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Generalized Geodesics

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Preface

On each smooth manifold M with an affine connection ∇ there is an important system of distinguished curves called geodesics. For instance, in the case of Riemannian structures, the geodesics induced by the Levi–Civita connection are very well studied and they provide plenty of applications. Geodesics are uniquely determined by a tangent vector in one point and each geodesic carries a preferred class of affine parametrizations, see e.g. [**10**], [**19**], [**22**]. The bundle like approach, presented by e.g. [**14**], allows us to define geodesics as projections of the flow lines of horizontal "constant" vector fields on P^1M according to the principal connection induced by the affine connection ∇. This is the content of Section 1 culminating in Proposition 1.4.

The latter approach generalizes immediately to any Cartan geometry which is split in the sense of 2.1, see Definition 2.4. Any Cartan geometry is modeled over a homogeneous space G/H and much of the geometry of the homogeneous model carries over all Cartan geometries of the same type. In particular, general properties of generalized geodesics can be read just on the level of homogeneous spaces G/H, where the generalized geodesics come to be cosets of shifted one–parameter subgroups in G. The crucial construction here is the development of curves in M into curves in the modeling space G/H (via the parallel transport in the Cartan's space) which forms the main point of Section 2. In the case of affine geometries, we obtain the classical development of curves into the tangent space in any fixed point, cf. [**14**], [**16**], [**24**]. Our original generalized approach is provided in 2.7.

The definition of generalized geodesics recovers some well known types of distinguished curves in particular geometries, for example, the conformal circles in conformal Riemannian geometries, see e.g. [**2**], and the chains in hypersurface CR geometries, see [**12**], [**15**], and others. Generalized geodesics behave just like affine geodesics if the geometry in question is reductive. Otherwise, there are more generalized geodesics through one point tangent to the given vector, since they are determined by a jet of higher order in general. Furthermore, there appear curves of various types on the base manifolds, which may behave rather different. The very well known instance of such curves are null–geodesics on conformal manifolds of indefinite signature, [**11**]. If we look at unparametrized images of the generalized geodesics then the definition above brings a class of preferred parametrizations to any geodesic in question. This may differ from that coming from a more classical definition in particular geometries, however, the freedom in possible reparametrizations forms a Lie group acting on the line. In fact, the freedom is either affine or projective in most of interesting geometries, see [**9**]. The main aim of this work is to decide *which initial conditions determine generalized geodesics of a given type uniquely and which are the preferred reparametrizations of those curves.* This is solved in Chapter II for parabolic Cartan geometries.

Restricting ourselves to the parabolic geometries, we may say much more about generalized geodesics, which become usual geodesics with respect to certain distinguished principal connections underlying the given Cartan connection, see Proposition 3.7. These are called generalized Weyl connections and they can be initially found in conformal Riemannian or projective geometries, see [**5**] or [**23**] for details. In particular, the notion of projective equivalence of affine connections, classically defined in e.g. [**10**], can be visibly reformulated in this framework, see 3.8.

Chapter II brings original results following the paper [**8**]. Contribution of the author includes the results of sections 5–7, however, the final form presented in the paper above, and so here, is the issue of fruitful collaborations with coauthors A. Čap and J. Slovák.

First, we consider (without lost of generality) generalized geodesics in the homogeneous space of a parabolic geometry. Then we develop some algebraic techniques in Section 4 in order to conclude that each generalized geodesic is always determined by a jet of finite order, Proposition 4.4. The general approach is applied to irreducible parabolic geometries in Section 5 so that we get a better estimate of the order of jet determining generalized geodesics—always two—and an explicit description of preferred reparametrizations which may be projective here, see Proposition 5.6. A similar process on reparametrizations does not extend easily in geometries of longer gradation, except the case of geodesics of type $\mathcal{C}_{\mathfrak{g}_{-k}}$ in $|k|$–gradings, Theorem 6.2. However, for an arbitrary parabolic geometry, there is an alternative way to prove that the only possible parametrizations of a curve to be a generalized geodesic are either projective or affine. This is presented in [**9**] and the result is very useful especially in applications, where we may identify the reparametrization φ just by the value of φ'' in one point. Next, in the case of irreducible geometries, we can express the subspace of jets in $T^r_1 M$ exhausted by geodesics of some fixed type in a very clear way up to the third order, see 5.3. This point of view ensures us the best understanding of the problem.

Section 6 resolves the problem of refinement of the rough estimate from 4.4 for generalized geodesics of some specific types in parabolic geometries with general length of grading. The most general result is the Theorem 6.4. In the final section we gather complete classifications of generalized geodesics and their properties in several $|2|$–graded geometries, which are represented by the homogeneous models in low dimensions of projective contact structures, Lagrangean contact structures, CR structures, and x—x—dot structures. Other examples in the case of $|1|$–gradings can be found in 5.7; more precisely, conformal Riemannian structures, Grassmannian and projective structures are discussed there. Most of technical computations was performed with the help of the computational system Maple, by Waterloo Maple Inc. (1981–2001), see [**29**].

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CHAPTER I

Introduction

First, we present the classical approach to the study of affine geodesics with the well known solution of the problem when do two affine connections have the same unparametrized geodesics, see Proposition 1.1. Next, we describe an alternative point of view on affine geodesics, Proposition 1.4, in order to generalize this notion for split Cartan geometries in Section 2. At the same time, we generalize the classical concept of developments of curves in order to attach an alternative definition of generalized geodesics in 2.8. In Section 3 we deal with generalized Weyl connections and we restrict ourselves to parabolic Cartan geometries for the rest of this work. In particular, the notion of Weyl connections allows us to understand the classical result of 1.1 in a very geometric way.

1. Affine connections

1.1. Affine geodesics. Let $\nabla : \mathfrak{X}M \times \mathfrak{X}M \to \mathfrak{X}M$ be an affine connection on a smooth manifold M. A curve $c : \mathbb{R} \to M$ is called the *affine geodesic* if it is a solution of the ordinary differential equation $\nabla_{\dot{c}}\dot{c} = 0$, where $\dot{c}(t) = \frac{d}{dt}c(t)$ is the velocity vector field of c. Affine geodesics are uniquely determined by the initial conditions $c(0) = x \in M$ and $\dot{c}(0) = \xi \in T_xM$, i.e. by the 1–jets in one point. After an arbitrary reparametrization φ of the affine geodesic c, a direct calculation yields that the value of the covariant derivative $\nabla_{\dot{c}}\dot{c}$ becomes the multiple of $\dot{c}$ by the function $-\frac{\varphi''}{\varphi'^2}$, see [**19**, eq. (7.4)]. Hence each affine geodesic c carries a preferred class of reparametrizations φ such that $c \circ \varphi$ is an affine geodesic too. Obviously, these are established by the differential equation $\varphi'' = 0$ whose general solutions are the functions $\varphi(t) = at+b$. Reparametrizations of this type are called *affine*.

Another well known fact on affine geodesics is that geodesics of a given affine connection coincide with the geodesics of its symmetrization. So let Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$ be the Christoffel symbols of two symmetric affine connections on M. Some manipulation with the coordinate expression of the equation $\nabla_{\dot{c}}\dot{c}$ leads to the following classical result, see e.g. [**19**, §7], [**22**, p. 72], or [**27**, p. 30].

Proposition. *Two torsion free affine connections have the same unparametrized geodesics if and only if the difference tensor* $P^i_{jk} = \tilde{\Gamma}^i_{jk} - \Gamma^i_{jk}$ *has the form*

$$P^i_{jk} = \delta^i_j \Upsilon_k + \delta^i_k \Upsilon_j, \tag{1}$$

where δ^i_j *is the Kronecker delta and* Υ_k *is an arbitrary covector.*

This equation is called the Levi–Civita equation and two affine connections which are related in this way are called *projectively equivalent*. In this framework, the projective geometry on M is understood as a class of projectively equivalent affine connections, [**10**, ch. III]. Otherwise put, affine connections ∇ and $\tilde{\nabla}$ are projectively equivalent if and only if there is a one–form $\Upsilon \in \Omega^1(M)$ such that

$$\tilde{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\,\eta + \Upsilon(\eta)\,\xi \tag{2}$$

for all $\xi, \eta \in \mathfrak{X}M$, compare with the result of 3.8.

1.2. Absolute parallelism. Any affine connection on M, i.e. a linear connection on TM, is induced by a principal connection on the bundle of linear frames P^1M and this correspondence is bijective. Let $\gamma \in \Omega^1(P^1M, \mathfrak{gl}(m,\mathbb{R}))$ be the corresponding connection form and let us define the one–form $\omega \in \Omega^1(P^1M, \mathfrak{a}(m,\mathbb{R}))$ as $\omega = \theta \oplus \gamma$, where $\theta \in \Omega^1(P^1M, \mathbb{R}^m)$ is the canonical (or soldering) form on P^1M. The one–form ω has several nice properties, cf. [**6**] or [**14**, ch. III],

- $\star$ $\omega(u) : T_uP^1M \to \mathfrak{a}(m,\mathbb{R})$ is a linear isomorphisms for all $u \in P^1M$,
- $\star$ $\omega(Tr^h \cdot \xi) = \mathrm{Ad}_{h^{-1}}(\omega(\xi))$ for each $\xi \in TP^1M$ and all $h \in GL(m,\mathbb{R})$,
- $\star$ and $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{gl}(m,\mathbb{R})$ and $u \in P^1M$.

In the sequel, the one–form ω determined by the affine connection ∇ will be called *affine connection* as well. Next, we are going to recover the definition of affine geodesics from this point of view, see Proposition 1.4.

First of all, we need the notion of *constant vector fields*, which are those vector fields $\xi \in \mathfrak{X}(P^1M)$ that $\omega(\xi(u))$ is constant for all $u \in P^1M$. Clearly, any $X \in \mathfrak{a}(m,\mathbb{R})$ defines a unique constant vector field which we denote by $\omega^{-1}(X)$. For $X \in \mathbb{R}^m \subset \mathfrak{a}(m,\mathbb{R})$ we obtain a horizontal vector field and this is used to write down the obvious identification $TM = P^1M \times_{GL(m,\mathbb{R})} \mathbb{R}^m$ by the formula $\{u, X\} \mapsto Tp \cdot \omega^{-1}(X)(u)$. The defining representation of $GL(m,\mathbb{R})$ on $\mathbb{R}^m$ is the identical one, which in fact corresponds to the Ad–representation of $A(m,\mathbb{R})$ restricted to $GL(m,\mathbb{R})$, see Example 1 in 2.2. The essential ingredient here is the invariance of the subalgebra $\mathbb{R}^m$ in $\mathfrak{a}(m,\mathbb{R})$. Now one can easily conclude that the latter map is well defined due to the equivariance of ω.

1.3. Absolute derivative. For a general connection on a fibered bundle $Y \to M$, there is a notion of absolute derivative as follows. Let $\xi \in \mathfrak{X}M$ be a vector field, $\gamma(\xi) \in \mathfrak{X}Y$ its horizontal lift, and let $s : M \to Y$ be an arbitrary section. The *absolute derivative* is an operation $\nabla : C^\infty(Y) \to C^\infty(T^*M \otimes VY)$ defined by the formula [**17**, 17.9(4)]

$$\nabla_\xi s = Ts \cdot \xi - \gamma(\xi) \circ s, \tag{3}$$

where we write $\nabla_\xi s$ instead of $\nabla s(\xi)$. In other words, $\nabla_\xi s$ is the Lie derivative of s in the direction of the projectable (horizontal) vector field $\gamma(\xi) \in \mathfrak{X}Y$. If $Y \to M$ is an associated bundle $Y = P \times_H S$ then we identify sections $s : M \to Y$ with H–equivariant mappings $\bar{s} : P \to S$, the corresponding frame forms. Similarly, sections of the vertical bundle $VY = P \times_H TS$ correspond to equivariant mappings $P \to TS$, and if we consider the connection on Y is induced by a principal connection γ on P, one can prove the following Lemma [**17**, 17.10].

Lemma. *Let $\bar{s} : P \to S$ be the frame form of a smooth section $s : M \to Y$. Then the frame form of the absolute derivative $\nabla_\xi s : M \to VY$ is*

$$\nabla_\xi \bar{s} = T\bar{s} \cdot \gamma(\xi). \tag{4}$$

In the case of associated vector bundles $E = P \times_H W$ we have $TW = W \oplus W$ and the frame form of a section $M \to E$ can be interpreted as an H–equivariant function $P \to W$, so the absolute derivative along a vector field ξ turns sections of E into the sections of E itself. The formula (4) then reads as the ordinary derivative of the vector valued function $\bar{s}$ in the direction of the horizontal lift of the vector field $\xi \in \mathfrak{X}M$. In particular, the concept of absolute derivative for tensor bundles coincides with the covariant derivative of tensor fields in the classical sense, cf. [**14**, p. 116].

1.4. Proposition. *Let M be a smooth manifold with an affine connection ∇ and let ω be the corresponding one–form on P^1M according to 1.2. Then the unique affine geodesic on M going through the point $x = p(u)$ with the tangent vector $\{u, X\} \in T_xM$ is the curve $c^{u,X}(t) = p\big(\operatorname{Fl}_t^{\omega^{-1}(X)}(u)\big)$.*

This Proposition provides an alternative view on affine geodesics which will serve as definition in more general cases of Cartan geometries below, cf. [**14**, p. 139].

Proof. Using the above identification of the tangent bundle TM, the velocity vector field $\dot{c}(t) = \frac{d}{dt}c(t)$ of the curve $c = c^{u,X}$ can be written as $\dot{c}(t) = \{\operatorname{Fl}_t^{\omega^{-1}(X)}(u), X\}$. Hence the corresponding frame form is constant along the horizontal curve in P^1M and the previous Lemma yields $\nabla_{\dot{c}}\dot{c} = 0$.

In order to prove the uniqueness, we have to show that another representative $\{uh, \operatorname{Ad}_{h^{-1}} X\}$ of the tangent vector $\{u, X\} \in T_{c(0)}M$ defines the same curve, i.e. $p(\operatorname{Fl}_t^{\omega^{-1}(X)}(u)) = p(\operatorname{Fl}_t^{\omega^{-1}(\operatorname{Ad}_{h^{-1}} X)}(uh))$ holds for all $h \in H$. Now, the equivariance of the affine connection ω can be rewritten as

$$Tr^h \cdot \omega^{-1}(X)(u) = \omega^{-1}(\operatorname{Ad}_{h^{-1}} X)(uh)$$

and the flow lines of vector fields $\omega^{-1}(X)$ and $Tr^h \cdot \omega^{-1}(X)$ starting at u and uh, respectively, have certainly the same projection. Hence the result follows. □

2. Cartan connections

In the first instance, we put some basic definitions and notions on Cartan and Klein geometries following [**6**], [**21**], and [**23**]. The main aim of this section is to generalize the notion of affine geodesics for general split Cartan geometries and the classical concept of the development of curves which is the crucial construction for our later purposes; see 2.4, 2.7, and consecutive paragraphs. The geometry in this view is modeled on a homogeneous space G/H and the construction of the Cartan's space consists just of the putting the model space to be "tangent" in each point of the base manifold M. Then any invariant system of distinguished curves in G/H gives rise to a system of distinguished curves in M or, more generally, on all manifolds endowed with a Cartan geometry of the same type. Here we follow the general approach of [**6**], [**14**], [**16**], and others.

2.1. Definitions. Let us consider a Lie group G with a closed subgroup H and let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras. A *Cartan geometry* of type (G, H) on a manifold M is a principal H–bundle $\mathcal{G} \to M$ with the *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which is a $\mathfrak{g}$–valued one–form satisfying the following properties:

- ⋆ ω is an absolute parallelism, i.e. the map $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for each $u \in \mathcal{G}$,
- ⋆ ω is H–equivariant, i.e. the condition $(r^h)^*\omega = \operatorname{Ad}(h^{-1}) \circ \omega$ holds for all $h \in H$,

- ⋆ ω reproduces the fundamental vector fields, i.e. $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{h}$.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Cartan geometries of the same type with Cartan connections ω_1 and ω_2, respectively. A homomorphism $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ of principle fiber bundles is a morphism of Cartan geometries if $\psi^*\omega_2 = \omega_1$.

The *curvature* of a Cartan geometry with the Cartan connection ω is the 2–form $K = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}, \mathfrak{g})$. The curvature form is strictly horizontal so it can be understood as a pullback of a $\mathfrak{g}$–valued 2–form on the base manifold with respect to the projection $p : \mathcal{G} \to M$. If $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ is a morphism of Cartan geometries then the corresponding curvature forms K_1 and K_2 are ψ–related.

Cartan geometry is called *split* if there is a subalgebra $\mathfrak{n} \subset \mathfrak{g}$ complementary to $\mathfrak{h}$, i.e. $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ on the level of vector spaces. Cartan geometry is called *reductive* if there is a complementary H–invariant subspace $\mathfrak{n} \subset \mathfrak{g}$, according to the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ restricted to H.

For any $X \in \mathfrak{g}$, the *constant vector field* $\omega^{-1}(X)$ is the unique vector field on $\mathcal{G}$ which satisfies the condition $\omega(\omega^{-1}(X)(u)) = X$ for all $u \in \mathcal{G}$. Next, we can identify the tangent bundle TM with the associated bundle $\mathcal{G} \times_H (\mathfrak{g}/\mathfrak{h})$ where the action of the structure group H on $\mathfrak{g}/\mathfrak{h}$ is induced by the Ad–representation on $\mathfrak{g}$. The identification is provided by the mapping $\{u, X + \mathfrak{h}\} \mapsto Tp{\cdot}\omega^{-1}(X + \mathfrak{h})(u)$, where p denotes the projection $\mathcal{G} \to M$.

