MASARYK UNIVERSITY

FACULTY OF SCIENCE

Schrödinger equation in curved space

Bachelor's Thesis

Miroslav Spáčil

Brno, Spring 2025

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Department of Theoretical Physics and Astrophysics

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Abstrakt

Zakřivené prostory hrají důležitou roli ve fyzice, zejména v obecné teorii relativity a fyzice pevných látek. Hlavním cílem této práce je formulace Schrödingerovy rovnice v zakřivených prostorech, což vyžaduje nalezení vhodného tvaru derivace v takových geometriích, a nalézt jejich řešení pro pár jednoduchých případů. Nakonec se zaměřujeme se na tři konkrétní plochy: Poincarého disk, stereografickou projekci sféry a plochu neznámého tvaru, definovanou pouze jejím metrickým tenzorem. Na rozdíl od prvních dvou případů se pro neznámý povrch podařilo nalézt analytické řešení Schrödingerovy rovnice.

Abstract

Curved spaces play an important role in physics, particularly in general relativity and solid-state physics. The main goal of this thesis is to formulate the Schrödinger equation in curved spaces, which requires identifying the appropriate form of the derivative in such geometries, and to find solution to them for a few simple cases. Finally, we consider three specific surfaces: the Poincaré disk, the stereographic projection of a sphere and a surface of unknown shape, defined only by its metric tensor. Unlike the first two cases, an analytical solution of the Schrödinger equation is obtained for the unknown surface.

Zadání bakalářské práce

Zakřivené prostory se objevují v mnoha kontextech fyziky. Zajímavé jevy vznikají při studiu kvantové mechaniky v zakřivených prostorech. Student bude zkoumat, jak může být Schrödingerova rovnice formulována v zakřiveném prostoru. Jednoduché případy, jako Poincarého disk, budou zkoumány do detailu.

Thesis assignment

Curved spaces appear in many contexts in physics. Interesting phenomena arise when considering quantum mechanics in curved spaces. The student will consider how the Schrödinger equation may be formulated in a curved space. Simple cases like the Poincaré disc will be considered in detail.

Poděkování

Tímto bych rád poděkoval všem, kteří mi pomohli při vypracování této bakalářské práce.

Děkuji především svému vedoucímu práce, doc. J. L. Wulffovi, za jeho cenné rady, podporu a trpělivost během celého procesu psaní. Jeho odborné vedení a pomoc byly pro mě neocenitelné.

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Prohlášení

Prohlašuji, že jsem svoji bakalářskou práci vypracoval samostatně pod vedením vedoucího práce s využitím informačních zdrojů, které jsou v práci citovány.

V Brně, dne 12. května 2025
Miroslav Spáčil

Declaration

I hereby declare that I have completed my bachelor's thesis independently under the supervision of my thesis advisor, using the information sources that are cited in the thesis.

Brno, 12th of May, 2025
Miroslav Spáčil

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1 Introduction

Curved spaces are of interest in cases of general relativity where we are concerned with a curved spacetime that manifests as gravity due to massive objects inhabiting that space. It also has other applications in solid state physics, like when we consider a particle in a potential field, for example charged particles in a magnetic field that follow magnetic field lines can be described as moving in a curved space, or when charges are bound to move on surfaces of different shapes.

Physical spaces can be described with the tools provided by linear algebra as vector spaces. The central objects in these are vectors, or more generally tensors that include vectors and scalars, which can describe positions in physical space, or a directed displacement from a chosen origin point, and quantities with the property of direction. Vectors are then given by coordinates, which depend on chosen coordinate system. To construct any vectors, basis vectors for given space are first defined according to chosen coordinates, from which all other vectors can be made that span the whole space.

While vectors themselves do not change with coordinate systems, their representations do since they are represented by coordinates. While investigating spaces in general, we will consider two distinct coordinate systems without further specifications, with their coordinates denoted by x and ξ and quantities pertaining to the coordinate system $\{\xi^i\}$ will be denoted with a tilde, i.e. a quantity g in $\{x^i\}$ has an equivalent in $\{\xi^i\}$, denoted $\tilde{g}$.

Curved spaces change the form of Schrödinger's equation, because it involves the Laplace operator. Since the Laplace operator is divergence of a gradient of a scalar function, that means it's the divergence of another vector function. This means the equation involves a vector, and so it depends on a coordinate system. On the other hand, if it was a derivative of a scalar, the outcome wouldn't change and this work wouldn't be necessary.

Tensors can be covariant and contravariant. When we change a basis, we don't want to change tensors, only their representations; we call this invariance. Contravariant tensors change against the transformation to compensate for the change of basis and remain invariant, like regular vectors that would stretch if we stretched the basis vectors without compressing the vector back. A vector $\tilde{u}^i$ (in an N dimensional space) follows the transformation relation

$$\tilde{u}^i(\xi) = \sum_{j=1}^{N} \frac{\partial \xi^i}{\partial x^j} u^j(x)$$

and all contravariant vectors are denoted with an upper index. On the other hand, covariant tensors change with a coordinate transformation and are re-

lated to derivatives, one of the examples is a gradient. They change in this manner, since the gradient gives a rate of change with respect to coordinates, so if the coordinates expand, we need to reflect it in the derivatives. We can use the chain rule

$$\frac{\partial}{\partial \xi^i} = \sum_j \frac{\partial x^j}{\partial \xi^i} \frac{\partial}{\partial x^j}$$

and generalize it for covariant vectors u_i to get

$$\tilde{u}_i(\xi) = \frac{\partial x^j}{\partial \xi^i} u_j(x).$$

Here we can see that, other than covariant vectors being denoted with a lower index, the factor $\frac{\partial \xi^i}{\partial x^j}$ for the contravariant vector gets inverted when compared to the covariant vector, hence their reciprocal relationship, so $\sum_i u^i v_i$ is invariant.

By looking at the Schrödinger's equation for a free particle,

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi, \tag{1}$$

we see that it has the scalar value $E\psi$ on one side, on the other a scalar multiplied by a Laplace operator. This boils down to finding the Laplace operator in a curved space. For clarity, the Laplace operator in the Euclidean space shall be henceforth denoted by ∇^2 and in non-Euclidean space Δ.

2 Calculus in curved spaces

2.1 The covariant derivative

2.1.1 Preliminaries

For convenience, we will use the index notation along with Einstein's summation convention in this section, which implies summation over a pair of lower and upper repeated indices, i.e.

$$\sum_{j=j_0}^{N} u^j v_j = u^j v_j,$$

while the summation bounds are not explicitly stated as it is assumed the reader already knows them. Suppose we take the derivative of a covariant vector u_i and change both the derivative and vector from coordinates $x \to \xi$. Using the coordinate transformation relation for a covariant vector

$$\tilde{u}_\mu(\xi) = \frac{\partial x^\eta}{\partial \xi^\mu} u_\eta$$

and for a derivative

$$\frac{\partial}{\partial x^\nu} = \frac{\partial \xi^\sigma}{\partial x^\nu} \frac{\partial}{\partial \xi^\sigma}$$

the derivative of a covariant vector after coordinate transformation becomes

$$\begin{aligned} \frac{\partial \tilde{u}_\mu(\xi)}{\partial \xi^\nu} &= \frac{\partial x^\sigma}{\partial \xi^\nu} \frac{\partial}{\partial x^\sigma} \left(\frac{\partial x^\eta}{\partial \xi^\mu} u_\eta(x) \right) \\ &= \frac{\partial x^\sigma}{\partial \xi^\nu} \frac{\partial}{\partial x^\sigma} \left(\frac{\partial x^\eta}{\partial \xi^\mu} \right) u_\eta + \frac{\partial x^\eta}{\partial \xi^\mu} \frac{\partial x^\sigma}{\partial \xi^\nu} \frac{\partial}{\partial x^\sigma} \left(u_\eta \right) \end{aligned}$$

The above term is supposed to be covariant, but the term on the left includes the derivative of a coordinate transformation, which isn't covariant. The easiest way to rectify this is to get rid of it by defining a special kind of derivative that is the "regular" derivative with some additional term that would cancel this unwanted term. We assume that it takes the specific form of

$$\nabla_\nu u_\mu = \partial_\nu u_\mu + \Gamma^\lambda_{\mu\nu} u_\lambda, \tag{2}$$

where $\Gamma^\lambda_{\mu\nu}$ is formally known as the Christoffel symbol and ∇ in index notation is the covariant derivative.

