)\to\mathcal{C}\mathrm{ont}(d)$ of π. The map $\beta=\Psi\circ s$ has the properties as in Proposition 3.3 (with trivial G). It therefore induces an isomorphism $\mathcal{G}r[\mathcal{K}](d)\cong H^0(\mathcal{G}r^*(d),\delta)$. In particular, the map Ψ is an epimorphism onto $Ker(\delta\colon \mathcal{G}r^0(d)\to \mathcal{G}r^1(d)) = H^0(\mathcal{G}r^*(d),\delta)$. This, along with Theorem 3.1, proves Theorem A.

Proof of Theorem B. One assigns to each graph $\Lambda\in\mathcal{G}r[\mathcal{K}](d)$ the (formal) vf-order defined by the summation

$$ord_{\mathrm{vf}}(\Lambda) := \sum_{v\in Vert(\Lambda)} ord_{\mathrm{vf}}(v)\,, \tag{3.13}$$

where

$$ord_{\mathrm{vf}}(v) := \begin{cases} 0, & \text{if } v \text{ is } \xi_n^i,\ (n,i)\in S, \text{ and}\\ n, & \text{if } v \text{ is } b_n,\ n\geq 0\,. \end{cases}$$

The vf-order of a contraction scheme $G\in\mathcal{C}\mathrm{ont}(d)$ can be defined similarly, with the role of vertices ξ_n^i played by d_n^i, and the role of vertices b_n by v_n. Therefore, if a contraction scheme has vertices $v_{p_1},\dots,v_{p_t}$ for some $p_1,\dots,p_t\geq 0$ (plus possibly some other vertices of either types), then

$$p_1+\cdots+p_t \leq ord_{\mathrm{vf}}(G)\,. \tag{3.14}$$

Finally, the vf-order of a graph Λ in $\mathcal{G}\mathrm{r}^0(d)$ is given by formula (3.13) in which we define now

$$ord_{\mathrm{vf}}(v) := \begin{cases} 0, & \text{if } v \text{ is a } \nabla\text{-vertex, and} \\ n, & \text{if } v \text{ is } b_n,\ n \geq 0\,. \end{cases}$$

The vf-order of an element of $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$ (resp. $\mathcal{C}\mathrm{ont}(d)$, resp. $\mathcal{G}\mathrm{r}^0(d)$) is then the maximum of vf-orders of its linear constituents. It is clear that the (formal) vf-order of a cocycle in $\mathcal{G}\mathrm{r}^0(d)$ equals the vf-order of the operator it represents.

As in [2] one shows that map $\beta = \Phi \circ s\colon \mathcal{G}\mathrm{r}[\mathcal{K}] \to H^0(\mathcal{G}\mathrm{r}^*(d), \delta)$ (which is an isomorphism, by the stability assumption $\dim(M) \geq 2d-1$) constructed in the proof of Theorem A preserves the vf-order. Let $\mathcal{O} \in \mathfrak{Nat}(Con \times T^{\otimes d}, T)$ be a differential operator represented by a cocycle $c \in \mathcal{G}\mathrm{r}^0(d)$, $y := \beta^{-1}(c)$ and $C := s(y)$. According to our constructions, $C \in \mathcal{C}\mathrm{ont}(d)$ describes an iteration of $\{D^i_n\}_{(n,i)\in S}$ and $\{V_n\}_{n\geq 1}$ representing $\mathcal{O}$. Since, as in [2] both β and s preserve the vf-order, one has $ord_{\mathrm{vf}}(C) = ord_{\mathrm{vf}}(\mathcal{O})$. Theorem C now immediately follows from formula (3.14).

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