In the case of a split Cartan geometry, with the splitting $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, we identify $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{n}$ and via this identification the subalgebra $\mathfrak{n}$ becomes an H–module. The action of the group H on $\mathfrak{n}$ is called the *truncated adjoint representation* and it will be denoted by $\underline{\mathrm{Ad}}$. Obviously, this action is determined by the condition $\underline{\mathrm{Ad}}(h) = \pi \circ \mathrm{Ad}(h)$ for all $h \in H$, where π is the projection $\mathfrak{g} \to \mathfrak{n}$ in the direction of $\mathfrak{h}$. Now we can write $TM = \mathcal{G} \times_H \mathfrak{n}$ with respect to this action and the identification above turns into the mapping $\{u, X\} \mapsto Tp{\cdot}\omega^{-1}(X)(u)$ with the inverse $\xi \mapsto \{u, \pi(\omega(\hat{\xi}))\}$, where $\hat{\xi} \in T_u\mathcal{G}$ is an arbitrary lift of the vector $\xi \in T_{p(u)}M$. Further, any fixed subalgebra $\mathfrak{n}$ gives rise to a horizontal distribution $\omega^{-1}(\mathfrak{n})$ on $\mathcal{G}$, hence each splitting enters a general connection on $\mathcal{G}$. The equivariance of ω yields that this connection is principal if and only if $\underline{\mathrm{Ad}} = \mathrm{Ad}$, i.e. the geometry in question is reductive.

Given a Cartan connection on the principle bundle $\mathcal{G} \to M$, there is a natural differential operator acting on sections of natural bundles $FM = \mathcal{G} \times_H S$ as follows. The *fundamental D–operator* is the mapping $D : C^\infty(FM) \to C^\infty(AM^* \otimes VFM)$ given by the transcription on the frame forms of sections of FM, i.e. $D : C^\infty(\mathcal{G}, S)^H \to C^\infty(\mathcal{G}, \mathfrak{g}^* \otimes TS)^H$, so that

$$Ds(u)(X) = Ts{\cdot}\omega^{-1}(X)(u). \tag{1}$$

The bundle AM is the vector bundle $\mathcal{G} \times_H \mathfrak{g}$ associated via the Ad–representation restricted to H. The map D is well defined, i.e. the image Ds of a H–equivariant mapping $s : \mathcal{G} \to S$ is really H–equivariant, see e.g. [**4**, 3.1].

In the case of split Cartan geometries, the fundamental D–operator restricts to an operation $\nabla^\omega : C^\infty(\mathcal{G}, S) \to C^\infty(\mathcal{G}, \mathfrak{n}^* \otimes TS)$ which we call the *invariant derivative*. The essential difference to the D–operator is that ∇^ω does not transform sections into sections, except $\mathfrak{n}$ is H–invariant, i.e. the geometry in question is reductive. In that case, the Cartan connection ω is principal, so ∇^ω is the usual covariant derivative induced on FM. Anyway, for a vector field ξ on M, the symbol $\nabla^\omega_\xi s$ is well understood as a map $\mathcal{G} \to TS$ such that

$$\nabla^\omega_\xi s(u) = Ts \cdot \omega^{-1}(\bar{\xi}(u))(u), \tag{2}$$

where $\bar{\xi} : \mathcal{G} \to \mathfrak{n}$ is the frame form of ξ.

Among the natural bundles F, there is one of a particular interest, the bundle coming from the left multiplication of the group G on itself. So we get $\tilde{\mathcal{G}} = \mathcal{G} \times_H G$, the principal G–bundle over M which is called the *extension* of the principal bundle $\mathcal{G}$ to the structure group G. The Cartan connection ω on $\mathcal{G}$ determines a unique principal connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ on the extension $\tilde{\mathcal{G}}$ which coincides with ω on the image of the obvious inclusion $\mathcal{G} \subset \tilde{\mathcal{G}}$ given as $u \mapsto \{u, e\}$, cf. [**6**] or [**13**, p. 128]. In particular, $\tilde{\omega}(u)|_{T_u\mathcal{G}} = \omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ for each $u \in \mathcal{G}$ and $\tilde{\omega}(\zeta_X(u)) = X$ for all $X \in \mathfrak{g}$, $u \in \tilde{\mathcal{G}}$. Hence the horizontal lift of the vector $\xi = \{u, X + \mathfrak{h}\} \in T_{p(u)}M$ is

$$\tilde{\xi}(u) = \omega^{-1}(X)(u) - \zeta_X(u) \in T_u\tilde{\mathcal{G}}, \tag{3}$$

which is really well defined, projects onto ξ, and $\tilde{\omega}(\tilde{\xi}) = X - X = 0$. Altogether, any associated bundle $\mathcal{G} \times_H S$ arising from a restricted G–action is also associated to $\tilde{\mathcal{G}}$, so we get canonical induced connections on all natural bundles coming from G–actions restricted to H. The most known examples of such bundles are the tractor bundles, presented in [**3**] or [**4**], and the Cartan's space $\mathcal{G} \times_H (G/H)$ defined in 2.7 below.

Examples. Easy examples of Cartan connections are affine connections in the sense of 1.2. Similarly, any first order H–structure $\iota : \mathcal{G} \to P^1M$ endowed with a principal connection γ induces the Cartan connection $\omega = \theta \oplus \gamma$, where θ is the pullback of the soldering form on P^1M with respect to the reduction ι. The connection form ω takes values in the Lie algebra $\mathbb{R}^m \oplus \mathfrak{h}$ seen as a subalgebra of $\mathfrak{a}(m, \mathbb{R}) = \mathbb{R}^m \oplus \mathfrak{gl}(m, \mathbb{R})$. Morphisms are those homomorphisms of the H–structure which keep the connection form γ invariant. Cartan geometries of these types are naturally split and reductive. Hence the operation of invariant derivative on associated bundles turns sections into sections and it is the usual covariant derivative.

Principal bundles $G \to G/H$ with the Maurer–Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ represent the most important examples of Cartan geometries. The Maurer–Cartan form fulfils the structure equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$, so the curvature of the geometry vanishes. Given a Cartan geometry of type (G, H), the bundle $G \to G/H$ with the Maurer–Cartan form is called the *flat* or *homogeneous model* of the Cartan geometry. One can easily show that Cartan geometry is locally isomorphic to its homogeneous model if and only if the curvature of the geometry vanishes, [**21**, 5.5.1].

2.2. Klein geometries. The notion of Klein geometries could precede the definition of Cartan geometries as a good motivation, see [**21**, ch. 4]. Number of general properties of Cartan geometries is visible just on the level of homogeneous spaces $G \to G/H$, which form the principal bundles over Klein geometries as follows below. This point of view is of a particular interest in the theory of Cartan geometries and all our techniques will be based on the homogeneous models only.

A *Klein geometry* is a pair (G, H), where G is a Lie group, $H \subset G$ is a closed subgroup, and the coset space G/H is connected. Since H is closed, there exists a unique smooth structure on G/H such that the natural projection $G \to G/H$ is a submersion. This is a principal H–bundle and the base manifold G/H is called the *homogeneous space* of G or, in our context, the *space of the Klein geometry*. The left multiplication on G induces a left transitive action of G on G/H and the only automorphisms of the Cartan geometry $G \to G/H$ are just of this form. This claim follows immediately from [**21**, 3.5.2], see also 2.3.

On the other hand, the original Felix Klein's approach to the geometry consists of a smooth manifold M endowed with a smooth transitive (and effective) action of a Lie group G. Studying a geometry in this framework means studying properties which are invariant with respect to the motions of group G.

Let us fix a point $x \in M$ and denote by $H_x = \{g \in G : gx = x\}$ the *stabilizer* of the point x. Clearly, H_x is the preimage of x according to the map $G \to M$ defined by $g \mapsto gx$, thus it is a closed subgroup in G. Further, the map $G/H_x \to M$ sending $gH \mapsto gx$ is a bijection compatible with the G–actions. So this picture corresponds to the Klein geometry (G, H_x) in the sense of the definition above. Another choice of the fixed point $y \in M$ leads to another subgroup H_y, which is the image of H_x with respect to the conjugation by an element $g \in G$ satisfying the condition $gx = y$.

Example 1. Affine geometry. In this case, $M = \mathbb{R}^m$ with the group of affine motions $A(m, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ Y & A \end{pmatrix} : Y \in \mathbb{R}^m, A \in GL(m, \mathbb{R}) \right\}$. Under the identification $X \mapsto \begin{pmatrix} 1 \\ X \end{pmatrix}$, the action is simply written as $\begin{pmatrix} 1 & 0 \\ Y & A \end{pmatrix} \cdot \begin{pmatrix} 1 \\ X \end{pmatrix} = AX + Y$. The affine action is transitive and effective and the stabilizer of the origin is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \in GL(m, \mathbb{R}) \right\} = GL(m, \mathbb{R})$. Hence, the affine geometry in the above point of view is the pair $(A(m, \mathbb{R}), GL(m, \mathbb{R}))$ and the homogeneous space is naturally identified with the normal abelian subgroup $\mathbb{R}^m = \left\{ \begin{pmatrix} 1 & 0 \\ X & E \end{pmatrix} : X \in \mathbb{R}^m \right\}$ of all translations. On the infinitesimal level, we have the canonical decomposition $\mathfrak{a}(m, \mathbb{R}) = \mathfrak{n} \oplus \mathfrak{gl}(m, \mathbb{R})$ into subalgebras, where $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} : X \in \mathbb{R}^m \right\}$ is an ideal in $\mathfrak{a}(m, \mathbb{R})$, hence the geometry is reductive. Moreover, the restricted Ad–representation of the group $GL(m, \mathbb{R})$ on $\mathfrak{n}$ is $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ AX & 0 \end{pmatrix}$ which is just the standard representation $GL(m, \mathbb{R}) \to GL(\mathbb{R}^m)$. The Maurer–Cartan form on $\mathfrak{a}(m, \mathbb{R})$ defines an invariant torsion–free affine connection on $\mathbb{R}^m$, the usual parallel transport.

Example 2. Projective geometry. Here we have $M = \mathbb{R}P^m$ and let us consider the group $SL(m+1, \mathbb{R})$ to be the principal group of the geometry. The standard action on $\mathbb{R}^{m+1}$ induces the action on $\mathbb{R}P^m$ which is transitive but not effective. The stabilizer of the ray represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^{1+m}$ is the group $H = \left\{ \begin{pmatrix} |A|^{-1} & Z \\ 0 & A \end{pmatrix} : A \in GL(m, \mathbb{R}), Z \in \mathbb{R}^{m*} \right\}$ which is the semidirect product $H = GL(m, \mathbb{R}) \rtimes \exp \mathbb{R}^{m*}$. The Lie algebra $\mathfrak{g}$ is naturally split into the sum $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, where $\mathfrak{n} = \left\{\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} : X \in \mathbb{R}^m\right\}$ and $\mathfrak{h} = \left\{\begin{pmatrix} -\mathrm{tr}(A) & Z \\ 0 & A \end{pmatrix} : A \in \mathfrak{gl}(m,\mathbb{R}), Z \in \mathbb{R}^{m*}\right\}$. The commutative subalgebra $\mathfrak{n}$ is not H–invariant and there is no H–invariant complementary subalgebra to $\mathfrak{h}$. Hence projective geometries are naturally split but never reductive.

2.3. Left logarithmic derivative. For later use, let us explain the notion of left logarithmic derivative in more details. Let G be a Lie group with the Maurer–Cartan form $\omega \in \Omega^1(G,\mathfrak{g})$. By definition, $\omega(\xi) = T\ell_{g^{-1}}\cdot\xi$ for $\xi \in T_gG$, which provides the identification $TG = G \times \mathfrak{g}$. For any smooth map $f : M \to G$, the *left logarithmic* or *Darboux derivative* of f is the $\mathfrak{g}$–valued one–form $\delta f : TM \to \mathfrak{g}$ defined as $\delta f = f^*\omega$. In particular, for $M = \mathbb{R}$ the map $\delta f(s,-) : T_s\mathbb{R} = \mathbb{R} \to \mathfrak{g}$, linear for each s, can be identified with the image of the unit vector $1 \in T_s\mathbb{R}$, and this convention will be kept hereafter. More precisely, for any curve $f : \mathbb{R} \to G$, the left logarithmic derivative δf is understood as a curve $\mathbb{R} \to \mathfrak{g}$ such that $\delta f(s) = \omega(T_sf\cdot 1) = T\ell_{f(s)^{-1}}\cdot f'(s)$.

If two smooth maps $f_1, f_2 : M \to G$ on connected M satisfy $\delta f_1 = \delta f_2$ then there is a unique element $c \in G$ such that $f_2 = \ell_c \circ f_1$. This is the essential property of the left logarithmic derivative which represents a nonabelian generalization of the uniqueness of the primitive function in elementary calculus, see [**21**, 3.5.2] for the proof and other comments.

For any smooth maps $f_1, f_2 : M \to G$, the Leibniz rule of the left logarithmic derivative has the form [**17**, 4.26]

$$\delta(f_1\cdot f_2)(x) = \delta f_2(x) + \mathrm{Ad}_{f_2(x)^{-1}}(\delta f_1(x)). \tag{4}$$

Further, we are interested in the left logarithmic derivative of the map $\exp : \mathfrak{g} \to G$. The formula for the right logarithmic derivative $\delta^R \exp : T\mathfrak{g} \to \mathfrak{g}$ deduced in [**17**, 4.27] can be easily adapted for the left one so that, for all $X \in T_Y\mathfrak{g}$, the condition

$$(\delta \exp)(X) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(\mathrm{ad}(-Y))^k(X) \tag{5}$$

is satisfied. If Y is a curve $Y : \mathbb{R} \to \mathfrak{g}$ then $\exp\circ Y : \mathbb{R} \to G$ and we understand $\delta(\exp\circ Y)$ to be a map $\mathbb{R} \to \mathfrak{g}$ as above. According to (5), the equality

$$\delta(\exp\circ Y)(t) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(\mathrm{ad}(-Y(t)))^k(Y'(t)) \tag{6}$$

is obviously satisfied for all t.

2.4. Distinguished curves. Here we present the promised generalization of affine geodesics following Proposition 1.4. The generalized geodesics are defined as projections of flow lines of horizontal constant vector fields:

Definition. Let $p : \mathcal{G} \to M$ be a Cartan geometry of type (G,H) split as $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ and let $A \subseteq \mathfrak{n}$ be an arbitrary subset. A smooth curve on the base manifold is called the *generalized geodesic of type* $\mathcal{C}_A$ if it has the shape $c^{u,X}(t) = p(\mathrm{Fl}_t^{\omega^{-1}(X)}(u))$ for some $u \in \mathcal{G}$ and $X \in A$.

Any choice of $A \subseteq \mathfrak{n}$ defines geodesics of type $\mathcal{C}_A$ on all manifolds equipped with a Cartan geometry of the same type. The type of geodesics will not be mentioned if $A = \mathfrak{n}$. Further, we always suppose the subset A is neither empty nor trivial. The generalized geodesic $c^{u,X}$ clearly goes through the point $p(u) = c(0)$ with the tangent vector $\{u, X\} \in T_{p(u)}M$ but, in contrast to the affine case, this initial condition does not determine the generalized geodesic uniquely. Otherwise put, another representative of the tangent vector may define another generalized geodesic. In general, generalized geodesics are uniquely given by tangent vectors if and only if the geometry in question is reductive, which is easily visible by the second part of the proof of Proposition 1.4.

All generalized geodesics with the common tangent vector $\{u, X\} \in T_{p(u)}M$ have the shape $c^{uh, \underline{\mathrm{Ad}}_h^{-1} X}$ for $h \in H$, so the set of all such curves is identified with a subset in H. In particular, all elements in H which keep $\mathfrak{n}$, or $A \subset \mathfrak{n}$ for geodesics of type $\mathcal{C}_A$, invariant define the same curve, since $\underline{\mathrm{Ad}}(h) = \mathrm{Ad}(h)$ is satisfied in that case. The answer further depends on the type of the tangent vector if there are distinguished ones. With respect to the identification $TM = \mathcal{G} \times_H \mathfrak{n}$ above, distinguished tangent vectors correspond to H–invariant subsets in $\mathfrak{n}$ and generalized geodesics of type $\mathcal{C}_A$ emanate in directions corresponding to the H–orbit of the subset $A \subset \mathfrak{n}$. In other words, for any vector $\xi \in T_xM$ there is at least one geodesic of type $\mathcal{C}_A$ tangent to ξ if and only if ξ lies in the image of the H–orbit of A in T_xM.

Anyway, the equality $\nabla^{\omega}_{\dot{c}} \dot{c} = 0$ is still satisfied in some specific sense due to the same arguments as in the proof of Proposition 1.4. More precisely, $\nabla^{\omega}_{\dot{c}} \dot{c}$ is a mapping $\mathcal{G}\big|_c \to \mathfrak{n}$ which is not equivariant in general, see 2.1, hence the vanishing of $\nabla^{\omega}_{\dot{c}} \dot{c}$ along some section does not imply the vanishing at all. This leads to the following formulation.