In curved spaces, specific geometry is defined via the metric tensor that follows the transformation relation

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\rho}{\partial \xi^\mu}\frac{\partial x^\sigma}{\partial \xi^\nu} g_{\rho\sigma} \tag{3}$$

or it's inverse following

$$\tilde{g}^{\mu\nu} = \frac{\partial \xi^\mu}{\partial x^\rho}\frac{\partial \xi^\nu}{\partial x^\sigma} g^{\rho\sigma}. \tag{4}$$

Therefore, it is natural to express the Christoffel symbol using only metric tensors. However, it is not easy to see which steps to take to derive it. For that purpose, we will first consider a one-dimensional case.

2.1.2 One-dimensional case

By constricting the problem to one-dimensional spaces, it simplifies the metric tensor transformation relation to

$$\tilde{g} = \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 g$$

and the covariant derivative of a covariant vector in coordinates ξ to x

$$\begin{aligned}\tilde{\nabla}\tilde{u} = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}\xi} + \tilde{\Gamma}\tilde{u} &= \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}u\right) + \tilde{\Gamma}\tilde{u}, \\ &= \frac{\mathrm{d}x}{\mathrm{d}\xi}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\xi}u + \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}x}u\right) + \tilde{\Gamma}\tilde{u}.\end{aligned}$$

We can simplify it further by trying to get a second derivative instead of $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\xi}$. We invert the derivative and use chain rule to get

$$\begin{aligned}\tilde{\nabla}\tilde{u} &= \frac{\mathrm{d}x}{\mathrm{d}\xi}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-1}\right)u + \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}x}u\right) + \tilde{\Gamma}\tilde{u} \\ &= \frac{\mathrm{d}x}{\mathrm{d}\xi}\left(\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-2}\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}u + \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}x}u\right) + \tilde{\Gamma}\tilde{u}, \\ &= \frac{\mathrm{d}x}{\mathrm{d}\xi}\left(\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{2}\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}u + \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}x}u\right) + \tilde{\Gamma}\tilde{u}.\end{aligned}$$

Here, we can try to change the regular derivative of u in x coordinates into a covariant derivative in hopes to get a transformation relation between covariant derivatives. So, we add and subtract $\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \Gamma u$ to get

$$\tilde{\nabla}\tilde{u} = -\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^3\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}u + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2\nabla u + \tilde{\Gamma}\frac{\mathrm{d}x}{\mathrm{d}\xi}u - \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2\Gamma u. \tag{5}$$

Now, there is a term $\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^3 \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} u$ that resembles something that is not so easily discovered and it is related to a trick with inverting the derivative inside another derivative, which we will do shortly.

To attempt to express the Christoffel symbol with metric tensors, it is natural to try and figure out what to do with the metric transformation relation for one-dimensional case such that it can be compared with eq. (5). We can try to differentiate it with respect to x to get

$$\frac{\mathrm{d}\tilde{g}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2\right) g + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \frac{\mathrm{d}g}{\mathrm{d}x}.$$

Were we now to just proceed normally, we would get

$$\frac{\mathrm{d}\tilde{g}}{\mathrm{d}x} = 2\frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right) g + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \frac{\mathrm{d}g}{\mathrm{d}x}.$$

The problem is that we have decreased the power of the first order term, but we would like to increase it. For that we will take a step back and use the earlier mentioned trick, that is invert the first derivative and evaluate the derivative after that.

$$\begin{aligned}\frac{\mathrm{d}\tilde{g}}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-2}\right) g + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \frac{\mathrm{d}g}{\mathrm{d}x}, \\ &= -2\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-3} \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} g + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \frac{\mathrm{d}g}{\mathrm{d}x},\end{aligned}$$

which leads us to the final form

$$\frac{\mathrm{d}\tilde{g}}{\mathrm{d}x} = -2\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^3 \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} g + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \frac{\mathrm{d}g}{\mathrm{d}x}. \tag{6}$$

We can now see that is shares the term $\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^3 \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}$ with eq. (5).

We can now proceed where we left off and try to simplify eq. (5) further. Since the covariant derivative in coordinates ξ is equal to $\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \nabla u$, we can get rid of them to get

$$0 = -\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^3 \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} u + \tilde{\Gamma}\frac{\mathrm{d}x}{\mathrm{d}\xi} u - \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \Gamma u.$$

Furthermore, we want to impose the condition that the equation is invariant

for all u, so the following must hold.

$$0 = -\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^3 \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} + \underbrace{\frac{\mathrm{d}x}{\mathrm{d}\xi}}_{\tilde{g}/g\frac{\mathrm{d}\xi}{\mathrm{d}x}} \tilde{\Gamma} - \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \Gamma,$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\tilde{g}}{g}\tilde{\Gamma} = \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^3 \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \Gamma,$$

which leads to the final equation

$$-2\frac{\mathrm{d}\xi}{\mathrm{d}x}\tilde{g}\tilde{\Gamma} = -2\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^3 \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} g - 2g\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 \Gamma. \tag{7}$$

Comparing with eq. (6) we infer

$$\frac{\mathrm{d}\xi}{\mathrm{d}x}\tilde{\Gamma} = -\frac{1}{2\tilde{g}}\frac{\mathrm{d}\tilde{g}}{\mathrm{d}x},$$
$$\tilde{\Gamma} = -\frac{1}{2\tilde{g}}\frac{\mathrm{d}\tilde{g}}{\mathrm{d}\xi},$$
$$\Gamma = -\frac{1}{2g}\frac{\mathrm{d}g}{\mathrm{d}x}.$$

Finally, the covariant derivative in 1D is

$$\nabla u = \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{2g}\frac{\mathrm{d}g}{\mathrm{d}x}.$$

2.2 General case

Now that we have the general idea what to do, we take the covariant derivative, given by eq. (2), and transform from coordinates $x \to \xi$

$$\begin{aligned}
\tilde{\nabla}_k \tilde{u}_i &= \frac{\partial \tilde{u}_i}{\partial \xi^k} + \tilde{\Gamma}^{\mu}_{ik}\tilde{u}_\mu, \\
&= \frac{\partial}{\partial \xi^k}\left(\frac{\partial x^\nu}{\partial \xi^i} u_\nu\right) + \tilde{\Gamma}^{\mu}_{ik}\tilde{u}_\mu, \\
&= \frac{\partial}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^i} u_\nu + \frac{\partial x^\nu}{\partial \xi^i}\frac{\partial u_\nu}{\partial \xi^k} + \tilde{\Gamma}^{\mu}_{ik}\tilde{u}_\mu, \\
&= \frac{\partial}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^i} u_\nu + \frac{\partial x^\nu}{\partial \xi^i}\frac{\partial x^\sigma}{\partial \xi^k}\frac{\partial u_\nu}{\partial x^\sigma} + \tilde{\Gamma}^{\mu}_{ik}\frac{\partial x^\rho}{\partial \xi^\mu} u_\rho.
\end{aligned}$$

