MODEL CATEGORIES VIA RELATIVE HOMOTOPY LIFTING: PRELIMINARY STATEMENT OF MAIN RESULT.

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We work in a complete and cocomplete category $\mathcal{K}$. Notation for weak factorization systems is as in [1]. When applying terminology of model categories to a weak factorization system $(\mathcal{L}, \mathcal{R})$ we identify the latter with the trivial model structure $(\mathcal{L}, \mathcal{K}, \mathcal{R})$ where every map is a weak equivalence.

Given a map $s\colon A \to X$ in a model category $\mathcal{K}$, a cylinder object for X relative to s is the codomain of a cylinder object for the object s in the model category $(A{\downarrow}\mathcal{K})$, i.e. an object P_s together with a factorization of the codiagonal $(X|X)\colon X +_A X \to X$ as $X +_A X \xrightarrow{\gamma_s} P_s \xrightarrow{\sigma_s} X$ such that σ_s is a weak equivalence. Here $X +_A X$ is the pushout of s with itself, which is the codomain of the coproduct of s and s in $(A{\downarrow}\mathcal{K})$. Terminology for cylinders follows [4, Definition 4.2], so we call the cylinder object and the resulting (left) homotopy relation on $(A{\downarrow}\mathcal{K})$ "very good" if the above σ_s is a trivial fibration. We use $\overset{s}{\sim}$ for this homotopy relation.

For a set I of maps we define $I^{\diamondsuit}$ as the class of all those maps f such that every square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle s}\downarrow & \overset{d}{\nearrow}\ \overset{s}{\sim} & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

with $s \in I$ has a diagonal where the upper triangle commutes strictly and the lower triangle commutes up to very good homotopy in $(A{\downarrow}\mathcal{K})$. This is the "relative homotopy lifting" of [3, Definition 3.1]

A cocylinder object (= path object) $X \xrightarrow{\tau_X} \Gamma X \xrightarrow{\pi_X} X \times X$ is called fibrant if the maps $\pi_X^0, \pi_X^1\colon \Gamma X \to X$ are fibrations. It is said to have the homotopy exchange property for $s\colon A \to X$ if it satisfies the following condition:

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Let $\{i,j\} = \{0,1\}$ and suppose that there are maps $g\colon B \to \Gamma X$ and $x\colon B \to X$ with $g\pi_X^j \overset{s}{\sim} x \pmod{P_s}$. Then there exists a $g'\colon B \to \Gamma X$ satisfying $sg' = sg$, $g\pi_X^i = g'\pi_X^i$ and $g'\pi_X^j = x$.

Theorem 1. *Let I be a set of maps in a complete and cocomplete category $\mathcal{K}$, such that all domains and codomains of maps in I are small with respect to* cell(I). *Consider the resulting weak factorization system $({}^{\square}(I^{\square}), I^{\square})$ and suppose that the following conditions are satisfied:*

(1) Every map from I has cofibrant domain (and hence cofibrant codomain).
(2) For all $s \in I$ the very good relative homotopy relation on $(A{\downarrow}\mathcal{K})$ is transitive.
(3) There is a functorial cocylinder (Γ, π, τ) for $({}^{\square}(I^{\square}), I^{\square})$ such that all its cylinder objects are fibrant and have the relative homotopy exchange property for all maps in I.

Then setting $\mathcal{C} = {}^{\square}(I^{\square})$ and $\mathcal{W} = I^{\diamond}$ gives a cofibrantly generated model structure, which moreover has the following properties:

(a) every object is fibrant.
(b) the trivial cofibrations are generated by the set $J = \{\gamma_s^0\colon \mathrm{cod}(s) \to P_s \mid s \in I\}$, obtained from very good cylinder objects $(P_s, \gamma_s, \sigma_s)$ for s.
(c) $\mathcal{W}$ is a smallest localizer for ${}^{\square}(I^{\square})$ in the sense of [2, Définition 3.4] *and hence the smallest possible choice of weak equivalences for a model structure where ${}^{\square}(I^{\square})$ are cofibrations.*

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