Proposition. *In the setting above, a curve c in M is a generalized geodesic if and only if there is a curve $\hat{c}$ in $\mathcal{G}$ covering c such that $\nabla^{\omega}_{\dot{c}} \dot{c}$ vanishes along $\hat{c}$.*

Proof. Let c be a generalized geodesic, i.e. there exist $u \in \mathcal{G}$ and $X \in \mathfrak{n}$ such that $c(t) = p(\mathrm{Fl}_t^{\omega^{-1}(X)}(u))$ for all t. Then the identification $TM = \mathcal{G} \times_H \mathfrak{n}$ yields the velocity vector field of c has got the form $\dot{c}(t) = \{\mathrm{Fl}_t^{\omega^{-1}(X)}(u), X\}$. Hence the frame form of this partially defined vector field is constant along the horizontal lift $\hat{c}(t) = \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$, so $\nabla^{\omega}_{\dot{c}} \dot{c}\,\big|_{\hat{c}} = 0$ by the definition of invariant derivative.

Conversely, let c be a curve on M and let $\hat{c}$ be any cover of c in $\mathcal{G}$ such that $\nabla^{\omega}_{\dot{c}} \dot{c}\,\big|_{\hat{c}} = 0$. Then the velocity vector field of the curve $c \subset M$ can be expressed as $\dot{c}(t) = \{\hat{c}(t), X(t)\}$, where X is the curve in $\mathfrak{n}$ given by the condition $X(t) = (\pi \circ \omega)\big(\frac{d}{dt}\hat{c}(t)\big)$. Now the assumption $\nabla^{\omega}_{\dot{c}} \dot{c}\,\big|_{\hat{c}} = 0$ implies that the value of X is constant, hence $\hat{c}$ is the flow line of the constant vector field $\omega^{-1}(X)$ which is horizontal by definition. This completes the proof. □

For later use, let us formulate a Proposition.

2.5. Proposition. *Let $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ be a morphism of Cartan geometries of type (G, H) which cover $\underline{\psi} : M_1 \to M_2$. A smooth curve c on M_1 is a geodesic of type $\mathcal{C}_A$ if and only if the curve $\underline{\psi} \circ c$ is a geodesic of the same type on M_2.*

Proof. Let ω_i be the Cartan connection on $\mathcal{G}_i$ and let p_i be the bundle projection $\mathcal{G}_i \to M_i$, for $i = 1, 2$, respectively. The assumptions yield $\psi^*\omega_2 = \omega_1$ and $\underline{\psi} \circ p_1 = p_2 \circ \psi$, in particular, the former equality implies $\psi \circ \mathrm{Fl}_t^{\omega_1^{-1}(X)} = \mathrm{Fl}_t^{\omega_2^{-1}(X)} \circ \psi$ for any $X \in \mathfrak{g}$ and all t where the flow is defined. Now it is obvious that $\underline{\psi} \circ p_1 \circ \mathrm{Fl}_t^{\omega_1^{-1}(X)} = p_2 \circ \mathrm{Fl}_t^{\omega_2^{-1}(X)} \circ \psi$. Hence the image $\underline{\psi} \circ c$ of a curve c on M_1 is a generalized geodesic of type $\mathcal{C}_A$ on M_2, i.e. $\underline{\psi} \circ c = c^{u_2,X}$ for some $u_2 \in \mathcal{G}_2$ and $X \in A$, if and only if $c = c^{u_1,X}$ where u_1 is the unique element in $\mathcal{G}_1$ such that $\psi(u_1) = u_2$. □

Remarks. In the homogeneous model $G \to G/H$, constant vector fields are just the left invariant ones and their flow lines are shifted one–parameter subgroups [**17**, 4.18]. Hence the generalized geodesics of type $\mathcal{C}_A$ look like $c^{g,X}(t) = g \exp tX \cdot H$ for any $g \in G$ and $X \in A$. All automorphisms of the Cartan geometry $G \to G/H$ are given by the left multiplication on G, so the latter Proposition is satisfied trivially. In the case of reductive geometries, all geodesics $c^{gh,\underline{\mathrm{Ad}}_h^{-1}X}$, $h \in H$, coincide with $c^{g,X}$ as follows. Because $\underline{\mathrm{Ad}}(h) = \mathrm{Ad}(h)$ holds for all $h \in H$, we write $c^{gh,\underline{\mathrm{Ad}}_h^{-1}X}(t) = gh \exp(t\,\mathrm{Ad}_{h^{-1}} X) \cdot H = ghh^{-1} \exp(tX) h \cdot H = c^{g,X}(t)$. This really agrees with the general discussion above.

2.6. Distinguished jets. Let us denote by $T_1^r M$ the bundle of r–velocities on M, i.e. the bundle of r–jets of curves $T_1^r M = J_0^r(\mathbb{R}, M)$. For a split Cartan geometry $\mathcal{G} \to M$ and a subset $A \subseteq \mathfrak{n}$, all r–jets of generalized geodesics of type $\mathcal{C}_A$ on M form a subset $T^r_{\mathcal{C}_A} M$ in $T_1^r M$. In particular, $T^1_{\mathcal{C}_A} M = \mathcal{G} \times_H H(A)$ where $H(A)$ is the H–orbit of A in $\mathfrak{n}$. Now, $T^r_{\mathcal{C}_A}$ is a functor on the category of Cartan geometries of type (G, H) split as $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, which takes values in the category of fibered spaces. Proposition 2.5 implies the functor is well defined, however, $T^r_{\mathcal{C}_A} M$ has not to be a smooth subbundle of $T_1^r M$ in general (see the description of the standard fiber in 5.3).

Let r be an order of jet which determines geodesics of type $\mathcal{C}_A$ uniquely. Now the space $T^r_{\mathcal{C}_A} M$ is seen as the space of initial conditions where the entire geodesic of an appropriate type is completely given by a single value in $T^r_{\mathcal{C}_A} M$. In this setting, all geodesics of type $\mathcal{C}_A$ with a common tangent vector ξ are in a bijective correspondence with the preimage of ξ with respect to the jet projection $\pi_1^r : T_1^r M \to T_1^1 M = TM$ restricted to $T^r_{\mathcal{C}_A} M$.

2.7. Cartan's space and developments. Let $p : \mathcal{G} \to M$ be a Cartan geometry of type (G, H). The *Cartan's space* of M is the associated bundle $SM = \mathcal{G} \times_H (G/H)$ with the action of H on G/H given by the left multiplication—this is one of the most prominent natural bundles discussed in 2.1. So we write $SM = \tilde{\mathcal{G}} \times_G (G/H)$ and this bundle is endowed with an induced general connection coming from the Cartan connection on $\mathcal{G}$. First of all, we have the canonical global section of the bundle projection $SM \to M$ given by the transcription $x \mapsto \{u, eH\}$ for any $u \in \mathcal{G}$ staying over $x \in M$. The definition does not depend on a chosen u and the section is denoted by O. Furthermore, due to the existence of the global section O, any fiber of the vertical bundle $VSM = \mathcal{G} \times_H T(G/H)$ restricted to $O(M)$ can be thought of an analogy of the tangent space, since the bundles $VSM|_{O(M)}$ and TM are canonically isomorphic (via the isomorphism $T_o(G/H) \cong \mathfrak{g}/\mathfrak{h}$). In this view, the base manifold M seems to be "osculated" in each point by the homogeneous space G/H on the level of tangent spaces. This approach goes really back to Cartan, see [**16**] for details and other comments. The construction above is briefly described also in [**13**, p. 128].

Let c be a parametrized curve on M with a fixed point $x = c(t_0)$. The *development of* c at the point x is a curve $\bar{c}$ in the fiber $S_xM \cong G/H$ defined as follows. For a time t, the value $\bar{c}(t)$ is obtained by moving the point $O(c(t_0+t))$ back to the fiber S_xM using the parallel transport. More precisely, if $\tilde{c}(s)$ is the parallel curve in SM determined by the initial condition $\tilde{c}(0) = O(c(t_0+t))$ and covering the path $s \mapsto c(t_0+t+s)$ in M, then $\bar{c}(t) = \tilde{c}(-t)$. For a fixed $u \in p^{-1}(x)$ the development of a curve c at x can be written as $\bar{c}(t) = \{u, \breve{c}(t)\}$ where $\breve{c}$ is a curve in G/H going through the origin eH. Another frame $uh \in p^{-1}(x)$ changes the curve $\breve{c}$ to $\ell_{h^{-1}} \circ \breve{c}$.

Proposition. *Let $\mathcal{G} \to M$ be the principal bundle of a Cartan geometry of type (G,H) split as $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, where $\mathfrak{n}$ is supposed to be nilpotent. Let c be a smooth curve in M, $\hat{c}$ be the parallel horizontal curve over c in $\mathcal{G}$ starting at $u \in \mathcal{G}$, and let X be the curve in $\mathfrak{n}$ determined by the condition $X(t) = \omega\big(\frac{d}{dt}\hat{c}(t)\big)$. Then the development of the curve c at $x = p(u)$ is the curve $\{u, \exp Y(t)\cdot H\} \subset S_xM$, where Y is the unique curve in $\mathfrak{n}$ satisfying*

$$\mathrm{Ad}_{\exp(-Y(t))}(X(t)) + \delta(\exp\circ Y)(t) = 0 \tag{7}$$

for all t.

Proof. We have to prove that the parallel curve over c in $SM = \tilde{\mathcal{G}} \times_G (G/H)$ starting at $\{u, \exp Y(t)\cdot H\}$ attains the point $O(c(t_0+t)) = \{\hat{c}(t), eH\}$ just in the time t, where clearly $c(t_0) = x = p(u)$. The essence of the proof is to describe the parallel curve $\tilde{c}$ over c in $\tilde{\mathcal{G}}$ starting at $u = \hat{c}(0) = \tilde{c}(0)$ with respect to the principal connection $\tilde{\omega}$ as described in 2.1. Then the above condition reads as $\{\tilde{c}(t), \exp Y(t)\cdot H\} = \{\hat{c}(t), eH\}$, so we may write $\tilde{c}(t) = \hat{c}(t)\cdot \exp Y(t)$, and we are going to find $Y : \mathbb{R} \to \mathfrak{n}$ so that $\tilde{c}$ is the appropriate parallel curve in $\tilde{\mathcal{G}}$, in particular, $Y(0) = 0$.

The definition of X gives $\frac{d}{dt}\hat{c}(t) = \omega^{-1}(X(t))(\hat{c}(t))$ and the vectors $\frac{d}{dt}\exp Y(t)$ are obviously written as $T_o\ell_{\exp Y(t)}\cdot\delta(\exp\circ Y)(t)$. Hence the derivative of $\tilde{c}(t) = \hat{c}(t)\cdot \exp Y(t)$ yields

$$\tfrac{d}{dt}\tilde{c}(t) = Tr^{\exp Y(t)}\cdot\omega^{-1}(X(t))(\hat{c}(t)) + \zeta_{\delta(\exp\circ Y)(t)}(\hat{c}(t)\exp Y(t)).$$

The request for vectors $\frac{d}{dt}\tilde{c}(t)$ to be horizontal means that $\tilde{\omega}\big(\frac{d}{dt}\tilde{c}(t)\big) = 0$ for all t. From the right equivariance of the principal connection $\tilde{\omega}$ we conclude that the latter condition is really equivalent to $\mathrm{Ad}_{\exp(-Y(t))}(X(t)) + \delta(\exp\circ Y)(t) = 0$, since $\omega^{-1}(X(t))(\hat{c}(t))$ belongs to $T_{\hat{c}(t)}\mathcal{G}$ where $\tilde{\omega}$ and ω coincide.

For the proof of uniqueness of Y, we are going to rewrite the equation (7) in another way. Firstly, for each $z \in \mathfrak{gl}(\mathfrak{g})$ let us denote by $g(z) = \frac{e^z-1}{z}$ the linear endomorphism $\sum_{k=0}^{\infty}\frac{1}{(k+1)!}z^k$, which is invertible if and only if no eigenvalue of $z : \mathfrak{g} \to \mathfrak{g}$ has the form $2k\pi i$ for $k \in \mathbb{Z}\setminus\{0\}$, see [**17**, 4.28]. Then the equation (7), using (6) and omitting the variable t, reads as

$$\mathrm{Ad}_{\exp(-Y)}X + g(\mathrm{ad}(-Y))(Y') = 0, \tag{8}$$

which is a first order ODE linear in Y'. By the assumption, the only eigenvalue of $\operatorname{ad}(-Y) : \mathfrak{n} \to \mathfrak{n}$ is 0, hence $g(\operatorname{ad}(-Y))$ is invertible in $\mathfrak{n}$. Now Y', lying in $\mathfrak{n}$ for all t, can be separated from (8) so that it is a function of X and Y and the initial condition $Y(0) = 0$ determines the unique solution of this equation. □

Further, we need another expression of equation (7). Because $\operatorname{Ad} : G \to GL(\mathfrak{g})$ is a homomorphism of Lie groups, the derivative $\operatorname{ad} = T_e \operatorname{Ad}$ and the exponential mappings are related according to the condition $\operatorname{Ad} \circ \exp = \exp \circ \operatorname{ad}$. The right exp goes from $\mathfrak{gl}(\mathfrak{g}) \to GL(\mathfrak{g})$, so we may write

$$\operatorname{Ad}_{\exp Y}(X) = e^{\operatorname{ad} Y}(X) = \sum_{k=0}^{\infty} \frac{1}{k!} (\operatorname{ad} Y)^k (X) \tag{9}$$

for all $X, Y \in \mathfrak{g}$, cf. [**17**, 4.25]. Together with the expression (6) of $\delta(\exp \circ Y)$, the equation (7) reads as

$$\sum_{k=0}^{\infty} \frac{1}{k!} \Big(\operatorname{ad}(-Y(t))\Big)^k \Big(X(t) + \frac{1}{k+1} Y'(t)\Big) = 0. \tag{10}$$

This we use heavily below in order to compare the higher derivatives of Y and X. Especially, in the case of the subalgebra $\mathfrak{n} \subset \mathfrak{g}$ to be abelian, the latter equation takes the very nice and easy form $X(t) + Y'(t) = 0$ and the solutions are obvious. This happens, for instance, in all irreducible parabolic geometries. On the other hand, any parabolic geometry yields the subalgebra $\mathfrak{n}$ nilpotent, so the assumption of Proposition above is satisfied and, in particular, the sum in (10) is always finite.

Corollary. *In the setting above, two smooth curves on M have the contact of r^{th} order in $x \in M$ if and only if their developments at x have the contact of r^{th} order in $o \in G/H$.*

Proof. Let c_1 and c_2 be two curves on M such that $j_0^r c_1 = j_0^r c_2$ and let $x \in M$ be the point $c_1(0) = c_2(0)$. For a fixed frame $u \in \mathcal{G}$ over x, the lifting of curves according to the connection ω is a local diffeomorphism which respects the order of contact. So $j_0^r c_1 = j_0^r c_2$ if and only if $j_0^r \hat{c}_1 = j_0^r \hat{c}_2$, where $\hat{c}_1, \hat{c}_2$ are the corresponding parallel curves starting at u. Following the latter Proposition, let X_1, X_2 be the curves in $\mathfrak{n}$ defined as $X_i(t) = \omega\big(\frac{d}{dt}\hat{c}_i(t)\big)$, $i = 1, 2$, and let Y_1, Y_2 be the curves in $\mathfrak{n}$ coming from the developments of curves c_1 and c_2, respectively. In particular, the condition (10) is satisfied for both couples X_i, Y_i. We have to prove that $j_0^r c_1 = j_0^r c_2$ if and only if $j_0^r \exp Y_1 = j_0^r \exp Y_2$. According to the above ideas, $j_0^1 \exp Y_i = \frac{d}{dt}\big|_0 \exp Y_i(t) = Y_i'(0)$ and $X_i(0)$ is determined by $j_0^1 c_i = \frac{d}{dt}\big|_0 c_i(t)$, hence the latter statement reduces to the equivalence $j_0^{r-1} X_1 = j_0^{r-1} X_2$ if and only if $j_0^r Y_1 = j_0^r Y_2$, which we prove as follows.

Let us we write the equation (10) as

$$X(t) + Y'(t) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} S_k(t) = 0,$$

where obviously $S_k(t) = \big(\mathrm{ad}(Y(t))\big)^k \big(X(t)+\frac{1}{k+1}Y'(t)\big)$. The n^{th} iterated derivative yields, omitting the variable t,

$$S_k^{(n)} = \sum_J c_J \Big(\mathrm{ad}(Y^{(j_1)}) \circ \cdots \circ \mathrm{ad}(Y^{(j_k)})\Big)\Big(X^{(n-|J|)} + \frac{1}{k+1}Y^{(n+1-|J|)}\Big) = 0,$$

where the coefficients are $c_J = \frac{n!}{j_1!\dots j_k!(n-|J|)!}$ and the sum runs over all k–tuples of nonnegative integers $J = (j_1, \dots, j_k)$ such that $|J| = j_1 + \cdots + j_k \le n$.