We could get the covariant derivative with respect to the coordinates x in the above equation if we add and subtract $\frac{\partial x^\nu}{\partial \xi^i}\frac{\partial x^\sigma}{\partial \xi^k}\Gamma^\lambda_{\nu\sigma}u_\lambda$, so

$$\tilde{\nabla}_k \tilde{u}_i = \frac{\partial}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^i}u_\nu + \frac{\partial x^\nu}{\partial \xi^i}\frac{\partial x^\sigma}{\partial \xi^k}\nabla_\sigma u_\nu + \tilde{\Gamma}^\mu_{ik}\frac{\partial x^\rho}{\partial \xi^\mu}u_\rho - \frac{\partial x^\nu}{\partial \xi^i}\frac{\partial x^\sigma}{\partial \xi^k}\Gamma^\lambda_{\nu\sigma}u_\lambda.$$

Just like in the one-dimensional case, the term $\frac{\partial x^\nu}{\partial \xi^i}\frac{\partial x^\sigma}{\partial \xi^k}\nabla_\sigma u_\nu$ is another, more explicit, form of the term $\tilde{\nabla}_k \tilde{u}_i$, so we can get rid of them.

$$0 = \frac{\partial}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^i}u_\nu + \tilde{\Gamma}^\mu_{ik}\frac{\partial x^\rho}{\partial \xi^\mu}u_\rho - \frac{\partial x^\nu}{\partial \xi^i}\frac{\partial x^\sigma}{\partial \xi^k}\Gamma^\lambda_{\nu\sigma}u_\lambda.$$

And, again, this equation must hold for all vectors u_μ, so we can simplify it to

$$\boxed{\tilde{\Gamma}^\mu_{ik}\frac{\partial x^\lambda}{\partial \xi^\mu} + \frac{\partial}{\partial \xi^k}\frac{\partial x^\lambda}{\partial \xi^i} = \frac{\partial x^\rho}{\partial \xi^i}\frac{\partial x^\sigma}{\partial \xi^k}\Gamma^\lambda_{\rho\sigma}.} \tag{8}$$

The above equation is quite significant, similarly to the "metric tensor transformation relation" in eq. (3) or eq. (4), as it defines how a Christoffel symbol changes with coordinate change. I should point out that it's apparent that the Christoffel symbol does not transform "neatly" with just some coefficient, but rather introduces an additional term $\frac{\partial}{\partial \xi^k}\frac{\partial x^\lambda}{\partial \xi^i}$ that will from now on be referred to as the problematic term or the second order term. Note that the indices i, k are symmetric, which implies that the Christoffel symbol is symmetric in its lower indices, here denoted by i, k. Since we want to express the Christoffel symbol in terms of the metric tensor, we should choose g_{ik} due to said symmetry of indices i, k.

Now, we take the derivative of the metric tensor transformation relation with respect to ξ

$$\frac{\mathrm{d}\tilde{g}_{ik}}{\mathrm{d}\xi^j} = \left(\frac{\partial}{\partial \xi^j}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k} + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial}{\partial \xi^j}\frac{\partial x^\nu}{\partial \xi^k}\right)g_{\mu\nu} + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k}\frac{\partial g_{\mu\nu}}{\partial \xi^j}, \tag{9}$$

then multiply by $\frac{\partial \xi^k}{\partial x^\rho}$

$$\frac{\mathrm{d}\tilde{g}_{ik}}{\mathrm{d}\xi^j}\frac{\partial \xi^k}{\partial x^\rho} = \left(\frac{\partial}{\partial \xi^j}\frac{\partial x^\mu}{\partial \xi^i}\delta^\nu_\rho + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial \xi^k}{\partial x^\rho}\frac{\partial}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^j}\right)g_{\mu\nu} + \frac{\partial x^\mu}{\partial \xi^i}\delta^\nu_\rho\frac{\partial g_{\mu\nu}}{\partial \xi^j}$$

and then by $\frac{\partial \xi^i}{\partial x^\sigma}$

$$\begin{aligned}
\frac{\mathrm{d}\tilde{g}_{ik}}{\mathrm{d}x^\sigma}\frac{\partial \xi^k}{\partial x^\rho} &= \left(\frac{\partial \xi^j}{\partial x^\sigma}\frac{\partial}{\partial \xi^i}\frac{\partial x^\mu}{\partial \xi^j}\delta^\nu_\rho + \delta^\mu_\sigma\frac{\partial \xi^k}{\partial x^\rho}\frac{\partial}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^j}\right)g_{\mu\nu} + \frac{\partial x^\mu}{\partial \xi^i}\delta^\nu_\rho\frac{\partial g_{\mu\nu}}{\partial x^\sigma},\\
\frac{\mathrm{d}\tilde{g}_{ik}}{\mathrm{d}x^\sigma}\frac{\partial \xi^k}{\partial x^\rho} &= \left(\frac{\partial}{\partial x^\sigma}\frac{\partial x^\mu}{\partial \xi^i}\delta^\nu_\rho + \delta^\mu_\sigma\frac{\partial}{\partial x^\rho}\frac{\partial x^\nu}{\partial \xi^j}\right)g_{\mu\nu} + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial g_{\mu\rho}}{\partial x^\sigma},\\
\frac{\mathrm{d}\tilde{g}_{ik}}{\mathrm{d}x^\sigma}\frac{\partial \xi^k}{\partial x^\rho} &= \frac{\partial}{\partial x^\sigma}\frac{\partial x^\mu}{\partial \xi^i}g_{\mu\rho} + \frac{\partial}{\partial x^\rho}\frac{\partial x^\nu}{\partial \xi^j}g_{\sigma\nu} + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial g_{\mu\rho}}{\partial x^\sigma}.
\end{aligned}$$

We have run into a problem. The second order terms in the above equation are comparable to the problematic term, but it has split into two terms multiplied by a metric tensor that cannot be simultaneously contracted for both terms, i.e. the relation

$$g^{\mu\lambda}g_{\nu\lambda} = \delta^\mu_\nu \tag{10}$$

cannot be used to get rid of the metric tensor for both terms. To remedy this, we go to eq. (9), subtract the latter second order term and change the order of the second derivative, that is, swap indices j,k for that term, which does not change the outcome. We can also subtract the remaining three terms with swapped indices j,k, which would in fact yield the derivative of metric tensor $\frac{\partial \tilde{g}_{ij}}{\partial \xi^k}$, which we will put on the LHS and which is in general distinct from $\frac{\partial \tilde{g}_{ik}}{\partial \xi^j}$ and would not eliminate it.

$$\begin{aligned}
\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} - \frac{\partial \tilde{g}_{ij}}{\partial \xi^k} = \left(\frac{\partial}{\partial \xi^j}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k} - \frac{\partial}{\partial \xi^k}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial}{\partial \xi^j}\frac{\partial x^\nu}{\partial \xi^k} - \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^j}\right)g_{\mu\nu} +\\
+ \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k}\frac{\partial g_{\mu\nu}}{\partial \xi^j} - \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial \xi^k}
\end{aligned}$$

$$\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} - \frac{\partial \tilde{g}_{ij}}{\partial \xi^k} = \left(\frac{\partial}{\partial \xi^j}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k} - \frac{\partial}{\partial \xi^k}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\right)g_{\mu\nu} + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k}\frac{\partial g_{\mu\nu}}{\partial \xi^j} - \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial \xi^k}$$

This, however, introduces another problematic term, so we try to use the same method and add the same terms with k,i swapped with the unwanted term.

$$\begin{aligned}
\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} - \frac{\partial \tilde{g}_{ij}}{\partial \xi^k} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} = \left(\frac{\partial}{\partial \xi^j}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k} - \frac{\partial}{\partial \xi^k}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} + \frac{\partial}{\partial \xi^i}\frac{\partial x^\mu}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^j} +\right.\\
\left. + \frac{\partial x^\mu}{\partial \xi^k}\frac{\partial}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\right)g_{\mu\nu} + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k}\frac{\partial g_{\mu\nu}}{\partial \xi^j} - \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial \xi^k} + \frac{\partial x^\mu}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial \xi^i},
\end{aligned}$$

which simplifies to

$$\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} - \frac{\partial \tilde{g}_{ij}}{\partial \xi^k} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} = \left(\frac{\partial}{\partial \xi^j}\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k} + \frac{\partial x^\mu}{\partial \xi^k}\frac{\partial}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} \right) g_{\mu\nu} + \\ + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^k}\frac{\partial g_{\mu\nu}}{\partial \xi^j} - \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial \xi^k} + \frac{\partial x^\mu}{\partial \xi^k}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial \xi^i},$$

which finally allows to manipulate it and compare it with eq. (8), due to both of the first order factors of the second order terms having the same index, k.

So, multiplying by $\frac{\partial \xi^k}{\partial x^\rho}$

$$\left(\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} \right)\frac{\partial \xi^k}{\partial x^\rho} - \frac{\partial \tilde{g}_{ij}}{\partial x^\rho} = \left(\frac{\partial}{\partial \xi^j}\frac{\partial x^\mu}{\partial \xi^i}\delta^\nu_\rho + \delta^\mu_\rho \frac{\partial}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} \right) g_{\mu\nu} + \\ + \frac{\partial x^\mu}{\partial \xi^i}\delta^\nu_\rho \frac{\partial g_{\mu\nu}}{\partial \xi^j} - \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial x^\rho} + \delta^\mu_\rho \frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial \xi^i},$$

$$\left(\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} \right)\frac{\partial \xi^k}{\partial x^\rho} - \frac{\partial \tilde{g}_{ij}}{\partial x^\rho} = \frac{\partial}{\partial \xi^j}\frac{\partial x^\mu}{\partial \xi^i} g_{\mu\rho} + \frac{\partial}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} g_{\rho\nu} + \\ + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial g_{\mu\rho}}{\partial \xi^j} + \frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\rho\nu}}{\partial \xi^i} - \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial x^\rho}.$$

The two second order terms are now the same, so

$$\left(\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} \right)\frac{\partial \xi^k}{\partial x^\rho} - \frac{\partial \tilde{g}_{ij}}{\partial x^\rho} = 2\frac{\partial}{\partial \xi^j}\frac{\partial x^\mu}{\partial \xi^i} g_{\mu\rho} + \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial g_{\mu\rho}}{\partial \xi^j} + \frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\rho\nu}}{\partial \xi^i} - \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial x^\rho}$$

Now, to isolate the second order term so it can be directly comparable to the problematic term, we multiply by $\frac{1}{2}g^{\lambda\rho}$

$$\frac{1}{2}g^{\lambda\rho}\left(\left(\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} \right)\frac{\partial \xi^k}{\partial x^\rho} - \frac{\partial \tilde{g}_{ij}}{\partial x^\rho} \right) = \frac{\partial}{\partial \xi^j}\frac{\partial x^\lambda}{\partial \xi^i} + \frac{1}{2}g^{\lambda\rho}\left(\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial g_{\mu\rho}}{\partial \xi^j} + \right. \\ \left. + \frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\rho\nu}}{\partial \xi^i} - \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\frac{\partial g_{\mu\nu}}{\partial x^\rho} \right).$$

By taking eq. (8), swapping indices j, k for consistence and comparing with the result, we can assume that for the transformed Christoffel symbol,

$$\tilde{\Gamma}^\mu_{ij}\frac{\partial x^\lambda}{\partial \xi^\mu} = \frac{1}{2}g^{\lambda\rho}\left(\left(\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} \right)\frac{\partial \xi^k}{\partial x^\rho} - \frac{\partial \tilde{g}_{ij}}{\partial x^\rho} \right),$$

multiplying by $\frac{\partial \xi^\sigma}{\partial x^\lambda}$

$$\begin{aligned}
\tilde{\Gamma}^\sigma_{ij} &= \frac{1}{2} g^{\lambda\rho} \left(\left(\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} \right) \frac{\partial \xi^k}{\partial x^\rho} - \frac{\partial \tilde{g}_{ij}}{\partial x^\rho} \right) \frac{\partial \xi^\sigma}{\partial x^\lambda}, \\
&= \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} - \frac{\partial \tilde{g}_{ij}}{\partial \xi^k} \right) \frac{\partial \xi^\sigma}{\partial x^\lambda} \frac{\partial \xi^k}{\partial x^\rho}, \\
\tilde{\Gamma}^\sigma_{ij} &= \frac{1}{2} \tilde{g}^{\sigma k} \left(\frac{\partial \tilde{g}_{ik}}{\partial \xi^j} + \frac{\partial \tilde{g}_{jk}}{\partial \xi^i} - \frac{\partial \tilde{g}_{ij}}{\partial \xi^k} \right).
\end{aligned}$$

For the non-transformed one,

$$\begin{aligned}
\frac{\partial x^\rho}{\partial \xi^i} \frac{\partial x^\sigma}{\partial \xi^j} \Gamma^\lambda_{\rho\sigma} &= \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial x^\mu}{\partial \xi^i} \frac{\partial g_{\mu\rho}}{\partial \xi^j} + \frac{\partial x^\nu}{\partial \xi^j} \frac{\partial g_{\rho\nu}}{\partial \xi^i} - \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j} \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right), \\
\frac{\partial x^\rho}{\partial \xi^i} \frac{\partial x^\sigma}{\partial \xi^j} \Gamma^\lambda_{\rho\sigma} &= \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial x^\mu}{\partial \xi^i} \frac{\partial g_{\mu\rho}}{\partial x^\omega} \frac{\partial x^\omega}{\partial \xi^j} + \frac{\partial x^\nu}{\partial \xi^j} \frac{\partial g_{\rho\nu}}{\partial x^\tau} \frac{\partial x^\tau}{\partial \xi^i} - \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j} \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right).
\end{aligned}$$

Multiplying by $\frac{\partial \xi^i}{\partial x^\alpha} \frac{\partial \xi^j}{\partial x^\beta}$ will finally give us the result for the Christoffel symbol,

$$\boxed{\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\alpha\rho}}{\partial x^\beta} + \frac{\partial g_{\rho\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\rho} \right)}.$$

2.3 Laplace operator in curved spaces

Covariant derivative acting on a scalar function is the same as a partial derivative, but a gradient of a scalar function is a vector function, so the "covariant laplacian" is not the same

$$\Delta\psi = \nabla^\mu \nabla_\mu \psi = \nabla^\mu \left(\partial_\mu \psi \right).$$

However, we have only defined a covariant derivative with a lower index. Here, we can use the relation

$$\nabla^\mu \nabla_\mu = g^{\mu\nu} \nabla_\nu \nabla_\mu,$$

so, the Laplace operator in curved spaces is

$$\Delta\psi = g^{\mu\nu} \nabla_\nu \partial_\mu \psi$$

and explicitly

$$\boxed{\Delta\psi = g^{\mu\nu} \left(\partial_\nu \partial_\mu \psi + \Gamma^\lambda_{\mu\nu} \partial_\lambda \psi \right).} \tag{11}$$

3 Schrödinger's equation in curved space

3.1 Isotropic tensor in 2D

An isotropic tensor is a tensor given by

$$g_{\mu\nu} = g\delta_{\mu\nu}$$

where g is in general some function of coordinates of respective space/coordinate system, i.e. it can be represented by a diagonal matrix with the same elements. The inverse metric is then

$$g^{\mu\nu} = \frac{1}{g}\delta^{\mu\nu}.$$

The relevance of this section will be clear in the following sections, as it is put here *a posteriori*, that is after deriving the Schrödinger's equation for Poincaré disk and then trying to generalize it as it looked like it could work. Because the metric on Poincaré disk and stereographic projection, among other metrics of different coordinate systems, have an isotropic metric tensor, it allows finding the form of Schrödinger's equation for spaces with isotropic metric more easily.