First of all, let us remark that the definition of Y ensures the value $Y(0) = 0$ and so all summands of $S_k^{(n)}(0)$ with some $j_i = 0$ vanishes. Then, for all $k > n$, each k–tuple J obviously contains some $j_i = 0$ and so $S_k^{(n)}(0)$ vanishes, hence the n^{th} iterated derivative of (10) evaluated in 0 is the sum

$$X^{(n)}(0) + Y^{(n+1)}(0) + \sum_{k=1}^{n} \frac{(-1)^k}{k!} S_k^{(n)}(0) = 0. \tag{11}$$

Now, since the nonzero summands correspond to those J's with all $j_i > 0$, the condition $|J| > 0$ is always satisfied and so each $S_k^{(n)}(0)$ is expressed in terms of the derivatives of X and Y in 0 of at most n^{th} order.

The rest is obvious from (11). First, let us assume that $j_0^r X_1 = j_0^r X_2$, i.e. $X_1^{(n)}(0) = X_2^{(n)}(0)$ for all $n = 0, \dots, r$. Clearly, $Y_1(0) = Y_2(0) = 0$ and, inductively, $Y_1^{(n)}(0) = Y_2^{(n)}(0)$ immediately implies $Y_1^{(n+1)}(0) = Y_2^{(n+1)}(0)$ for all $n = 0 \dots, r$. So we have proved $j_0^{r+1} Y_1 = j_0^{r+1} Y_2$. Conversely, for $n = 0$, the condition $Y_1'(0) = Y_2'(0)$ is obviously equivalent to $X_1(0) = X_2(0)$. Similarly, the assumption $Y_1^{(n)}(0) = Y_2^{(n)}(0)$ for all $n < r$ yields $X_1^{(n)}(0) = X_2^{(n)}(0)$ for all $n < r - 1$, which completes the proof. □

For a fixed frame over some point $x_0 \in M$, the construction of developments establishes a correspondence between smooth curves on M which map 0 to the fixed point $x_0 \in M$ and smooth curves in the homogeneous space G/H mapping 0 to the origin $o = eH$. Further, this is a bijection onto the image compatible with taking jets in 0 in the sense of Corollary above. The injectivity can be seen as follows. Let c_1 and c_2 be two curves in M which develop into the same curve at some point. In accord with the Proposition 2.7, we suppose that the curves X_1 and X_2 gives the same $Y_1 = Y_2$ and the equality (7) is satisfied for both couples X_1, Y_1 and X_2, Y_2, respectively. This immediately leads to the condition $\mathrm{Ad}_{\exp(-Y(t))}(X_1(t)) = \mathrm{Ad}_{\exp(-Y(t))}(X_2(t))$ provided that we denote $Y_1 = Y_2$ by Y. Hence the curves X_1 and X_2 coincide since the map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is invertible for any $g \in G$.

The notion of the development is often used to distinguish curves on manifolds endowed with a Cartan geometry of some fixed type by means of distinguished curves in the homogeneous model. The following Proposition may provide an alternative definition of generalized geodesics.

2.8. Proposition. *In the setting above, a curve through the point $x \in M$ is a geodesic of type $\mathcal{C}_A$ if and only if it develops at x into the curve $\{u, \exp tX \cdot H\} \subset S_xM$ for some $X \in A$ and $u \in \mathcal{G}$.*

Proof. According to Proposition 2.7 with the notation therein, we have only to prove that the curve $X(t)$ is constant if and only if $Y(t) = tX$ for some $X \in A$. If $X(t) = X$ is the constant curve, we define $Y(t) = -tX$, and, conversely, if Y has the form $Y(t) = tX$ then we put $X(t) = -X$. However, the equation (10) is still satisfied, since the sum $\sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{ad}(-tX))^k(-X + \frac{1}{k+1}X)$ vanishes for all t. The uniqueness of the development completes the proof. □

Remark. Our definition of developments generalizes the classical concept of the development of curves on a manifold with an affine connection. In that case, the homogeneous model is a pair $(A(m,\mathbb{R}), GL(m,\mathbb{R}))$ and the model space $G/H = \mathbb{R}^m$ is globally identified with $\mathfrak{n}$ so that the two actions of the structure group $H = GL(m,\mathbb{R})$ coincide. Then the Cartan's space SM equals to the tangent bundle TM and a curve is an affine geodesic if and only if it develops into a straight line within the tangent space of any single point, cf. [**14**, p. 138].

3. Weyl connections

This section studies certain more familiar structures which underlie a given Cartan geometry. However, our aims require to consider parabolic geometries only and this convention will be kept in the rest of work.

Weyl connections form a preferred class of connections induced by the Cartan connection of a given parabolic geometry. The modeling examples could be conformal Riemannian structures where a lot of referred concepts have got their preimage. In concluding paragraphs we look for a thread between generalized geodesics of the given Cartan connection and geodesics of induced Weyl connections. In particular, we illustrate this approach in the homogeneous model of projective geometries in order to understand the definition of the projective equivalence of affine connections in 1.1 more geometrically. All facts without references can be checked in [**5**], [**6**], [**23**], or [**26**].

3.1. Parabolic geometries. Cartan geometry of type (G, P) is called *parabolic* if G is a semisimple Lie group and P a parabolic subgroup. Let $\mathfrak{p}$ and $\mathfrak{g}$ be the corresponding Lie algebras. The names come from the structure theory of Lie algebras, where a subalgebra $\mathfrak{p}$ in semisimple $\mathfrak{g}$ is called parabolic if it contains a Borel subalgebra, i.e. maximal solvable subalgebra of $\mathfrak{g}$. The nilradical $\mathfrak{p}_+$ of the parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ provides a filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^k$ so that $\mathfrak{p} = \mathfrak{g}^0$ and $\mathfrak{p}_+ = \mathfrak{g}^1$. A choice of the Levi factor $\mathfrak{g}_0$ in $\mathfrak{p}$ induces a grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ such that $\mathfrak{g}^j = \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k$ for all j. This is an algebraic analogy of the construction of Weyl structures, see below. Conversely, for any semisimple Lie algebra $\mathfrak{g}$ endowed with a $|k|$–grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$, the subalgebra $\mathfrak{g}^0 = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ is parabolic. Anyway, the subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is reductive. The subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ forms a natural complement to $\mathfrak{p}$ in $\mathfrak{g}$. Obviously, the subalgebra $\mathfrak{g}_-$ is not ideal and there is no complementary ideal to $\mathfrak{p}$ in $\mathfrak{g}$. Hence parabolic geometries are never reductive but always naturally split.

Any parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is isomorphic by an inner automorphism of $\mathfrak{g}$ to some standard parabolic subalgebra, which is simply described by a subset of simple roots of $\mathfrak{g}$ containing all of the standard Borel subalgebra. In other words, for any parabolic subalgebra $\mathfrak{p}$, one can choose the Cartan subalgebra and the system of positive roots in such a way that $\mathfrak{p}$ is standard parabolic. This is clearly described by crosses in corresponding nodes of the Dynkin diagram of $\mathfrak{g}$ and, moreover, the number of crosses coincides with the dimension of the center of $\mathfrak{g}_0$. Then the semisimple part of $\mathfrak{g}_0$ is represented by the Dynkin diagram which is obtained from the initial one by removing all crossed nodes and all edges which emanate from them.

On the level of Lie groups, P is the semidirect product $G_0 \rtimes \exp \mathfrak{p}_+$, where G_0 is the subgroup of P which consists of all elements keeping the gradation of $\mathfrak{g}$ invariant with respect to the Ad–representation.

3.2. Weyl structures. Let $\mathcal{G} \to M$ be the principal bundle of a parabolic geometry. By $\mathcal{G}_0$ we denote the quotient $\mathcal{G}/\exp \mathfrak{p}_+$ which is a principle G_0–bundle over M. A *Weyl structure* or a *Weyl geometry* of the given parabolic geometry is a reduction of $\mathcal{G}$ to the structure group G_0.

Any Weyl structure is given by a G_0–equivariant section of the projection $p_0 : \mathcal{G} \to \mathcal{G}_0$. Such a section always exists and all Weyl structures on M form an affine bundle modeled over the vector bundle T^*M. More precisely, if σ_1 and σ_2 are Weyl structures then there is a unique G_0–equivariant map $\Upsilon : \mathcal{G}_0 \to \mathfrak{p}+$ such that $\sigma_1(u) = \sigma_2(u) \cdot \exp \Upsilon(u)$ for all $u \in \mathcal{G}_0$. Due to the Cartan–Killing form we have got the canonical identification $\mathfrak{p}_+ = (\mathfrak{g}_-)^*$ and the cotangent bundle is then written as $T^*M = \mathcal{G} \times_P \mathfrak{p}_+$. Now any G_0–equivariant map $\Upsilon : \mathcal{G}_0 \to \mathfrak{p}_+$, extends to a P–equivariant map on $\mathcal{G}$, which is the frame form of some one–form on M. Conversely, for a Weyl structure σ and any one–form with the frame form Υ there is another Weyl structure $\sigma + \Upsilon$ defined by $(\sigma + \Upsilon)(u) = \sigma(u) \cdot \exp \Upsilon(u)$. Altogether, all Weyl structures form an affine bundle modeled over T^*M.

Given a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ and the Cartan connection ω, split as $\omega_- \oplus \omega_0 \oplus \omega_+ \in \Omega^1(\mathcal{G}, \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+)$, the pullback $\sigma^*\omega_0$ defines a principal connection on $\mathcal{G}_0$ called the *Weyl connection* of the Weyl structure σ. Each Weyl connection induces connections on all bundles associated to $\mathcal{G}_0$.

3.3. Underlying structures. The filtration of the Lie algebra $\mathfrak{g}$ gives rise to a filtration of the tangent bundle $TM = \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ by projecting the subspaces $\omega^{-1}(\mathfrak{g}^j) \subset T\mathcal{G}$. The filtration is written as $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ where $T^jM = \mathcal{G} \times_P (\mathfrak{g}^j/\mathfrak{p})$ with respect to the described mapping. If the length of grading k equals to 1 then the filtration of TM is trivial and so the $|1|$–graded parabolic geometries are called *irreducible*.

By analogy to the principal G_0–bundle $\mathcal{G}_0 = \mathcal{G}/\exp \mathfrak{p}_+$, one can construct other principal bundles underlying the bundle $\mathcal{G}$. So we define $\mathcal{G}_j = \mathcal{G}/\exp \mathfrak{g}^{j+1}$ which is a principle bundle over M with the structure group $P/\exp \mathfrak{g}^{j+1}$. On the other hand each $\mathcal{G} \to \mathcal{G}_j$ is a principal bundle with the structure group $\exp \mathfrak{g}^{j+1}$ and together we get the so called *tower* of principal bundles $\mathcal{G} = \mathcal{G}_k \to \mathcal{G}_{k-1} \to \cdots \to \mathcal{G}_0 \to M$. The last but one level in the tower, the bundle $\mathcal{G}_{k-1}$, is of the particular interest since it can be understood as a classical first order $(P/\exp \mathfrak{g}_k)$–structure on M. The reduction $\iota : \mathcal{G}_{k-1} \to P^1M$ is constructed due to the Cartan connection ω on $\mathcal{G}$ descending to a well defined one–form on $\mathcal{G}_{k-1}$ with values in $\mathfrak{g}/\mathfrak{p} = \mathfrak{g}_-$, which plays the role of soldering form. Since the kernel of the representation $\underline{\mathrm{Ad}} : P \to GL(\mathfrak{g}_-)$ is the normal subgroup $\exp \mathfrak{g}_k$, the induced mapping $P/\exp \mathfrak{g}_k \to GL(\mathfrak{g}_-)$ is an embedding of groups over which the reduction ι is constructed.

For a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ and any quotient projection $p_j : \mathcal{G} \to \mathcal{G}_j$, the composition is again G_0–equivariant, so $p_j \circ \sigma : \mathcal{G}_0 \to \mathcal{G}_j$ is a reduction of $\mathcal{G}_j$

to the structure group G_0. Now the Weyl connection $\sigma^*\omega_0$ extends to a principal connection on all $\mathcal{G}_j$. In particular, any choice of a Weyl structure defines an affine connection on $\mathcal{G}_{k-1}$ belonging to the $(P/\exp\mathfrak{g}_k)$–structure in the sense of Example 2.1.

Moreover, any Weyl structure provides a reduction $\iota:\mathcal{G}_0\to P^1M$ of the linear frame bundle to the structure group G_0 in such a way that $\iota^*\theta=\sigma^*\omega_-$. At the same time we get an identification of the tangent bundle $TM=\mathcal{G}\times_P\mathfrak{g}_-$ with the associated graded vector bundle $\mathrm{gr}(TM)=\bigoplus_{j=-k}^{-1}T^jM/T^{j+1}M$ whose standard fiber is a sum of $\mathfrak{g}^j/\mathfrak{g}^{j+1}=\mathfrak{g}_j$. The later bundle can be written as $\mathrm{gr}(TM)=\mathcal{G}_0\times_{G_0}(\bigoplus_j\mathfrak{g}_j)$ and the promised identification $\mathrm{gr}(TM)\cong TM$ looks like $\{u,X\}\mapsto\{\sigma(u),X\}$. Now $\sigma^*(\omega_-\oplus\omega_0)$ is an affine connection which belongs to the G_0–structure and preserves the grading.

Remark. For $|1|$–graded parabolic geometries, both the reduction $\iota:\mathcal{G}_0\to P^1M$ and the identification $\mathcal{G}_0\times_{G_0}\mathfrak{g}_-\cong\mathcal{G}\times_P\mathfrak{g}_-$ does not depend on a chosen Weyl structure. The underlying first order G_0–structure with a preferred class of its affine connections generalizes the notion of Weyl connections in conformal Riemannian geometries. More comments and motivations can be found in the introduction section of [**23**].

3.4. Changes of Weyl connections. Now we consider a vector bundle $\mathcal{G}_0\times_{G_0}E$ and two Weyl structures σ, $\tilde\sigma$ in order to compare induced linear connections ∇^σ and $\nabla^{\tilde\sigma}$ corresponding to given Weyl structures. Let s be any section of the vector bundle with the frame form denoted by the same symbol. Following 1.3, the covariant derivative on the level of frame forms is $\nabla^\sigma_\xi s=Ts\cdot\gamma^\sigma(\xi)$, where $\gamma^\sigma(\xi)$ denotes the horizontal lift of $\xi\in\mathfrak{X}M$ with respect to the principal connection $\sigma^*\omega_0$ on $\mathcal{G}_0$. The two lifts which correspond to Weyl structures σ and $\tilde\sigma$ may differ only in a vertical direction, so there is a mapping $A:\mathcal{G}_0\to\mathfrak{g}_0$ such that $\gamma^{\tilde\sigma}(\xi)(u)=\gamma^\sigma(\xi)(u)+\zeta_{A(u)}(u)$ for all $u\in\mathcal{G}_0$.

In order to describe this mapping we have to deal with the tangent map to the section $\tilde\sigma=\sigma+\Upsilon$. By definition, for each $u\in\mathcal{G}_0$ we have $\tilde\sigma(u)=\sigma(u)\cdot\exp\Upsilon(u)$ and a direct computation [**5**, 3.9] yields $T\tilde\sigma\cdot\xi=Tr^{\exp\Upsilon(u)}T\sigma\cdot\xi+\zeta_{\Phi(\xi)}(\tilde\sigma(u))$ for all $\xi\in T_u\mathcal{G}_0$ and suitable $\Phi(\xi)\in\mathfrak{p}_+$. Hence, by the equivariance of the Cartan connection ω we get

$$\tilde\sigma^*\omega(\xi)=\omega(T\tilde\sigma\cdot\xi)=\mathrm{Ad}_{\exp(-\Upsilon)}(\sigma^*\omega(\xi))+\Phi(\xi). \tag{1}$$

Now let us write $\hat\xi$ instead of ξ and let us assume the vector $\hat\xi\in T_u\mathcal{G}_0$ is horizontal with respect to the principal connection $\sigma^*\omega_0$, i.e. $\sigma^*\omega_0(\hat\xi)=0$. In other words, there are $X\in\mathfrak{g}_-$ and $Z\in\mathfrak{p}_+$ such that $T\sigma\cdot\hat\xi=\omega^{-1}(X)(\sigma(u))+\zeta_Z(\sigma(u))$, which implies that $\hat\xi=Tp_0\cdot\omega^{-1}(X)(\sigma(u))$ and, obviously, $\hat\xi$ is the horizontal lift of the vector $\xi=\{\sigma(u),X\}\in T_xM$. If the Weyl structure is changed to $\tilde\sigma=\sigma+\Upsilon$ then $\tilde\sigma^*\omega_0(\hat\xi)=\pi_0(\tilde\sigma^*\omega(\hat\xi))=\pi_0(\mathrm{Ad}_{\exp(-\Upsilon)}X)$ by equality (1), where $\pi_0:\mathfrak{g}\to\mathfrak{g}_0$ is the $\mathfrak{g}_0$–part, i.e. the projection in the direction of $\mathfrak{g}_-\oplus\mathfrak{p}_+$. Hence, in general, horizontal lifts of a vector field $\xi\in\mathfrak{X}M$ with respect to the Weyl connections $\sigma^*\omega_0$ and $(\sigma+\Upsilon)^*\omega_0$ differ by the vertical vector field ζ_A where $A:\mathcal{G}_0\to\mathfrak{g}_0$ is given by $A=\pi_0(\mathrm{Ad}_{\exp(-\Upsilon)}(\bar\xi))$. Here $\bar\xi:\mathcal{G}_0\to\mathfrak{g}_-$ is the frame form of ξ with respect to σ, i.e. $\bar\xi=\sigma^*\omega_-(\hat\xi)$ where $\hat\xi$ is an arbitrary lift of ξ. In this notation we may write

$\gamma^{\sigma+\Upsilon}(\xi) = \gamma^\sigma(\xi) - \zeta_A$, which in particular implies that the induced connections differ as

$$\nabla_\xi^{\sigma+\Upsilon} s = \nabla_\xi^\sigma s + \lambda'(A) \circ s, \tag{2}$$

where $\lambda' : \mathfrak{g}_0 \to \mathfrak{gl}(E)$ is the derivative of the representation defining the associated bundle $\mathcal{G}_0 \times_{G_0} E$.