First, let's consider the part of the Laplace eq. (11) with the Christoffel symbol. An isotropic metric multiplied by a vector $V_\lambda = \partial_\lambda \psi$ is

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} V_\lambda &= \frac{1}{2} g^{\lambda k}\left(-\frac{\partial g_{\mu\nu}}{\partial x^k} + \frac{\partial g_{\mu k}}{\partial x^\nu} + \frac{\partial g_{k\nu}}{\partial x^\mu}\right) V_\lambda, \\ &= \frac{1}{2g}\delta^{\lambda k}\left(-\frac{\partial g}{\partial x^k}\delta_{\mu\nu} + \frac{\partial g}{\partial x^\nu}\delta_{\mu k} + \frac{\partial g}{\partial x^\mu}\delta_{k\nu}\right) V_\lambda.\end{aligned}$$

Multiplying by $g^{\mu\nu}$ to get the whole part with the Christoffel symbol

$$\begin{aligned}g^{\mu\nu}\Gamma^\lambda_{\mu\nu} V_\lambda &= \frac{1}{2g}\frac{1}{g}\delta^{\mu\nu}\delta^{\lambda k}\left(-\frac{\partial g}{\partial x^k}\delta_{\mu\nu} + \frac{\partial g}{\partial x^\nu}\delta_{\mu k} + \frac{\partial g}{\partial x^\mu}\delta_{k\nu}\right) V_\lambda, \\ &= \frac{1}{2g^2}\delta^{\lambda k}\left(-\dim N\frac{\partial g}{\partial x^k} + \frac{\partial g}{\partial x^\nu}\delta^\nu_k + \frac{\partial g}{\partial x^\mu}\delta^\mu_k\right) V_\lambda,\end{aligned}$$

where $\delta^{\mu\nu}\delta_{\mu\nu} = \dim N$ is the dimension of N-dimensional space

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} V_\lambda = \frac{1}{2g^2}\left(-\dim N\frac{\partial g}{\partial x^k}\delta^{\lambda k} + 2\frac{\partial g}{\partial x^k}\delta^{\lambda k}\right) V_\lambda.$$

In the case of surfaces, $\dim N = 2$. So, for a metric given by a diagonal matrix with equal elements defined in 2-dimensional space

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} V_\lambda = 0$$

We are then left only with

$$\Delta\psi = g^{\mu\nu}\partial_\nu\partial_\mu\psi$$
$$\Delta\psi = \frac{1}{g}\delta^{\mu\nu}\partial_\nu\partial_\mu\psi$$

Therefore, the Laplace operator in a space given by a 2-dimensional isotropic metric

$$\boxed{\Delta\psi = \frac{1}{g}\nabla^2\psi.} \tag{12}$$

3.2 Deriving the metric tensor

To find the metric tensor, we need to first parameterize a space by coordinates, from which we can find the infinitesimal displacement ds on the surface in terms of Euclidean coordinates. Metric tensor can, in fact, be defined with

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu.$$

which also tells us how to find it from the infinitesimal displacement.

3.3 Unknown surface

3.3.1 The equation

The following is an surface with explicitly unspecified shape and is only defined via an infinitesimal element

$$ds^2 = \frac{dx^2 + dy^2}{1 - x^2 - y^2}$$

for the purpose of having a "training" surface with existing solution for Schrödinger's equation. The metric tensor for given length element is thus given by

$$g_{\mu\nu} = \frac{1}{1 - x^2 - y^2}\delta_{\mu\nu}$$

and for brevity

$$g = \frac{1}{1 - x^2 - y^2}.$$

The inverse metric is then

$$g^{\mu\nu} = \frac{1}{g}\delta^{\mu\nu}.$$

Since the metric is given by an isotropic tensor, the Schrödinger's equation is

$$\boxed{E\psi = -\frac{\hbar^2}{2m}\left(1-x^2-y^2\right)\nabla^2\psi.} \tag{13}$$

We can isolate the Laplacian

$$\nabla^2\psi = -\frac{2mE}{\hbar^2}\frac{1}{1-x^2-y^2}\psi \tag{14}$$

and use the non-relativistic formula for momentum $p^2 = 2mE$

$$\nabla^2\psi = -\left(\frac{p}{\hbar}\right)^2\frac{1}{1-x^2-y^2}\psi.$$

By using the Laplacian in polar coordinates [1]

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} \tag{15}$$

and transforming the equation accordingly

$$-\frac{p^2}{\hbar^2}\psi\frac{1}{1-r^2} = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} \tag{16}$$

we can use the method of separation of variables by assuming that the solution takes the form of

$$\psi(r,\varphi) = F(\varphi)R(r).$$

3.3.2 The angular part

Using the method of separation of variables for the Schrödinger's equation, we get

$$-\frac{p^2}{\hbar^2}\psi\frac{1}{1-r^2} = F(\varphi)\frac{\partial^2R(r)}{\partial r^2} + \frac{1}{r}F(\varphi)\frac{\partial R(r)}{\partial r} + \frac{1}{r^2}R(r)\frac{\partial^2F(\varphi)}{\partial\varphi^2}$$

and multiplying by $\frac{r^2}{FR}$ gives us

$$-r^2\frac{p^2}{\hbar^2}\frac{1}{1-r^2} = \frac{r^2}{R(r)}\frac{\partial^2R(r)}{\partial r^2} + \frac{r}{R(r)}\frac{\partial R(r)}{\partial r} + \frac{1}{F(\varphi)}\frac{\partial^2F(\varphi)}{\partial\varphi^2}.$$

Now, we have achieved the desired result of getting two ODEs instead of one PDE. Relying on the assumption that F is a function of φ only, we get

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\varphi^2} = -\omega^2 F(\varphi). \tag{17}$$

where ω is some constant. This is the known equation for a harmonic oscillator,

$$F(\varphi) = Ae^{i\omega\varphi} + Be^{-i\omega\varphi}. \tag{18}$$

It was decided that the RHS of eq. (17) should be negative, since the resulting function $F(\varphi)$ would be a periodic function (as opposed to an exponential), which is quite expected for a function of an angle.

As it was said, the function must be periodic with period 2π, that is

$$F(\varphi) = F(\varphi + 2\pi).$$

This means that

$$\begin{aligned} e^{i\omega\varphi} &= e^{i\omega\varphi + i\omega 2\pi}, \\ &= e^{i\omega\varphi} e^{i\omega 2\pi}, \\ &= e^{i\omega\varphi} \left(\cos(2\pi\omega) + i\sin(2\pi\omega)\right). \end{aligned}$$

This sets the condition

$$\omega = n, \quad n \in \mathbb{N} \tag{19}$$

Similarly,

$$e^{-i\omega\varphi + i\omega 2\pi} = e^{-i\omega\varphi} \left(\cos(2\pi\omega) + i\sin(2\pi\omega)\right)$$

which sets the same condition for the second, negative, exponential. To find the coefficients A, B, we would impose the boundary conditions.

3.3.3 The radial part

For the second equation, we plug the solution from eq. (17) to get

$$\begin{aligned} -r^2 \frac{p^2}{\hbar^2} \frac{1}{1-r^2} &= \frac{r^2}{R(r)} \frac{\mathrm{d}^2 R(r)}{\mathrm{d}r^2} + \frac{r}{R(r)} \frac{\mathrm{d}R(r)}{\mathrm{d}r} - \omega^2 \\ -R(r) \frac{p^2}{\hbar^2} \frac{r}{1-r^2} &= r \frac{\mathrm{d}^2 R(r)}{\mathrm{d}r^2} + \frac{\mathrm{d}R(r)}{\mathrm{d}r} - \frac{\omega^2}{r} R(r) \end{aligned}$$

using product rule

$$-R(r)\frac{p^2}{\hbar^2}\frac{r}{1-r^2}+\frac{\omega^2}{r}R(r)=\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R(r)}{\mathrm{d}r}+R(r)\right)-\frac{\mathrm{d}R(r)}{\mathrm{d}r}$$
$$R(r)\left(\omega^2-\frac{p^2}{\hbar^2}\frac{r^2}{1-r^2}\right)=r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right)$$

Here are two things worth mentioning. First, the RHS contains $r\frac{\mathrm{d}}{\mathrm{d}r}$ twice, so it would be best if we could use some substitution with u such that $\frac{\mathrm{d}}{\mathrm{d}r}=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}u}$. It's clear to see that $u=\ln r$, so $\frac{\mathrm{d}}{\mathrm{d}r}=\frac{\mathrm{d}u}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}u}=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}u}$. We can now rewrite the equation as

$$-R(u)\left(\frac{p}{\hbar}\right)^2\frac{e^{2u}}{1-e^{2u}}=\frac{\mathrm{d}^2R(u)}{\mathrm{d}u^2} \tag{20}$$

Second, the $\frac{1}{1-r^2}=\frac{1}{1-e^{2u}}$ is the sum of geometric series, assuming that $r<1$, that is

$$\frac{1}{1-e^{2u}}=\sum_{n=0}^{\infty}e^{2nu}$$

multiplying by e^{2u} gives us

$$e^{2u}\left(1+e^{2u}+e^{4u}+e^{6u}+\ldots\right)=\left(e^{2u}+e^{4u}+e^{6u}+e^{8u}+\ldots\right)=\sum_{n=1}^{\infty}e^{2nu}$$

We are then left with

$$-R(u)\left(\frac{p}{\hbar}\right)^2\sum_{n=1}^{\infty}e^{2nu}=\frac{\mathrm{d}^2R(u)}{\mathrm{d}u^2} \tag{21}$$

The main problem is to reconstruct the term

$$R(u)\sum_{n=1}^{\infty}e^{2nu}$$

using the second derivative. We can reconstruct the geometric series by assuming that the solution is

$$R_1(u)=e^{e^{2u}}$$

so the first derivative is

$$\frac{\mathrm{d}R_1(u)}{\mathrm{d}u}=2R(u)e^{2u}$$

and the second derivative is

$$\frac{\mathrm{d}^2 R_1(u)}{\mathrm{d}u^2} = R(u)_1 \left(4e^{2u} + 4e^{4u}\right).$$

By assuming the solution is

$$R_1(u) + R(u)_2 = e^{e^{2u}} + e^{e^{4u}}$$

we get

$$\frac{\mathrm{d}^2 R_1(u) + R_2(u)}{\mathrm{d}u^2} = e^{e^{2u}} \left(4e^{2u} + 4e^{4u}\right) + e^{e^{4u}} \left(16e^{8u} + 16e^{4u}\right)$$

From this we can notice a pattern that would lead to the solution

$$R_j(u) = \sum_{j=1}^{\infty} e^{e^{a_j u}},$$

which might look odd, but let's not forget that $u = \ln r$, so $R_j(r) = \sum_{j=1}^{\infty} e^{r^{a_j}}$. The problem is to notice a pattern that would allow us to identify the coefficients $a_2, \ldots, a_j$. The general formula for the derivative is

$$\frac{\mathrm{d}^2 R_j(u)}{\mathrm{d}u^2} = \frac{\mathrm{d}}{\mathrm{d}u} \left(e^{e^{2ju}} \left(2ne^{2ju}\right)\right) = e^{e^{2ju}} \left(4j^2 e^{4ju} + 4j^2 e^{2ju}\right) = 4j^2 R_j(u) \left(e^{4ju} + e^{2ju}\right)$$

To get rid of the coefficients, we just divide by $4j^2$. Taking the sum for first few R_j we get

$$R_1(u) \left(e^{4u} + e^{2u}\right) + R_2(u) \left(e^{8u} + e^{4u}\right) + R_3(u) \left(e^{12u} + e^{6u}\right)$$

For each term e^{2nu}, except for the first one $n = 1$, we get them two times, so we would divide further by 2 all terms with $n > 1$. However, there is a problem, since what we get from the chain rule corresponds to the geometric series for $\frac{1}{1-e^{2u}}$, but each n-th term is multiplied by $R_n(u)$, not a common $R(u)$, so this, sadly, does not seem to lead to a solution.

The actual solution seems to be more complicated. The eq. (21) can be constructed from linear combinations of the type of equation

$$y_n"(x) + ae^{2nx} y_n(x) = D_n, \tag{22}$$

where D_n is a constant such that

$$\sum_{n_1}^{\infty} D_n = 0.$$

The homogeneous solution to eq. (22) is[2]

$$y = C_1 J_0(z) + C_2 Y_0(z), \quad z = \frac{2}{2n}\sqrt{a}e^{\frac{2nx}{2}} \tag{23}$$

where C_1, C_2 are integration constants and J_0, Y_0 are the Bessel functions, that is the Bessel function of the first kind of order zero is

$$J_0 = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1)} \left(\frac{z}{2}\right)^{2m} \tag{24}$$

and the Bessel function of the second kind of order zero is

$$Y_0 = \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} \cos(z\cos\theta) \left(\gamma + \ln\left(2z\sin^2\theta\right)\right) d\theta \tag{25}$$

where γ is the Euler's constant (not to be confused with Euler's number).

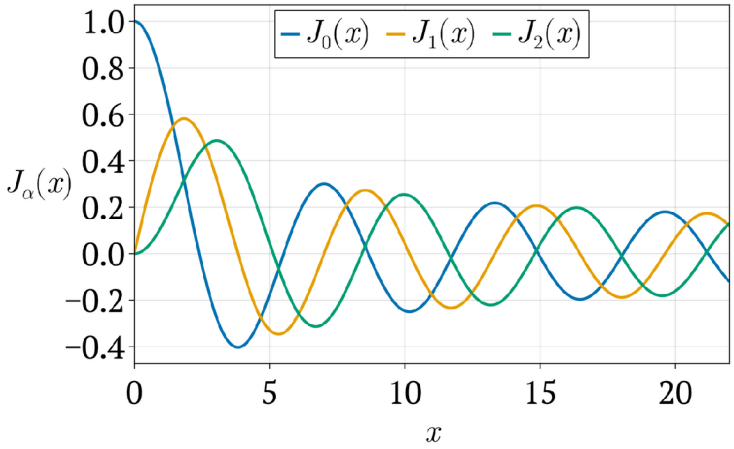

Fig. 1: Plot of Bessel function of the first kind, taken from Wikipedia[3].

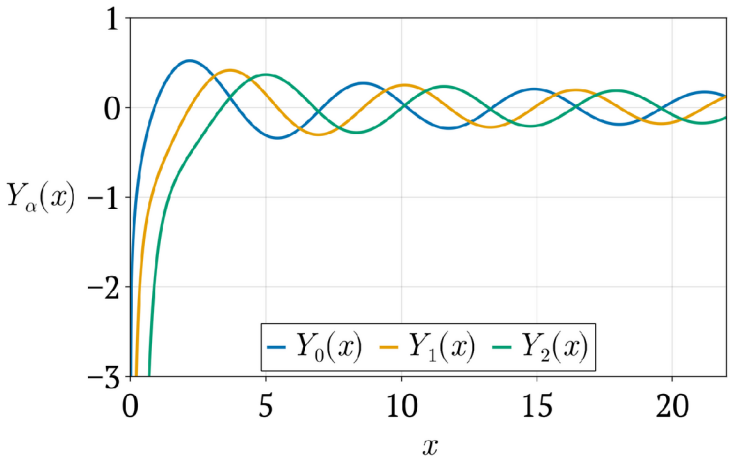

Fig. 2: Plot of Bessel function of the second kind, taken from Wikipedia[4].