Remark. In conclusion, in the case of $|1|$–gradings it is easy to verify that $\pi_0(\mathrm{Ad}_{\exp(-\Upsilon)}(\xi)) = -[\Upsilon, \xi]$. Hence the formula above takes the following nice shape

$$\nabla_\xi^{\sigma+\Upsilon} s = \nabla_\xi^\sigma s + \lambda'([\xi, \Upsilon]) \circ s, \tag{3}$$

where the ξ in brackets means the frame form of the vector field ξ. See 3.8 for an application.

3.5. Rho–tensor. The *rho–tensor* of a Weyl structure σ is defined as $\mathrm{P}^\sigma = \sigma^*\omega_+ \in \Omega^1(\mathcal{G}_0, \mathfrak{p}_+)$. Obviously, $\mathrm{P}^\sigma(\zeta_A) = 0$ for each $A \in \mathfrak{g}_0$, so the value of P^σ does not depend on a lift of vector $\xi \in TM$ and so P^σ can be understood as a pullback of a one–form on the base manifold with values in $\mathfrak{p}_+$ according to the projection $\mathcal{G}_0 \to M$. Together with the canonical identification $\mathfrak{p}_+ = (\mathfrak{g}_-)^*$, the rho–tensor can be seen as a 2–form on M. In the case of conformal Riemannian structures, the rho–tensor is a well known object usually expressed in terms of the metric and Ricci tensors. We refer to [**7**] where authors have explicitly shown that these two concepts are equivalent.

The rho–tensor P^σ is used to compare the Cartan connection ω and the Weyl connection $\sigma^*\omega_0$ as follows. The principal connection $\sigma^*\omega_0$ on $\mathcal{G}_0$ extends to a principal connection on $\mathcal{G}$ and in the image $\sigma(\mathcal{G}_0) \subset \mathcal{G}$ we can compare the horizontal lifts with respect to these connections. Let $\hat{\xi} \in T_{\sigma(u)}\mathcal{G}$ be the horizontal lift of the vector $\xi = \{\sigma(u), X\}$ with respect to the connection $\sigma^*\omega_0$. Now $\omega(\hat{\xi}) = X + \omega_+(\hat{\xi})$, which yields $\hat{\xi} = \omega^{-1}(X)(\sigma(u)) + \zeta_{\mathrm{P}^\sigma(\xi)}(\sigma(u))$, where $\omega^{-1}(X)(\sigma(u))$ is the horizontal lift of ξ with respect to the Cartan connection ω. Altogether, in the image $\sigma(\mathcal{G}_0) \subset \mathcal{G}$, the following holds,

$$\nabla_\xi^\sigma s = \nabla_\xi^\omega s - \lambda'(\mathrm{P}^\sigma(\xi)) \circ s. \tag{4}$$

3.6. Normal Weyl structures. Among general Weyl structures there is an interesting class of them which are related to the notion of normal coordinates as follows. For any $u \in \mathcal{G}$, the mapping $\Phi_u : \mathfrak{g}_- \to M$, $X \mapsto p(\mathrm{Fl}_1^{\omega^{-1}(X)}(u))$, is a local diffeomorphism around 0 and this is called the *normal coordinates* at u. Obviously, generalized geodesics on M are images of straight lines in suitable normal coordinates and, further, our definition agrees with the notion of normal coordinates in Riemannian and affine geometries.

Each normal coordinates define a normal Weyl structure as follows. Let Φ_u be normal coordinates defined on a neighbourhood $U \subset \mathfrak{g}_-$ of 0. Obviously, over the image $\Phi_u(U) \subset M$ there is a unique G_0–equivariant section $\sigma_u : \mathcal{G}_0 \to \mathcal{G}$ such that $\mathrm{Fl}_1^{\omega^{-1}(U)}(u) \subset \sigma_u(\mathcal{G}_0)$. Although the section σ_u is not defined globally, we call σ_u the *normal Weyl structure* at $u \in \mathcal{G}$.

3.7. Proposition. *Locally, a curve c is a generalized geodesic if and only if there is a Weyl structure σ such that the curve is a geodesic of the corresponding Weyl connection and the rho–tensor vanishes along c. In particular, all geodesics of normal Weyl structures are generalized geodesics.*

Proof. [**23**, 2.37–39] If c is a geodesic curve of a Weyl connection $\sigma^*\omega_0$ then it is the projection of a horizontal flow $c_0(t)$ with respect to the Weyl connection on $\mathcal{G}_0$. Moreover, $\mathrm{P}^\sigma(\dot{c}) = 0$ implies that $\sigma(c_0(t))$ is a horizontal flow of the Cartan connection ω, so c is a generalized geodesic. Using the formula (4), this claim is briefly written as $\nabla^\omega_{\dot{c}}\dot{c} = \nabla^\sigma_{\dot{c}}\dot{c} + \lambda'(\mathrm{P}^\sigma(\dot{c}))\circ\dot{c} = 0$. Conversely, let us denote $c_1(t) = \mathrm{Fl}^{\omega^{-1}(X)}_t(u)$ and consider the generalized geodesic $c = p(c_1)$. Let σ be the unique normal Weyl structure such that $\sigma\circ p_0\circ c_1 = c_1$. Such a σ always exists but not globally if the curve $p_0(c_1)$ has got nonisolated points of self–intersection. Then, at least locally, the corresponding rho–tensor fulfils $\mathrm{P}^\sigma(\dot{c}) = 0$ and so $\nabla^\sigma_{\dot{c}}\dot{c} = \nabla^\omega_{\dot{c}}\dot{c} = 0$.

The second statement is obvious from the construction of normal Weyl structures. More precisely, P^{σ_u} is identically zero at u and further P^{σ_u} vanishes along all geodesic of the normal Weyl structure σ_u. □

3.8. Example. Here we consider the most frequent associated vector bundle—the tangent bundle—in order to describe the changes of Weyl connections as discussed in 3.4 in the case of projective geometries. Now, $TM = \mathcal{G}_0\times_{G_0}\mathfrak{g}_-$, where the defining representation $\lambda : G_0\to GL(\mathfrak{g}_-)$ is the Ad–representation restricted to G_0. Hence the derivative is $\lambda' = \mathrm{ad}|_{\mathfrak{g}_0} : \mathfrak{g}_0\to\mathfrak{gl}(\mathfrak{g}_-)$ and the formula (3) reads as $\nabla^{\sigma+\Upsilon}_\xi\eta = \nabla^\sigma_\xi\eta + [[\xi,\Upsilon],\eta]$. All brackets are the Lie brackets in $\mathfrak{g}$ and ξ, η, and Υ are frame forms of vector fields and a one–form on M, respectively.

Let M be a manifold with a structure of the projective Cartan geometry. According to Example 2 in 2.2, one can easily verify the bracket equals to $[[\xi,\Upsilon],\eta] = \Upsilon(\xi)\,\eta + \Upsilon(\eta)\,\xi$, provided that we write $\xi = \left(\begin{smallmatrix}0&0\\\xi&0\end{smallmatrix}\right)$, $\eta = \left(\begin{smallmatrix}0&0\\\eta&0\end{smallmatrix}\right)$, and $\Upsilon = \left(\begin{smallmatrix}0&\Upsilon\\0&0\end{smallmatrix}\right)$. This recovers the definition of the projective equivalence from 1.1, so we conclude that two affine connections are projectively equivalent if and only if they belong to the same projective structure. By [**13**, 7.1], torsion free affine connections on M, i.e. principal connections on P^1M, are in a bijective correspondence with global sections of the bundle projection $P^2M/G^1_m\to M$ where $G^1_m = GL(m,\mathbb{R})$ is the structure group of P^1M. Any such section provides a reduction of P^2M to the subgroup $G^1_m\subset G^2_m$, i.e. a G^1_m–equivariant section of $P^2M\to P^1M$, which reminds the notion of Weyl structures. Indeed, the principal bundle $\mathcal{G}$ of a projective structure whose $\mathfrak{g}_-$ part of the curvature vanishes can be given as a reduction $\iota : \mathcal{G}\to P^2M$ such that $\iota^*\theta = \omega_-\oplus\omega_0$, where θ is the canonical form on P^2M. Then $\mathcal{G}_0 = P^1M$ and all Weyl structures induce projectively equivalent affine connections without torsion on M.

Let c be a generalized geodesic on M. By Proposition 3.7, there exists a Weyl structure σ such that c is a geodesic of the corresponding Weyl connection, so we can write $\nabla^\sigma_{\dot{c}}\dot{c} = 0$. Any other Weyl structure determined by $\Upsilon\in\Omega^1(M)$ yields $\nabla^{\sigma+\Upsilon}_{\dot{c}}\dot{c} = [\dot{c},[\Upsilon,\dot{c}]] = 2\Upsilon(\dot{c})\,\dot{c}$, which is a multiple of the velocity vector field of c. Hence all Weyl connections underlying a projective Cartan geometry have got common geodesics as unparametrized curves. Moreover, from 1.1 we know that a reparametrization φ of the geodesic c acts on the result of the covariant derivative $\nabla_{\dot{c}}\dot{c}$ by adding the value $-\frac{\varphi''}{\varphi'^2}\dot{c}$, so all reparametrizations of c coming from the changes of Weyl connections must satisfy the condition

$$\varphi'' = -2\varphi'^2\Upsilon(\dot{c}). \tag{5}$$

Let us verify this equality in the homogeneous model. For the sake of simplicity, we consider only geodesics through the origin $o = eH$ which are tangent to the vector $\{e, X\} \in T_o(G/P)$, i.e. we have to deal with the curves of the shape $c^{h,\underline{\mathrm{Ad}}_h^{-1}X}(t) = h\exp(t\,\underline{\mathrm{Ad}}_h^{-1}X)\cdot P$ where $h \in P$ and $X \in \mathfrak{g}_{-1}$. However, these restrictions describe the geodesics in general due to the notion of developments of curves and the fact that there are no distinguished tangent vectors in $T(G/P)$. With respect to 2.4, all geodesics of the above shape with $h \in G_0$ coincide with the curve $c^{e,X}$, so we consider $h = \exp Z \in \exp\mathfrak{g}_1$ in order to get new generalized geodesics with the given tangent vector. In that case, $\mathrm{Ad}_{\exp Z} X = X + [ZX] + \frac{1}{2}[Z[ZX]]$ and $\underline{\mathrm{Ad}}_{\exp Z} X = X$, so we will compare generalized geodesics $c^{e,X}$ and $c^{\exp Z,X}$ for all $Z \in \mathfrak{g}_1$. Obviously, these two curves coincide if and only if there is a curve $u : \mathbb{R} \to P$ such that $\exp tX \cdot u(t) = \exp Z \exp tX$. Anyway, this equality defines a curve in G,

$$u(t) = \begin{pmatrix} 1 & 0 \\ -tX & E \end{pmatrix} \cdot \begin{pmatrix} 1+tZ(X) & Z \\ tX & E \end{pmatrix} = \begin{pmatrix} 1+tZ(X) & Z \\ -t^2 Z(X)X & -tX\otimes Z + E \end{pmatrix}, \tag{6}$$

and u belongs to P for all t if only if $Z(X) = 0$. Otherwise, we get new generalized geodesics tangent to the vector $\{e, X\} = \{\exp Z, X\}$ and the family of all such curves is 1–dimensional according to the values of $Z(X)$.

Let us describe the mentioned curves in more details. First of all, we are going to verify the equality

$$c^{\exp Z,X}(t) = \begin{pmatrix} 1+tZ(X) & Z \\ tX & E \end{pmatrix} \cdot P = \begin{pmatrix} 1 & 0 \\ \frac{tX}{1+tZ(X)} & E \end{pmatrix} \cdot P \tag{7}$$

which clearly recovers the result above, i.e. curves $c^{\exp Z,X}$ and $c^{e,X}$ coincide if and only if $Z(X) = 0$. Otherwise, we get $c^{\exp Z,X}(t) = c^{e,X}(\varphi(t))$ where $\varphi(t) = \frac{t}{1+tZ(X)}$ is a local reparametrization. This in particular implies that all generalized geodesics with the given tangent vector parametrize the same curve and so each tangent direction determines a unique unparametrized generalized geodesic—this is the essential property of projective geometries. Now, equality (7) really holds as follows. Denote by c_1 the curve $\exp Z \exp tX = \begin{pmatrix} 1+tZ(X) & Z \\ tX & E \end{pmatrix}$ in G. The projection $c_1 \cdot P = c^{\exp Z,X}$ is certainly represented by $c_0(t) = c_1(t) \cdot \exp\left(-\frac{Z}{1+tZ(X)}\right) = \begin{pmatrix} |A|^{-1} & 0 \\ tX & A \end{pmatrix}$ where $A = -\frac{t}{1+tZ(X)} X \otimes Z + E$ and $|A|^{-1} = 1 + tZ(X)$. Finally, the representative $c_0(t) \cdot \begin{pmatrix} |A| & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ |A|tX & E \end{pmatrix}$ yields the formula (7).

Next, following the previous sections, we are going to describe the Weyl connections which have the curves $c^{e,X}$ and $c^{\exp Z,X}$ as geodesics, respectively. Obviously, there is plenty of Weyl connections for which some curve c is a geodesic but all of them must coincide along c, so we may always describe the defining Weyl structure along the curve and consider the "constant" extension elsewhere. This actually corresponds to the choice of the normal Weyl structure as will be visible later on. First, let us remind that the principal bundle $\mathcal{G}$ is $G = SL(m+1,\mathbb{R})$ and, by definition, $\mathcal{G}_0 = G/\exp\mathfrak{g}_1 \cong A(m,\mathbb{R})$ is the underlying principal bundle with the structure group $G_0 = GL(m,\mathbb{R})$. The cosets in $\mathcal{G}_0$ will be denoted by brackets, so the quotient projection $p_0 : \mathcal{G} \to \mathcal{G}_0$ is written as $p_0(g) = [g]$.

Let us consider the curve $\exp tX = \begin{pmatrix} 1 & 0 \\ tX & E \end{pmatrix}$ in G which projects to the generalized geodesic $c^{e,X}$ in G/P. According to Proposition 3.7, the curve $c^{e,X}$ is a geodesic of the Weyl connection $\sigma^*\omega_0$ if and only if the defining Weyl structure σ satisfies $\sigma([\exp tX]) = \exp tX$. Due to the G_0–equivariance of σ, the transcription $\left[\begin{pmatrix} 1 & 0 \\ X & E \end{pmatrix}\right] \mapsto \begin{pmatrix} 1 & 0 \\ X & E \end{pmatrix}$, for all $X \in \mathfrak{g}_{-1}$, fully determines a Weyl structure with the property in request and we denote it by σ_0. For general elements of $\mathcal{G}_0$, the transcription must look like $\left[\begin{pmatrix} a & Z \\ X & A \end{pmatrix}\right] \mapsto \begin{pmatrix} a & 0 \\ X & -\frac{1}{a}X\otimes Z + A \end{pmatrix}$ in order the map to be well defined. Indeed, σ_0 is G_0–equivariant and $p_0 \circ \sigma_0 = \mathrm{id}$ and, moreover, $\exp X \subset \sigma_0(\mathcal{G}_0)$ for all $X \in \mathfrak{g}_{-1}$, so σ_0 is the normal Weyl structure for $e \in \mathcal{G}$. Now, let c_1 be the curve $\exp Z \exp tX = \begin{pmatrix} 1+tZ(X) & Z \\ tX & E \end{pmatrix}$ in G whose projection to G/P is the generalized geodesic $c^{\exp Z,X}$. We look for a Weyl structure σ_Z such that $\sigma_Z([c_1]) = c_1$ which is equivalent to finding a G_0–equivariant map $\Upsilon : \mathcal{G}_0 \to \mathfrak{g}_1$ satisfying $\sigma_0([c_1])\cdot \exp\Upsilon([c_1]) = c_1$. Following the notation and ideas of the paragraph just below the equality (7), we may rewrite the latter condition as $\sigma_0([c_0])\cdot\exp\Upsilon([c_0]) = c_1$ since $[c_1] = [c_0]$. Hence we get $\Upsilon([c_0(t)]) = \begin{pmatrix} 0 & \frac{Z}{1+tZ(X)} \\ 0 & 0 \end{pmatrix}$ which defines the Weyl structure σ_Z along the curve. Similarly to the previous case, we define Υ (and so σ_Z) globally and σ_Z is the normal Weyl structure for $\exp Z \in \mathcal{G}$. Further, one can verify that changes of the frame in $\mathcal{G}_0$ yield $\Upsilon([\exp X]) = (1 + Z(X))Z$ for all X.