As you can see from fig. 2, the Bessel function of second kind diverges for $r \to 0$, which is problematic for our space, or for physics in general. For that reason, we exclude this term by setting $C_2 = 0$, so the solution to eq. (22) is

$$y = C_1 J_0(z), \quad z = \frac{2}{2n}\sqrt{a}e^{\frac{2nx}{2}} \tag{26}$$

This form of solution is difficult to work with, since the wavefunction is a double sum of Bessel functions. It should be pointed out that an alternative form of eq. (20) can be found using the hyperbolic functions. Expanding the LHS of eq. (20) by $\frac{e^{-x}}{e^{-x}}$ gives us

$$\begin{aligned} -R(u)\left(\frac{p}{\hbar}\right)^2 \frac{e^u}{e^{-u}-e^u} &= \frac{\mathrm{d}^2R(u)}{\mathrm{d}u^2} \\ -R(u)\left(\frac{p}{\hbar}\right)^2 \frac{\sinh u + \cosh u}{2\sinh u} &= \frac{\mathrm{d}^2R(u)}{\mathrm{d}u^2} \\ -R(u)\left(\frac{p}{\hbar}\right)^2 \frac{1}{2}(1+\coth u) &= \frac{\mathrm{d}^2R(u)}{\mathrm{d}u^2} \end{aligned}$$

An attempt to find an alternative solution can be made by going back to the unsubstituted equation, as it looks similar to an equation that was solved by A. D. Polyanin[5]

$$x^2\left(ax^n-1\right)y'' + x\left(apx^n+q\right)y' + \left(arx^n+s\right)y = 0, \tag{27}$$

which we will now attempt to match.

$$R(r)\left(\frac{p^2}{\hbar^2}\frac{r}{1-r^2} + \frac{\omega^2}{r}\right) + r\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{\mathrm{d}R(r)}{\mathrm{d}r} = 0$$

multiplying by $(1-r^2)$ and rearranging

$$r(1-r^2)\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+(1-r^2)\frac{\mathrm{d}R(r)}{\mathrm{d}r}+\left(\frac{p^2}{\hbar^2}r+\frac{\omega^2}{r}(1-r^2)\right)R(r)=0$$

and multiplying by $-r$

$$r^2(r^2-1)\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+r(r^2-1)\frac{\mathrm{d}R(r)}{\mathrm{d}r}+\left(-\frac{p^2}{\hbar^2}r^2+\omega^2(r^2-1)\right)R(r)=0,$$
$$r^2(r^2-1)\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+r(r^2-1)\frac{\mathrm{d}R(r)}{\mathrm{d}r}+\left(\left(\omega^2-\frac{p^2}{\hbar^2}\right)r^2-\omega^2\right)R(r)=0.$$

So, the constants in eq. (27) are $n=2, a=p=q=1, r=\omega^2-\frac{p^2}{\hbar^2}$ and $s=-\omega^2$. We are to find the solution to two quadratic equations, first being

$$0=A^2-(q+1)A-s=A^2-2A+\omega^2$$

with solution

$$A=1\pm\sqrt{1-\omega^2},$$

and second being (with r not being confused with radial coordinate)

$$0=B^2-(p-1)B+r=B^2+\omega^2-\frac{p^2}{\hbar^2}$$

with solution

$$B=\pm\sqrt{\frac{p^2}{\hbar^2}-\omega^2}$$

For roots of the quadratic equations A_1, A_2, B_1, B_2, we define four parameters

$$c=A_1=1+\sqrt{1-\omega^2},$$
$$\alpha=\frac{A_1+B_1}{n}=\frac{1+\sqrt{1-\omega^2}+\sqrt{\frac{p^2}{\hbar^2}+\omega^2}}{2},$$
$$\beta=\frac{A_1+B_2}{n}=\frac{1+\sqrt{1-\omega^2}-\sqrt{\frac{p^2}{\hbar^2}-\omega^2}}{2},$$
$$\gamma=1+\frac{A_1-A_2}{n}=1+\frac{1+\sqrt{1-\omega^2}-1+\sqrt{1-\omega^2}}{2}=1+\sqrt{1-\omega^2}.$$

The solution to the original equation has the form

$$y = x^c u(ax^n)$$

where $u(z)$ is the general solution of hypergeometric equation

$$z(z-1)u''_{zz} + \left[\left(\alpha+\beta+1\right)z-\gamma\right]u'_z + \alpha\beta u = 0,$$

which was also solved by Polyanin for some cases[6]. However, our cases is not included. The solution with double sum of Bessel function will have to suffice.

3.4 Stereographic projection

3.4.1 The equation

Suppose we have a sphere cut in half by a plane with north pole N and south pole S. For any point on the sphere P (excluding N, S), there exists exactly one line that goes through both P and N, and there exists only one intersection P' of the line with the plane. The point P' is defined as the stereographic projection of the point P.

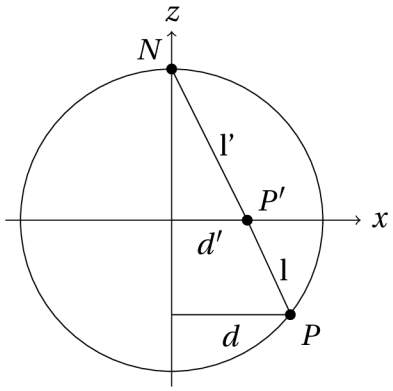

Fig. 3: Diagram of a stereographic projection from a side, own work.

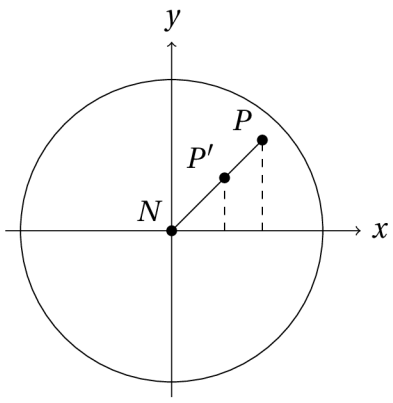

Fig. 4: Diagram of stereographic projection from above, where the distance between NP is d and between NP' is d' from fig. 3, own work.

We can use the law of similar triangles to get the relation from fig. 3

$$\frac{d'}{l'} = \frac{d}{L} \tag{28}$$

where $L = l + l'$.