Now we can summarize the achieved results. First, we have proved that generalized geodesics $c^{e,X}$ and $c^{\exp Z,X}$ always parametrize the same curve and the unique reparametrization which satisfies the condition $c^{\exp Z,X} = c^{e,X} \circ \varphi$ is $\varphi(t) = \frac{t}{1+tZ(X)}$. Certainly, $\varphi'(t) = \frac{1}{(1+tZ(X))^2}$, $\varphi''(t) = -2\frac{Z(X)}{(1+tZ(X))^3}$, and so the frame form of the vector field $-\frac{\varphi''}{\varphi'^2}\dot{c}^{e,X}$ with respect to the Weyl structure σ_0 maps $[\exp tX] \mapsto 2Z(X)(1+tZ(X))X \in \mathbb{R}^m$ because the frame form of $\dot{c}^{e,X}$ maps $[\exp tX] \mapsto X$ for all t. Indeed, this agrees with the frame form of the vector field $2\Upsilon(\dot{c}^{e,X})\,\dot{c}^{e,X}$ and so the formula (5) follows.

Of course, the equation (5) represents a necessary condition for the function φ to be a reparametrization which turns generalized geodesics into generalized geodesics, but we have found the possible reparametrizations very explicitly. Happily, we can perform the same for all irreducible parabolic geometries, see Proposition 5.6.

3.9. Exercise. Compute the bracket $[[\xi,\Upsilon],\eta]$ in order to express the transformation rules for changes of Weyl connections in the case of conformal Riemannian structures. Compare the result with the well known transformations of $\nabla_\xi\eta$ according to the change of metric in the conformal class. Prove that all null–geodesics tangent to the given common vector parametrize the same curve. (See 5.7 for the description of the Lie algebra in question).

CHAPTER II

Conclusion

The main aim of this chapter is to find conditions which determine generalized geodesics of a given type uniquely. For any parabolic geometry, all generalized geodesics are determined by a jet of finite order as Proposition 4.4 shows. Section 5 is devoted to detailed discussions on irreducible parabolic geometries. In those cases, the set of jets distinguished by generalized geodesics in T_1^r can be expressed explicitly and, moreover, the class of preferred parametrizations is obtained in a very visible way, Proposition 5.6. Section 6 brings some refinements for geodesics of specific types in general parabolic geometries, see Theorem 6.3. The concluding section contains complete classifications of generalized geodesics in several particular geometries.

The chapter is structured following the article [**8**] and a lot of paragraphs are just reformulated here. All of the achieved results generalize some well known facts especially from conformal, projective, and CR geometries; cf. [**2**], [**15**], [**16**], and others.

4. General results

Let $\mathcal{G} \to M$ be the principal bundle of a parabolic geometry of type (G, P) in the sense of 3.1. We will prove the order of jet determining generalized geodesics of an arbitrary type uniquely is finite. In other words, for any $A \subseteq \mathfrak{g}_-$ there is a finite r such that generalized geodesics of type $\mathcal{C}_A$ are given by a single value in $T^r_{\mathcal{C}_A}M$. The functor $T^r_{\mathcal{C}_A}$ is defined in 2.6. Obviously, the order must be at least 2 since parabolic geometries are never reductive.

Further, we deduce some technical lemma on reparametrizations which will serve us to describe possible reparametrizations for $|1|$–graded geometries in Section 5 and for other particular examples in Section 7.

4.1. Setup. With regard to the ideas of 2.7–2.8, we may restrict ourselves only to curves in the homogeneous space G/P going through the origin $o = eP$. In that case, generalized geodesics of type $\mathcal{C}_A$ look like $c^{b,X}(t) = b\exp(tX)\cdot P$ for any $b \in P$ and $X \in A \subseteq \mathfrak{g}_-$. Further we can restrict our attention to specific expressions for generalized geodesics as described in the following paragraphs.

First of all, let us remind that $\mathrm{Conj}_b \circ \exp = \exp \circ \mathrm{Ad}_b$ for all $b \in P$, so $b\exp(tX) = \exp(t\,\mathrm{Ad}_b\,X)b$ and any generalized geodesic through the origin can be written as $c^{b,X}(t) = \exp(t\,\mathrm{Ad}_b\,X)\cdot P$. Further, for any element $b \in P$ there are unique elements $b_0 \in G_0$ and $Z \in \mathfrak{p}_+$ such that $b = b_0\exp(Z)$. Thus, the same element can be written as well as $b = \exp(\mathrm{Ad}_{b_0} Z)b_0$ and from the definition of generalized geodesics one can conclude that curves $c^{b_0\exp Z, X}$ and $c^{\exp(\mathrm{Ad}_{b_0} Z), \mathrm{Ad}_{b_0} X}$ coincide. Hence any generalized geodesic can be expressed in the form $c^{\exp Z, Y}$ for some $Z \in \mathfrak{p}_+$ and $Y \in \mathfrak{g}_-$. If we consider geodesics of type $\mathcal{C}_A$, $A \subset \mathfrak{g}_-$, we have to suppose the subset A is G_0–invariant for the above elimination to be valid. This convention will be kept hereafter.

Below we will ask when two generalized geodesics c^{b_1,X_1}, c^{b_2,X_2} coincide and have a contact of some order in the origin, respectively. Obviously, the above two curves coincide if and only if the same is true for the curves c^{e,X_1} and $c^{b_1^{-1}b_2,X_2}$. Similarly, r–jets of the former curves in 0 equal if and only if r–jets in 0 of the later curves equal. So we may suppose $b_1 = e$ in the sequel. Altogether, if we can prove that for a G_0–invariant subset $A \subseteq \mathfrak{g}_-$ and some r the curves $c^{e,X}$ and $c^{\exp Z,Y}$ (with $X, Y \in A$ and $Z \in \mathfrak{p}_+$) having the same r–jet in 0 are equal, then this will mean that all geodesics of type $\mathcal{C}_A$ are uniquely determined by its r–jet in one point.

4.2. Technicalities. With respect to the above setting, let us compare generalized geodesics $c^{e,X}$ and $c^{\exp Z,Y}$. The two curves coincide if and only if the curve $u : \mathbb{R} \to G$, uniquely defined by the equation

$$\exp(tX) = \exp(t \operatorname{Ad}_{\exp Z} Y) \cdot u(t), \tag{1}$$

takes values in P. Since the map exp is analytic, the curve u is analytic too for arbitrary entries X, Y, and Z, so the previous statement is equivalent to the requirement that all derivatives $u^{(i)}(0) = \frac{d^i}{dt^i}\big|_0 u(t)$ are tangent to P. In order to make the latter requirement more precise, we shall differentiate the map δu instead of u, where $\delta u : T\mathbb{R} \to \mathfrak{g}$ is the left logarithmic derivative of u defined in 2.3 by $\delta u = u^*\omega$. Let us remind the convention that δu is actually understood as a map $\mathbb{R} \to \mathfrak{g}$ so that $\delta u(s) = \omega(T_s u \cdot 1) = T\ell_{u(s)^{-1}} \cdot u'(s)$ for all s. This obviously implies that $u^{(i)}(0) \in \mathfrak{p}$ for all $i \leq r$ if and only if $(\delta u)^{(i)}(0) \in \mathfrak{p}$ for all $i \leq r - 1$. Hence the Lemma follows.

Lemma. *Let u be the curve defined by (1) and r be arbitrary. Then curves $c^{e,X}$ and $c^{\exp Z,Y}$ have the same r–jet in 0 if and only if the derivatives $(\delta u)^{(i)}(0)$ belong to $\mathfrak{p}$ for all $i \leq r - 1$.*

Next, according to the general formula (6) in 2.3, we get for curves of the shape $Y(t) = tY$ the derivative $\delta(\exp \circ Y)(t) = Y$ constant. Hence, applying the left logarithmic derivative to equation (1), taking into account the Leibniz rule 2.3(4), yields

$$\delta u(t) = X - \operatorname{Ad}_{u(t)^{-1}} \operatorname{Ad}_{\exp Z} Y. \tag{2}$$

In particular, the condition $\delta u(0) \in \mathfrak{p}$ is satisfied if and only if X and $\operatorname{Ad}_{\exp Z} Y$ represent the same class in $\mathfrak{g}/\mathfrak{p}$, i.e. the curves really have the same tangent vector in the origin. Further, $(\delta u)'(0) = -\operatorname{ad}(-u'(0)) \operatorname{Ad}_{\exp Z} Y = [u'(0), \operatorname{Ad}_{\exp Z} Y]$ and for $u'(0) = \delta u(0) = X - \operatorname{Ad}_{\exp Z} Y$ we obtain $(\delta u)'(0) = [\delta u(0), X]$. The same equality holds for all $t \in \mathbb{R}$ and, moreover, there is a general formula for $(\delta u)^{(i)}(t)$ of any order, as the following Lemma shows.

4.3. Lemma. *In the setting above, the following equality holds for all $i \geq 1$,*

$$(\delta u)^{(i)}(t) = \operatorname{ad}(-X)^i \delta u(t). \tag{3}$$

Proof. Let us start with the first order derivative, so we have to prove $(\delta u)'(t) = [\delta u(t), X]$. In order to do this we have to compute the derivative of $\operatorname{Ad} \circ \nu \circ u : \mathbb{R} \to GL(\mathfrak{g})$ which maps $t \mapsto \operatorname{Ad}_{u(t)^{-1}}$. Clearly, the derivative of this map is $(T \operatorname{Ad} \circ T\nu)(u'(t))$, so we are going to express $T_g\nu$ and $T_g \operatorname{Ad}$ in general.

From $r_g \circ \nu \circ \ell_g = \nu$ we have $T_{g^{-1}} r_g \circ T_g \nu \circ T_e \ell_g = T_e \nu$, thus $T_g \nu = -T_e r_{g^{-1}} \circ T_g \ell_{g^{-1}}$. Similarly, $\operatorname{Ad} \circ \ell_g = \operatorname{Ad}_g \circ \operatorname{Ad}$ implies $T_g \operatorname{Ad} \circ T_e \ell_g = \operatorname{Ad}_g \circ T_e \operatorname{Ad}$, so $T_g \operatorname{Ad} = \operatorname{Ad}_g \circ \operatorname{ad} \circ T_g \ell_{g^{-1}}$. Altogether,

$$\tfrac{d}{dt} \operatorname{Ad}_{u(t)^{-1}} = (\operatorname{Ad}_{u(t)^{-1}} \circ \operatorname{ad} \circ T\ell_{u(t)}) \circ (-Tr_{u(t)^{-1}} \circ T\ell_{u(t)^{-1}})(u'(t)).$$

Since $\operatorname{Ad}_g = T_e(\ell_g \circ r_{g^{-1}})$ and $\delta u(t) = T\ell_{u(t)^{-1}} \circ u'(t)$, the latter expression reads as $\frac{d}{dt} \operatorname{Ad}_{u(t)^{-1}} = (-\operatorname{Ad}_{u(t)^{-1}} \circ \operatorname{ad} \circ \operatorname{Ad}_{u(t)})(\delta u(t))$ and so

$$(\delta u)'(t) = \operatorname{Ad}_{u(t)^{-1}}[\operatorname{Ad}_{u(t)} \delta u(t), \operatorname{Ad}_{\exp Z} Y] = [\delta u(t), \operatorname{Ad}_{u(t)^{-1}} \operatorname{Ad}_{\exp Z} Y].$$

Substituting $\operatorname{Ad}_{u(t)^{-1}} \operatorname{Ad}_{\exp Z} Y = X - \delta u(t)$ from (2), the claim follows.

Now, let $i > 1$ and assume that the formula is valid for all orders less then i. Then

$$(\delta u)^{(i)}(t) = \tfrac{d}{dt} \operatorname{ad}(-X)^{i-1} \delta u(t)$$

and since $\operatorname{ad}(-X)^{i-1}$ is linear and $(\delta u(t))'$ is already computed, we arrive at

$$(\delta u)^{(i)}(t) = \operatorname{ad}(-X)^{i-1}(\delta u(t))' = \operatorname{ad}(-X)^{i} \delta u(t),$$

which is the required formula. □

An easy consequence of this Lemma is the finiteness of the order of jets determining generalized geodesics. An estimate depending on the length of grading of the Lie algebra is deduced in the following Proposition but it is not sharp at all. Better estimates depend especially on the type of generalized geodesics and they will be improved for a number of cases in next sections.

4.4. Proposition. *Let k be the length of the grading of a Lie algebra $\mathfrak{g}$. If two geodesics of an arbitrary type have the same $(k+2)$–jet in one point then they coincide.*

Proof. Let us consider $A = \mathfrak{g}_-$. Clearly this choice provides the estimate for all $A \subset \mathfrak{g}_-$. With respect to the setting in 4.1 we have to show that curves $c^{e,X}$ and $c^{\exp Z,Y}$ coincide if they share the same $(k+2)$–jet in 0. Let $u : \mathbb{R} \to G$ be the curve determined by equation 4.2(1). By Lemmas 4.3 and 4.2, the assumption on the $(k+2)$–jet in 0 implies that $\operatorname{ad}(-X)^i(\delta u(0)) \in \mathfrak{p}$ for all $i \le k+1$. Since $X \in \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ and $\delta u(0) \in \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$, compatibility of Lie bracket with the grading implies that $\operatorname{ad}(-X)^i \delta u(0)$ lies in $\mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{k-i}$. In particular, for $i = k+1$ we see that $\operatorname{ad}(-X)^{k+1}(Z)$ has to lie both in $\mathfrak{g}_-$ and in $\mathfrak{p}$, so it must be zero. This implies that $(\delta u)^{\ell}(0) = 0 \in \mathfrak{p}$ for all $\ell > k+1$ and thus curves $c^{e,X}$ and $c^{\exp Z,Y}$ coincide. □

4.5. Reparametrizations. Here we shall discuss when are two generalized geodesics equal up to a change of parametrization, i.e. we consider unparametrized geodesics of a certain type on which a class of preferred parametrizations appears. We will prove a general technical lemma which will allow us to find the group of admissible reparametrizations in some particular cases.

First we have to modify the basic equation (1) as follows. Generalized geodesics $c^{e,X}$ and $c^{\exp Z,Y}$ coincide up to some reparametrization if and only if there exist mappings $u : \mathbb{R} \to P$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\exp(\varphi(t)X) = \exp(t\,\mathrm{Ad}_{\exp Z} Y) \cdot u(t). \tag{4}$$

In order to φ be a local reparametrization we require $\varphi'(t) \neq 0$ and, for simplicity, $\varphi(0) = 0$.

First of all, the left logarithmic derivative of $\delta(\exp \circ Y)(t)$ for $Y(t) = \varphi(t)X$ equals to $\varphi'(t)X$ by the formula (6) in 2.3. Now, with the Leibniz rule as above, the left logarithmic derivative of (4) gives

$$\delta u(t) = \varphi'(t)X - \mathrm{Ad}_{u(t)^{-1}} \mathrm{Ad}_{\exp Z} Y. \tag{5}$$

Obviously, the condition $\delta u(0) \in \mathfrak{p}$ is satisfied if and only if tangent vectors of the two curves in the origin equal up to a scalar multiple $\varphi'(0)$. In the proof of 4.3 we have shown the equality $\frac{d}{dt} \mathrm{Ad}_{u(t)^{-1}} W = -[\delta u, \mathrm{Ad}_{u(t)^{-1}} W]$ holds for any $W \in \mathfrak{g}$. Now the derivative $(\delta u)'$, after some substituting from (5), looks like

$$(\delta u)'(t) = \varphi''(t)X - \varphi'(t)[X, \delta u(t)]. \tag{6}$$

Inductively, we get a recursive formula without any heavy effort,

$$(\delta u)^{(i)}(t) = \varphi^{(i+1)}(t)X - \sum_{k=1}^{i} \binom{i-1}{k-1} \varphi^{(k)}(t)[X, (\delta u)^{(i-k)}(t)]. \tag{7}$$

Another way of expressing $(\delta u)^{(i)}$ is to substitute $(\delta u)'$ from (6) in each step of the induction, which leads to the following Lemma.

Lemma. *With the notation as above, omitting the variable t, the following holds for all $i \geq 1$,*

$$(\delta u)^{(i)} = \varphi^{(i+1)}X + \sum_{k=1}^{i} \sum_{J,A} (-1)^k c_{J,A} (\varphi^{(j_1)})^{a_1} \dots (\varphi^{(j_s)})^{a_s} \,\mathrm{ad}(X)^k(\delta u), \tag{8}$$

where the internal sum runs over all s–tuples of positive integers $J = (j_1, \dots, j_s)$ satisfying $j_1 < j_2 < \dots < j_s$ and s–tuples of arbitrary positive integers $A = (a_1, \dots, a_s)$ such that $a_1 j_1 + \dots + a_s j_s = i$ and $a_1 + \dots + a_s = k$. The coefficients $c_{J,A}$ are

$$c_{J,A} = \frac{i!}{(j_1!)^{a_1} \dots (j_s!)^{a_s} a_1! \dots a_s!}.$$

Let us describe in words what the individual terms in the general formula mean. The value of k says how many times φ occurs in the term in question (and so many times X hits δu via the adjoint action and the sign is set appropriately), while the coefficients $c_{A,J}$ express in how many different ways we may split i derivatives onto k copies of φ's in order to achieve the result $(\varphi^{(j_1)})^{a_1} \dots (\varphi^{(j_s)})^{a_s}$. Now, the differentiation of this formula and substitution from (6) means that we perform the last derivative on one of the φ's in the individual terms in the formula, or we attach a new φ, which is differentiated only once, to the existing terms.

But this is exactly how all splittings of $i+1$ (distinguishable) hits of k (undistinguishable) targets are obtained from the answers to the same question for i derivatives and k or $k-1$ targets. Either the last hit has been added to some existing one among k targets, i.e. we use the answer with i hits and k targets, or we have had to introduce a new target which was hit once, i.e. we used the answer with i hits and $k-1$ targets.

The above ideas provide a way to count the coefficients $c_{J,A}$. Now we are ready to prove the above Lemma by induction.

Proof of Lemma. First of all, let us rewrite the formula (8) as

$$(\delta u)^{(i)} = \varphi^{(i+1)}X + \sum_B c_B(\varphi')^{b_1}\dots(\varphi^{(i)})^{b_i}\,\mathrm{ad}(X)^{|B|}(\delta u), \tag{9}$$

where the sum runs over all i–tuples of nonnegative integers $(b_1,\dots,b_i)=B$ which satisfy $b_1+2b_2+\cdots+ib_i=i$. In particular, b_i can take only two values, either 1 or 0. Symbol $|B|$ denotes the sum $b_1+\cdots+b_i$ and

$$c_B = \frac{(-1)^{|B|}\; i!}{(2!)^{b_2}\dots(i!)^{b_i}\,b_1!\dots b_i!}.$$

Let us start with the case $i=1$. Then the entire sum has just one possible term for $b_1=1$, which gives the correct formula (6). Inductively, let us assume the formula holds for $i\geq 1$. We are going to prove the same is true for $i+1$.

Let $W = c_B\cdot(\varphi')^{b_1}\dots(\varphi^{(i+1)})^{b_{i+1}}\,\mathrm{ad}(X)^{|B|}(\delta u)$ be a summand of $(\delta u)^{(i+1)}$ expressed according to (9), i.e. $\sum_{\ell=1}^{i+1}\ell\, b_\ell = i+1$ and $|B|=\sum_{\ell=1}^{i+1}b_\ell$. Any such W, written as $W=c_B\cdot W_B$, arises by assembling certain terms from the derivative of $(\delta u)^{(i)}$. If $b_{i+1}=1$ then $b_1=\cdots=b_i=0$, $c_B=1$, and so $W=\varphi^{(i+1)}[X,\delta u]$. In that case, W_B appears only in the derivative of $\varphi^{(i)}[X,\delta u]$ and the coefficient agrees.

For $b_{i+1}=0$, all possible contributions to W occur as follows. For any $\ell=2,\dots,i$ such that $b_\ell>0$, the term W_B appears in the derivative of the power

$$c_{B_\ell}(\varphi')^{b_1}\dots(\varphi^{(\ell-1)})^{b_{\ell-1}+1}(\varphi^{(\ell)})^{b_\ell-1}\dots(\varphi^{(i)})^{b_i}\,\mathrm{ad}(X)^{|B_\ell|}(\delta u)$$

with the coefficient $(b_{\ell-1}+1)c_{B_\ell}$. Obviously, $|B_\ell|=|B|$ for all mentioned ℓ. If $b_1>0$, the last contribution comes from

$$c_{B_1}(\varphi')^{b_1-1}(\varphi'')^{b_2}\dots(\varphi^{(i)})^{b_i}\,\mathrm{ad}(X)^{|B_1|}(\delta u)',$$

where clearly $|B_1|=|B|-1$. Substituting $(\delta u)'$ from (6) really yields W_B with the coefficient $-c_{B_1}$.

In order to complete the proof we have to show that the sum of all coefficients obtained above equals to c_B. By the inductive presumption, all mentioned coefficients have the appropriate form, i.e.

$$c_{B_1} = \frac{(-1)^{|B_1|}\; i!}{(2!)^{b_2}\dots(i!)^{b_i}\;(b_1-1)!\dots b_i!}$$

and, for each $\ell = 2, \dots, i$,

$$c_{B_\ell} = \frac{(-1)^{|B|}\ i!}{(2!)^{b_2} \dots ((\ell-1)!)^{b_{\ell-1}+1} (l!)^{b_\ell - 1} \dots (i!)^{b_i}\ b_1! \dots (b_{\ell-1}+1)!(b_\ell - 1)! \dots b_i!}.$$

Now we can summarize the coefficients

$$-c_{B_1} + \sum_{\ell=2}^{i} (b_{\ell-1}+1) c_{B_\ell} = \frac{(-1)^{|B|}\ i!}{(2!)^{b_2} \dots (i!)^{b_i}\ b_1! \dots b_i!} \left(b_1 + \sum_{\ell=2}^{i} \frac{\ell!\, b_\ell}{(\ell-1)!} \right).$$

Obviously, the sum in brackets is $\sum_{\ell=1}^{i} \ell\, b_\ell$ which equals to $i+1$ due to the condition $b_{i+1} = 0$. Hence the right hand side is really c_B, which completes the proof. □

5. Irreducible geometries

Irreducible parabolic geometries enjoy the most understandable systems of generalized geodesics. The main consequence of the $|1|$–grading of the Lie algebra $\mathfrak{g}$ is that the subgroup $\exp \mathfrak{p}_+ \subset P$ acts trivially on $\mathfrak{g}_-$ so the isotropy representation of the structure group P on $\mathfrak{g}/\mathfrak{p} = \mathfrak{g}_-$ factorizes over G_0. Indeed, for any $X \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{p}_+ = \mathfrak{g}_1$, the $\mathfrak{g}_-$ part of $\mathrm{Ad}_{\exp Z} X = X + [ZX] + \frac{1}{2}[Z[ZX]]$ is just X. This fact brings a lot of simplifications in computations which allow us to refine the estimate from Proposition 4.4, to find the possible reparametrizations of any geodesic of an arbitrary type, and moreover, to express the space of initial conditions very explicitly.

5.1. Proposition. *Each generalized geodesic in a $|1|$–graded parabolic geometry is uniquely determined by its 2–jet in one point.*

Proof. Let $u : \mathbb{R} \to G$ be the curve determined by geodesics $c^{e,X}$, $c^{\exp Z, Y}$, and equation 4.2(1). For a $|1|$–graded geometry the condition $\delta u(0) \in \mathfrak{p}$ reads as $X = Y$ and so $\delta u(0) = -[ZX] - \frac{1}{2}[Z[ZX]]$. Following Lemma 4.3 we get

$$(\delta u)'(0) = [X[ZX]] + \tfrac{1}{2}[X[Z[ZX]]],$$

which belongs to $\mathfrak{p}$ if and only if $[X[ZX]] = 0$. This implies $[X[Z[ZX]]] = [[XZ][ZX]] + [Z[X[ZX]]] = 0$, thus $(\delta u)'(0) = 0$. Altogether, presumptions $\delta u(0) \in \mathfrak{p}$ and $(\delta u)'(0) \in \mathfrak{p}$ imply $(\delta u)^{(i)}(0) = 0$ for all $i \geq 2$ so the statement holds true. □

Next paragraphs provide an explicit description of spaces $T^r_{\mathcal{C}_A} M$ up to the third order which enables us, in addition, to recover the above refinement in a very visible way, see 5.3. Unfortunately, a similar process is much more complicated for longer gradings.

5.2. Jet bundles. Let us begin with an alternative description of the jet bundle $T^r_1 M = J^r_0(\mathbb{R}, M)$ in the case of homogeneous spaces $M = G/P$. Recall that $T^r_1 M = P^r M \times_{G^r_m} J^r_0(\mathbb{R}, \mathbb{R}^m)_0$, where $P^r M = J^r_0(\mathbb{R}^m, M)$ is the bundle of r–frames on M and both actions of the structure group $G^r_m = \mathrm{inv} J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ are given by the jet compositions. For an arbitrary local diffeomorphism $u : \mathbb{R}^m \to M$

and any path $c : \mathbb{R} \to M$ with $u(0) = c(0)$, the above identification is indicated as $j^r_0 c \mapsto \{j^r_0 u, j^r_0(u^{-1} \circ c)\}$. Further, the standard fiber $J^r_0(\mathbb{R}, \mathbb{R}^m)_0$ is obviously identified with $(\mathbb{R}^m)^r$ due to the transcription $j^r_0 c \mapsto (c'(0), \dots, c^{(r)}(0))$. Now we can write $T^r_1 M = P^r M \times_{G^r_m} (\mathbb{R}^m)^r$.

Next, for the homogeneous bundle $p : G \to G/P$ of a parabolic geometry, there is the standard principal bundles' homomorphism $\iota : G \to P^r(G/P)$ over the homomorphism of Lie groups $i : P \to G^r_m$ given by the formulae

$$\iota(g) = j^r_0(\ell_g \circ \varepsilon) \text{ and } i(b) = j^r_0(\varepsilon^{-1} \circ \ell_b \circ \varepsilon).$$

The map ε is the local diffeomorphism $p \circ \exp|_{\mathfrak{g}_-} : \mathfrak{g}_{-1} \to G/P$, i.e. the normal coordinates in $e \in G$ in the sense of 3.6. If r is at least 2 then the homomorphism ι is a reduction of the bundle $P^r(G/P)$ to the structure group P, see [**18**] for the proof. In that case we can write $T^r(G/P) = G \times_P (\mathfrak{g}_{-1})^r$, where the left action $\lambda : P \times (\mathfrak{g}_{-1})^r \to (\mathfrak{g}_{-1})^r$ is fully determined by the action of G^r_m on $J^r_0(\mathbb{R}, \mathbb{R}^m)_0$. Below we describe the action λ explicitly up to the third order. In particular, for $r = 1$ we will recover the truncated adjoint action of P on $\mathfrak{g}_{-1}$ defining the tangent bundle $T(G/P) = G \times_P \mathfrak{g}_{-1}$.

For any $b \in P$, let us denote by $\psi_b = \varepsilon^{-1} \circ \ell_b \circ \varepsilon$ the local diffeomorphism around $0 \in \mathfrak{g}_{-1}$ which appears in the definition of the homomorphism $i : P \to G^r_m$. Then $\lambda(b)$ is completely given by $j^r_0 \psi_b$ so that

$$\lambda(b)(j^r_0 c) = j^r_0(\psi_b \circ c). \tag{1}$$

Obviously, for any $X \in \mathfrak{g}_{-1}$ we have $p(b \exp X) = b \exp X \cdot P = \exp(\mathrm{Ad}_b X) \cdot P$ and so $\psi_b(X) = \varepsilon^{-1}(\exp(\mathrm{Ad}_b X) \cdot P)$. In other words, $\psi_b(X)$ is the unique $Y \in \mathfrak{g}_{-1}$ satisfying

$$\exp(\mathrm{Ad}_b X) \cdot P = \exp(Y) \cdot P. \tag{2}$$

It is not easy to find the image $\psi_b(X)$ in general, but we need the r–jet of this map only. This is performed for $r = 3$ in the following Lemma.

Lemma. *The action $\lambda : P \times (\mathfrak{g}_{-1})^3 \to (\mathfrak{g}_{-1})^3$ defined above is given by the formula*

$$\lambda(b_0 \exp Z) \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \mathrm{Ad}(b_0) \begin{pmatrix} X_1 \\ X_2 - [X_1[ZX_1]] \\ X_3 - 3[X_1[ZX_2]] + \frac{3}{2}[[X_1[ZX_1]][ZX_1]] \end{pmatrix}, \tag{3}$$

where $b_0 \in G_0$, $Z \in \mathfrak{g}_1$, and $\mathrm{Ad}(b_0)(X_1, X_2, X_3) = (\mathrm{Ad}_{b_0} X_1, \mathrm{Ad}_{b_0} X_2, \mathrm{Ad}_{b_0} X_3)$.

Proof. Inspired by [**23**, 2.35]. Recall that $\lambda(b)(j^3_0 c) = j^3_0(\psi_b \circ c)$ and let us assume the 3–jet of c in 0 is (X_1, X_2, X_3). For any $b \in P$, uniquely written as $b = b_0 \exp Z$ with $b_0 \in G_0$ and $Z \in \mathfrak{g}_1$, we have $\psi_b = \psi_{b_0} \circ \psi_{\exp Z}$. The understanding of the action of b_0 is effortless, since $\psi_{b_0} = \mathrm{Ad}(b_0)$ is linear and keeps $\mathfrak{g}_{-1}$ invariant. Hence, $\lambda(b_0)(X_1, X_2, X_3) = (\mathrm{Ad}_{b_0} X_1, \mathrm{Ad}_{b_0} X_2, \mathrm{Ad}_{b_0} X_3)$.

Now we are going to describe the action of $\exp Z$, so we are dealing with an expression

$$\exp(\mathrm{Ad}_{\exp Z} X) = \exp\big(X + [ZX] + \tfrac{1}{2}[Z[ZX]]\big).$$

Let us denote $A = \mathrm{Ad}_{\exp Z} X$ and write $\exp A = \exp X \cdot \exp(-X) \cdot \exp A$. Then using the Baker–Campbell–Hausdorff formula [**17**, 4.29] yields

$$\exp A = \exp X \cdot \exp\big(-X + A - \tfrac{1}{2}[XA] + \tfrac{1}{12}([X[XA]] - [A[AX]]) + r(\mathfrak{p}) + r(4)\big),$$

where $r(\mathfrak{p})$ represents terms in $\mathfrak{p}$ and $r(4)$ represents terms containing at least 4 copies of X. Expanding this equality gives

$$\begin{aligned}\exp A = \exp X \cdot \exp\big(&-\tfrac{1}{2}[X[ZX]] + \tfrac{1}{12}(2[X[X[ZX]]] + [X[X[Z[ZX]]]] + \\ &+ [[ZX][X[ZX]]]) + [ZX] + \tfrac{1}{2}[Z[ZX]] + r(\mathfrak{p}) \cap r(2) + r(4)\big).\end{aligned}$$

Now, since $[X[X[ZX]]] = 0$ and $[X[Z[ZX]]] = [Z[X[ZX]]]$, taken into account the Jacobi identity, the sum $[X[X[Z[ZX]]]] + [[ZX][X[ZX]]]$ vanishes. Hence we can write

$$\exp A = \exp X \cdot e \cdot \exp\big(-\tfrac{1}{2}[X[ZX]] + [ZX] + \tfrac{1}{2}[Z[ZX]] + r(\mathfrak{p}) \cap r(2) + r(4)\big).$$

Next we substitute $e = \exp(-\frac{1}{2}[X[ZX]]) \cdot \exp(\frac{1}{2}[X[ZX]])$ and consecutively $e = \exp(\frac{1}{4}[[X[ZX]][ZX]]) \cdot \exp(-\frac{1}{4}[[X[ZX]][ZX]])$ in order to get

$$\begin{aligned}\exp(\mathrm{Ad}_{\exp Z} X) &= \exp\big(X - \tfrac{1}{2}[X[ZX]]\big) \cdot e \cdot \exp\big(\tfrac{1}{4}[[X[ZX]][ZX]] + r(\mathfrak{p}) + r(4)\big) = \\ &= \exp\big(X - \tfrac{1}{2}[X[ZX]] + \tfrac{1}{4}[[X[ZX]][ZX]]\big) \cdot \exp\big(r(\mathfrak{p}) + r(4)\big).\end{aligned}$$

Finally, any 3–jet (X_1, X_2, X_3) is represented by the curve $Y(t) = tX_1 + \frac{1}{2}t^2 X_2 + \frac{1}{6}t^3 X_3$ which, plugged into the latter equality instead of X, gives

$$\begin{aligned}\exp(\mathrm{Ad}_{\exp Z} Y(t)) = \exp\big(&tX_1 + t^2(\tfrac{1}{2}X_2 - \tfrac{1}{2}[X_1[ZX_1]]) + \\ &+ t^3(\tfrac{1}{6}X_3 - \tfrac{1}{2}[X_1[ZX_2]] + \tfrac{1}{4}[[X_1[ZX_1]][ZX_1]]) + r(t^4)\big). \exp\big(r(\mathfrak{p}) + r(t^4)\big).\end{aligned}$$

Hence

$$j^3_0(\psi_{\exp Z} \circ Y) = \big(X_1, X_2 - [X_1[ZX_1]], X_3 - 3[X_1[ZX_2]] + \tfrac{3}{2}[[X_1[ZX_1]][ZX_1]]\big),$$

which completes the proof. □

5.3. Distinguished jets. Let $c : \mathbb{R} \to G/P$ be a curve with $c(0) = gP$. The 3–jet of the curve in 0 is clearly written as $j^3_0 c = \{j^3_0(\ell_g \circ \varepsilon), j^3_0(\varepsilon^{-1} \circ \ell_{g^{-1}} \circ c)\} \in P^3(G/P) \times_{G^3_m} J^3_0(\mathbb{R}, \mathfrak{g}_{-1})_0$. With respect to the reduction $\iota : G \to P^3(G/P)$ and other constructions in 5.2, we can write $j^3_0 c = \{g, (X_1, X_2, X_3)\} \in G \times_P (\mathfrak{g}_{-1})^3$, where $X_i = \frac{d^i}{dt^i}\big|_0(\varepsilon^{-1} \circ \ell_{g^{-1}} \circ c)$. Another representative of $c(0) = gP$ certainly defines the same 3–jet. If c is a generalized geodesic $c = c^{g,X}$ then $c(0) = gP$ and $(\varepsilon^{-1} \circ \ell_{g^{-1}} \circ c)(t) = tX$. Hence the 3–jet has got a rather nice form

$$j^3_0 c^{g,X} = \{g, (X, 0, 0)\}.$$

Let $c^{e,X}$ and $c^{\exp Z,Y}$ be two generalized geodesics as in 4.1. The corresponding 3–jets in 0 are $j^3_0 c^{e,X} = \{e, (X, 0, 0)\}$ and $j^3_0 c^{\exp Z,Y} = \{\exp Z, (Y, 0, 0)\} =$

$\{e, \lambda(\exp Z)(Y, 0, 0)\}$, respectively. Due to Lemma 5.2, one can easily recompute the coordinates if the frame has been changed. So we obtain

$$j_0^3 c^{e,X} = \left\{ e, \begin{pmatrix} X \\ 0 \\ 0 \end{pmatrix} \right\} \text{ and } j_0^3 c^{\exp Z,Y} = \left\{ e, \begin{pmatrix} Y \\ -[Y[ZY]] \\ \frac{3}{2}[[Y[ZY]][ZY]] \end{pmatrix} \right\}. \tag{4}$$

Now the curves share the same 2–jet in 0 if and only if $X = Y$ and $[X[ZX]] = 0$. But then the 3–jets equal too, thus by Proposition 4.4 the two curves coincide. So we have recovered the refinement from 5.1 in a very explicit way.