Since the origin is in N and the radius of the circle is $r \equiv 1$, then the z coordinate must change to $1 - z$, i.e. the relation for Z measured from the point N as opposed to z measured from the origin point is

$$Z = 1 - z. \tag{29}$$

By taking the coordinates of both points as $P = [X, Z]$ and $P' = [x, Z']$, we get

$$\begin{gathered} d' = x, \quad d = X \\ l'^2 = d'^2 + Z'^2 \end{gathered} \tag{30}$$

By rewriting eq. (28) and substituting what we can, we get

$$\begin{gathered} d' = d\frac{l'}{L} \\ x = X\frac{l'}{L} \\ x = X\frac{\sqrt{Z'^2 + x^2}}{\sqrt{Z^2 + X^2}} \end{gathered}$$

but since we are interested in the intersection of a line starting at N with a plane at some distance h', then the Z' coordinate will always be equal to h', which is

for a circle of radius $r \equiv 1$ simply $Z' = 1$. Substituting it back

$$x = X\frac{\sqrt{1+x^2}}{\sqrt{Z^2+X^2}},$$

squaring

$$x^2 = X^2\frac{1+x^2}{Z^2+X^2},$$

factoring out $\frac{x}{X}$ for RHS and dividing by x

$$1 = \frac{\frac{1}{x^2}+1}{\frac{Z^2}{X^2}+1},$$
$$\frac{1}{x^2}+1 = \frac{Z^2}{X^2}+1,$$

which finally yields

$$x = \frac{X}{Z} \tag{31}$$

Or, alternatively, it can be seen from fig. 3 by using the law of similar triangles as

$$\frac{x}{1} = \frac{X}{Z}.$$

Nevertheless, using eq. (29) we get for the x coordinate of projection as x and x coordinate of a point on the sphere as X

$$X = x\,(1-z)\,. \tag{32}$$

By following the same procedure for the y coordinate, we would get the same thing, that is

$$Y = y\,(1-z)\,. \tag{33}$$

We have the required sets of coordinates for stereographic projection to find the infinitesimal length, which is in Euclidean space

$$ds^2 = dX^2 + dY^2 + dZ^2. \tag{34}$$

To convert it for stereographic projection we use the chain rule to get

$$dX = \frac{\partial X}{\partial x}dx = \frac{\partial}{\partial x}\left(\frac{x}{1-z}\right)dx,$$
$$dY = \frac{\partial Y}{\partial y}dy = \frac{\partial}{\partial y}\left(\frac{y}{1-z}\right)dy.$$

However, the z coordinate depends on both x, y, so changing from $X \to x$ and $Y \to y$ requires to express z as such. How would the z coordinate look like depends on the specific geometry used, so for that we can use the equation of a sphere with radius of 1, which is

$$X^2 + Y^2 + z^2 = 1$$

from which we get

$$z = \pm\sqrt{1 - X^2 - Y^2} \tag{35}$$

First, it could be helpful to do it for 2-dimensional case for $x - z$ plane, so substituting back to eq. (32) and setting $Y = 0$ gives

$$X = x\left(1 \mp \sqrt{1 - X^2}\right).$$

Expressing X in terms of x

$$\begin{aligned}
\pm x\sqrt{1 - X^2} &= x - X \\
x^2\left(1 - X^2\right) &= x^2 + X^2 - 2xX \\
x^2\left(-X^2\right) &= X^2 - 2xX \\
(1 + x^2)X^2 - 2xX &= 0
\end{aligned}$$

Since in stereographic projection the north pole is excluded, that means that the condition $x = 0 \wedge y = 0$ cannot be true, so we can divide the equation by x. From eq. (31) we can get the condition $\forall Z \in \mathbb{R}\backslash\{0\}, x = 0 \iff X = 0$. Therefore, it is possible to divide the last equation by X. We can then rewrite it as

$$\begin{aligned}
(1 + x^2)X - 2x &= 0, \\
X &= \frac{2x}{1 + x^2}.
\end{aligned}$$

Similarly can be done for Y to get

$$Y = \frac{2y}{1 + y^2}$$

Going back to the 3D case, we can assume that by setting $y = 0$ we simply turned $1 + x^2 + y^2$ to $1 + x^2$, so by substituting it back we get

$$X = \frac{2x}{1 + x^2 + y^2}$$

and similarly for Y as

$$Y = \frac{2y}{1+x^2+y^2}.$$

Before going to the line element in Euclidean space in eq. (34), we first need to find the differentials. For dX we get

$$dX = \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy = 2\frac{\left(1+x^2+y^2-2x^2\right)dx - 2xydy}{\left(1+x^2+y^2\right)^2},$$
$$dX^2 = \frac{\left(2\left(1-x^2+y^2\right)dx - 4xydy\right)^2}{\left(1+x^2+y^2\right)^4},$$

for dY we get

$$dY = \frac{\left(2+2x^2+2y^2-4y^2\right)dy - 4xydx}{\left(1+x^2+y^2\right)^2},$$
$$dY^2 = \frac{4\left(1+r^2-2y^2\right)^2 dy^2 - 16xy\left(1+r^2-2y^2\right)dxdy + 16x^2y^2dx^2}{\left(1+r^2\right)^4}$$

and for dZ we get

$$dZ = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)\left(\sqrt{1-X^2-Y^2}\right),$$
$$= \frac{\sqrt{\left(1+x^2+y^2\right)^2 - 4x^2 - 4y^2}}{1+x^2+y^2},$$
$$dZ^2 = \frac{1-x^2-y^2}{1+x^2+y^2}$$

Substituting these into eq. (34) and simplifying would yield the line element for stereographic projection as[7]

$$ds^2 = 4\frac{dx^2+dy^2}{1+x^2+y^2}$$

which defines the metric tensor as

$$g_{\mu\nu} = \frac{4}{1+x^2+y^2}\delta_{\mu\nu}.$$

This has after some attempts proven quite difficult and, due to time constraints, will unfortunately be skipped.

3.5 Poincaré disk

Poincaré disk has the geometry of a "hyperbolic saddle" and, along with a sphere, has constant curvature. The infinitesimal length on Poincaré disk is defined as

$$ds^2 = 4\frac{dx^2 + dy^2}{\left(1 - x^2 - y^2\right)^2}, \quad x^2 + y^2 < 1$$

so the metric tensor is

$$g_{\mu\nu} = \frac{4}{\left(1 - x^2 - y^2\right)^2}\delta_{\mu\nu}$$

and the inverse being

$$g^{\mu\nu} = \frac{1}{4}\left(1 - x^2 - y^2\right)^2 \delta^{\mu\nu}.$$

The Poincaré disk is then given by an isotropic metric tensor, so the Schrödinger's equation is

$$\boxed{E\psi = -\frac{\hbar^2}{8m}\left(1 - x^2 - y^2\right)^2 \nabla^2\psi.} \tag{36}$$

Which gives the same result as stereographic projection, apart from a sign.

We will now try to solve the equation for both stereographic projection and Poincaré disk, so the equation in question is

$$E\psi = -\frac{\hbar^2}{8m}\left(1 \pm \left(x^2 + y^2\right)\right)^2 \nabla^2\psi. \tag{37}$$

where + holds for stereographic projection, − for Poincaré disk. This equation does not seem to have an exact solution, so we will expand the problematic term into Taylor series around $r \to r_0$ for second order.

$$\left(1 \pm r^2\right)^2 \approx \left(1 \pm r_0^2\right)^2 + 2\left(1 \pm r_0^2\right)(\pm 2r_0)(r - r_0) \pm 2\left(1 \pm 3r_0^2\right)(r - r_0)^2$$

By setting $r_0 \to 0$ we get

$$\left(1 \pm r^2\right)^2 \approx 1 \pm 2r^2 = 1 \pm \left(x^2 + y^2\right)$$

Substituting back gives the approximated equation as

$$E\psi = -\frac{\hbar^2}{8m}\left(1 \pm \left(x^2 + y^2\right)\right)\nabla^2\psi \tag{38}$$

Which is for the case of Poincaré disk the same as the equation for unknown surface eq. (13). For stereographic projection only a sign differs, so it changes the angular part of equation such that it is no longer exponential with imaginary argument, but a real one, i.e. the eq. (18) would become

$$F(\varphi) = Ae^{\omega\varphi} + Be^{-\omega\varphi}$$

and the radial part would lead to the equation eq. (22) having a replaced with $-a$ so eq. (26) would become

$$y = C_1 J_0(z), \quad z = \frac{2}{2n} i\sqrt{a} e^{\frac{2nx}{2}}.$$

4 References

All links in this section are guaranteed to be working on 13th of May, 2025.

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