At the same time, for a G_0–invariant subset $A \subseteq \mathfrak{g}_{-1}$ we can easily describe the space $T^r_{\mathcal{C}_A}(G/P) = \{j_0^r c^{g,X} : g \in G, X \in A\}$ defined in 2.6. In the setting above, $T^r_{\mathcal{C}_A}(G/P) = \big\{\{g, (X, 0, \dots, 0)\} : g \in G, X \in A\big\}$, so

$$T^r_{\mathcal{C}_A}(G/P) = G \times_P S^r_A$$

where the standard fiber S^r_A is just the P–orbit of the subset $A \times \{0\} \times \dots \times \{0\} \subset (\mathfrak{g}_{-1})^r$ with respect to the action λ defined in 5.2. Especially, the standard fiber of the functor $T^3_{\mathcal{C}_A}$ can be written as

$$S^3_A = \left\{ \begin{pmatrix} X \\ [X[XZ]] \\ \frac{3}{2}[[X[ZX]][ZX]] \end{pmatrix} : Z \in \mathfrak{g}_1, X \in A \right\}$$

due to the G_0–invariance of A. This description shows most clearly that the subspace $T^r_{\mathcal{C}_A}(G/P) \subseteq T^r(G/P)$ is not a smooth submanifold in general since its fibers are algebraic submanifolds in $(\mathfrak{g}_{-1})^r$ only. We shall meet explicit examples in a while. An alternative way to the same result, based on the techniques developed in Section 4, can be found in [**8**].

Let $\mathcal{C}^\xi_A$ be the set of geodesics of type $\mathcal{C}_A$ with a common tangent vector ξ as suggested in 2.6. Without lost of generality we may suppose $\xi = \{e, X \in A\}$. For each $r > 2$, an arbitrary element of $T^r_{\mathcal{C}_A}(G/P)$ is completely determined by its projection to $T^2_{\mathcal{C}_A}(G/P)$, so the set $\mathcal{C}^\xi_A$ is naturally parametrized by the preimage of ξ with respect to the jet projection $\pi^2_1 : T^2_{\mathcal{C}_A}(G/P) \to T_{\mathcal{C}_A}(G/P)$. With respect to the setting above, the projection π^2_1 is given by the identity on G and the P–equivariant projection $S^2_A \to S^1_A = A$ onto the first slot. Hence the preimage $(\pi^2_1)^{-1}\{e, X\}$ is identified with the image of the mapping $(\operatorname{ad} X)^2\big|_{\mathfrak{g}_1} : \mathfrak{g}_1 \to \mathfrak{g}_{-1}$, which is isomorphic to $\mathfrak{g}_1 / \ker(\operatorname{ad} X)^2$. For similar discussions in more–graded geometries see 7.1 and consecutive examples.

5.4. Reparametrizations. Further, we can easily find necessary and sufficient conditions for two generalized geodesics $c^{e,X}$ and $c^{\exp Z,Y}$ to coincide up to a change of parametrization. With respect to Proposition 5.1 this is equivalent to the existence of a reparametrization φ such that $j_0^2(c^{e,X} \circ \varphi) = j_0^2 c^{\exp Z,Y}$, so we have to describe how reparametrizations effect jets of generalized geodesics.

Let us remind that $j_0^r c^{g,X} = \{g, (X, 0, \dots, 0)\}$, which has been obtained in 5.3 by differentiating of the curve $(\varepsilon^{-1} \circ \ell_{g^{-1}} \circ c^{g,X})(t) = tX$ in 0. Similarly, for a reparametrization $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(0) = 0$, the corresponding curve in $\mathfrak{g}_{-1}$ is

$(\varepsilon^{-1} \circ \ell_{g^{-1}} \circ c^{g,X} \circ \varphi)(t) = \varphi(t)X$ so the r–jet of the reparametrized geodesic looks like

$$j_0^r(c^{g,X} \circ \varphi) = \left\{ g, \begin{pmatrix} \varphi'(0)X \\ \varphi''(0)X \\ \vdots \end{pmatrix} \right\}.$$

Together with (4), the equality $j_0^2(c^{e,X} \circ \varphi) = j_0^2 c^{\exp Z,Y}$ holds if and only if

$$\varphi'(0)X = Y \text{ and } \varphi''(0)X = -[Y[ZY]]. \tag{5}$$

So we get the following Lemma.

Lemma. *The generalized geodesics $c^{e,X}$ and $c^{\exp Z,Y}$ in a $|1|$–graded parabolic geometry parametrize the same curve if and only if there is a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $c^{e,X} \circ \varphi$ is a generalized geodesic, $\varphi(0) = 0$, $\varphi'(0)X = Y$, and $\varphi''(0)X = -[Y[ZY]]$.*

Moreover, the same arguments as above yields that $j_0^3(c^{e,X} \circ \varphi) = j_0^3 c^{\exp Z,Y}$ if and only if we add

$$\varphi'''(0)X = \tfrac{3}{2}[[Y[ZY]][ZY]] \tag{6}$$

to the two conditions in (5). Substitution from (5) into (6) leads to the formula

$$\varphi'''(0) = \frac{3}{2} \frac{\varphi''(0)^2}{\varphi'(0)} \tag{7}$$

which reminds the Schwartzian differential equation well known from conformal geometries, see [**2**].

In order to describe the distinguished reparametrizations which appear in this way, we have to compute explicitly the consequences of all conditions $(\delta u)^{(i)}(0) \in \mathfrak{p}$, where u is determined by the equality (4) in 4.5.

5.5. Lemma. *If the generalized geodesics $c^{e,X}$ and $c^{\exp Z,Y}$ in a $|1|$–graded parabolic geometry parametrize the same curve then the corresponding reparametrization φ satisfies the following conditions,*

$$\varphi'(0)X = Y, \tag{8}$$

$$\frac{\varphi''(0)}{\varphi'(0)^2} X = [X[XZ]], \text{ and} \tag{9}$$

$$\varphi^{(i+1)}(0) = \frac{(i+1)!}{2^i} \frac{\varphi''(0)^i}{\varphi'(0)^{i-1}} \text{ for all } i \geq 2. \tag{10}$$

Proof. We shall evaluate explicitly the consequences of $(\delta u)^{(i)}(0) \in \mathfrak{p}$ for all i. The evaluation of 4.5(5) in 0 yields $\delta u(0) = \varphi'(0)X - (Y + [ZY] + \frac{1}{2}[Z[ZY]])$. Thus, $\delta u(0) \in \mathfrak{p}$ if and only if $\varphi'(0)X = Y$ and so (8) follows. In that case,

$$\delta u(0) = -\varphi'(0)\big([ZX] + \tfrac{1}{2}[Z[ZX]]\big). \tag{11}$$

Similarly, from 4.5(6) we get $(\delta u)'(0) = \varphi''(0)X - \varphi'(0)[X, \delta u(0)]$. After the substitution of (11) one can easily verify that the condition (9) is correct. Let us remind that conditions (8) and (9) just recover the two conditions in (5).

In order to complete the proof we need the general formula (8) from Lemma 4.5. Since our Lie algebra $\mathfrak{g}$ is $|1|$–graded, all iterated adjoint actions by X on $\delta u(0)$ vanish if the order is more then two. Thus only terms with $k \leq 2$ may survive and the general formula reads as

$$(\delta u)^{(i)} = \varphi^{(i+1)}X - \varphi^{(i)}[X, \delta u] + \frac{1}{2}\sum_{\ell=1}^{i-1} \tfrac{i!}{\ell!(i-\ell)!}\varphi^{(\ell)}\varphi^{(i-\ell)}[X[X, \delta u]],$$

where the variable t is omitted for the sake of lucidity. Now, evaluating in 0 and substituting from (11) yields

$$\begin{aligned}(\delta u)^{(i)} &= \varphi^{(i+1)}X + \varphi^{(i)}\varphi'[X[ZX]] - \\ &- \frac{1}{4}\sum_{\ell=1}^{i-1} \tbinom{i}{\ell} \varphi^{(\ell)}\varphi^{(i-\ell)}\varphi'[X[X[Z[ZX]]]] + \text{term in } \mathfrak{g}_0.\end{aligned}$$

Finally, due to the symmetry $[X[Z[ZX]]] = [Z[X[ZX]]]$ and the equality (9), for $i \geq 2$ we get

$$(\delta u)^{(i)} = \left(\varphi^{(i+1)} - \frac{\varphi^{(i)}\varphi''}{\varphi'} - \frac{1}{4}\sum_{\ell=1}^{i-1} \tbinom{i}{\ell} \varphi^{(\ell)}\varphi^{(i-\ell)}\frac{\varphi''^2}{\varphi'^3}\right)X + \text{ term in } \mathfrak{g}_0.$$

In particular, $(\delta u)^{(i)} \in \mathfrak{p}$ if and only if

$$\varphi^{(i+1)} = \frac{\varphi^{(i)}\varphi''}{\varphi'} + \frac{1}{4}\sum_{\ell=1}^{i-1} \tbinom{i}{\ell} \varphi^{(\ell)}\varphi^{(i-\ell)}\frac{\varphi''^2}{\varphi'^3}, \tag{12}$$

all evaluated in 0. For $i = 2$, the latter equation looks like $\varphi''' = \frac{3}{2}\frac{\varphi''^2}{\varphi'}$, which really agrees with the formula (10). Let us inductively suppose that $i > 2$ and the equality (10) holds up to $i - 1$, i.e. for all $\varphi^{(k)}$ with $k \leq i$. We have to show the same is true for i.

Obviously, $\varphi^{(i+1)}$ is determined uniquely in terms of $\varphi^{(k)}$ with $k \leq i$, so we can substitute $\varphi^{(k)} = \frac{k!}{2^{k-1}}\frac{\varphi''^{k-1}}{\varphi'^{k-2}}$ into (12) by the inductive assumption. Hence

$$\begin{aligned}\varphi^{(i+1)} &= \frac{i!}{2^{i-1}}\frac{\varphi''^{1+i-1}}{\varphi'^{1+i-2}} + \frac{1}{4}\sum_{\ell=1}^{i-1}\binom{i}{\ell}\frac{\ell!}{2^{\ell-1}}\frac{(i-\ell)!}{2^{i-\ell-1}}\frac{\varphi''^{2+\ell-1+i-\ell-1}}{\varphi'^{3+\ell-2+i-\ell-2}} = \\ &= \left(\frac{i!}{2^{i-1}} + \frac{1}{4}\sum_{\ell=1}^{i-1}\frac{i!}{\ell!(i-\ell)!}\frac{\ell!(i-\ell)!}{2^{i-2}}\right)\frac{\varphi''^i}{\varphi'^{i-1}}\end{aligned}$$

and the coefficient in the bracket simplifies to

$$\frac{i!}{2^{i-1}} + \frac{i-1}{4}\frac{i!}{2^{i-2}} = \frac{i!(2+i-1)}{2^i} = \frac{(i+1)!}{2^i}.$$

Hence the formula in (10) is correct. $\square$

Let us summarize what we have achieved so far. If the conditions of (8) and (9) are satisfied then $\varphi'(0)$ and $\varphi''(0)$ are determined by the choice of the tangent vectors to the curve and by the element $Z \in \mathfrak{g}_1$, respectively. All other derivatives of φ may be defined by the formula (10). In particular, the special case $i = 2$ yields

$$\varphi'''(0) = \frac{3}{2}\frac{\varphi''(0)^2}{\varphi'(0)},$$

which recovers the equality (7). We shall see that the formulae for $\varphi^{(i)}(0)$ determine an analytic local solution of the Schwartzian differential equation $\varphi''' = \frac{3}{2}\frac{\varphi''^2}{\varphi'}$.

5.6. Proposition. *Suppose that $\mathfrak{g}$ is $|1|$–graded. If two generalized geodesics coincide as unparametrized curves then the corresponding local reparametrization φ has the form $\varphi(t) = \frac{At+B}{Ct+D}$ where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. Conversely, if $c = c^{b,X}$ is a parametrized geodesic then all curves $c\circ\varphi$ with reparametrizations φ of the latter form are again geodesics if and only if there is $Z \in \mathfrak{g}_1$ such that $[X[XZ]] = X$.*

Proof. Let two generalized geodesics coincide up to a reparametrization φ. According to Lemma 5.5, all derivatives of the order greater than 3 of the function φ are expressed in terms of $\varphi'(0)$ and $\varphi''(0)$. Now the Taylor development of the function φ in 0 is

$$\varphi(t) = \varphi'(0)t + \frac{1}{2}\varphi''(0)t^2 + \frac{1}{6}\frac{3}{2}\frac{\varphi''(0)^2}{\varphi'(0)}t^3 + \dots + \frac{1}{(i+1)!}\frac{(i+1)!}{2^i}\frac{\varphi''(0)^i}{\varphi'(0)^{i-1}}t^{i+1} + \cdots$$

and the geometric series $\varphi(t) = \varphi'(0)t\sum_{i=0}^{\infty}\left(\frac{\varphi''(0)}{2\varphi'(0)}t\right)^i$ locally converges around 0 with the value

$$\varphi(t) = \varphi'(0)t\left(1 - \frac{\varphi''(0)}{2\varphi'(0)}t\right)^{-1}.$$

If we want to allow reparametrizations with $\varphi(0) \neq 0$, we have just to replace equation 4.5(4) by

$$\exp((\varphi(t) - \varphi(0))X) = \exp(t\,\mathrm{Ad}_{\exp Z}\, Y)\cdot u(t)$$

and the result differs only by adding the value $\varphi(0)$ to the fraction above. In that case the reparametrization takes the form

$$\varphi(t) = \frac{At+B}{Ct+D}, \text{ where } A = \varphi'(0) - \frac{\varphi''(0)}{2\varphi'(0)}\varphi(0),\ B = \varphi(0),\ C = -\frac{\varphi''(0)}{2\varphi'(0)},\ D = 1.$$

In particular, the solution with $\varphi''(0) = 0$ yields the affine reparametrization of the curve which of course have to be a geodesic too. Moreover, the determinant of the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is $\varphi'(0)$ which must be nonzero, so we may normalize this just to 1 and the first statement of the Proposition is proved.

For the second statement we may obviously restrict ourselves to the case when $\varphi(0) = 0$ and $b = e$. Let us fix $\varphi'(0)$ and $\varphi''(0)$ and choose Z with $[X[XZ]] =$

$\varphi''(0)X$ and $Y = \varphi'(0)X$. Then there is a solution of the Schwartzian differential equation with these initial data and so φ satisfies all conditions from Lemma 5.5. In particular, $\delta u^{(i)}(0) \in \mathfrak{p}$ for all i with this choice for φ and so φ is a reparametrization of $c^{e,X}$ and $c^{\exp Z,Y}$. □

In all cases we know, there always exists an element $Z \in \mathfrak{g}_1$ for each $X \in \mathfrak{g}_{-1}$ such that $[X[XZ]] = X$. This provides an invariant class of preferred reparametrizations on each generalized geodesic with the freedom of $SL(2,\mathbb{R})$, the projective group of line. Therefore reparametrizations of this type are called *projective*—in fact, they exhaust all nonsingular solutions of the Schwartzian differential equation.

Sometimes, for a fixed $X \in \mathfrak{g}_{-1}$ all elements $Z \in \mathfrak{g}_1$ satisfy $[X[XZ]] = X$, which implies that all geodesics in the given direction parametrize the same curve. In particular, all vectors in projective or null vectors in conformal geometries have got this property, see examples in 5.7.

Corollary. *The generalized geodesics $c^{e,X}$ and $c^{\exp Z,Y}$ in a $|1|$–graded parabolic geometry parametrize the same curve if and only if there are $a \neq 0$ and b such that $Y = aX$, and $[Y[YZ]] = bX$.*

Proof. The statement follows immediately from the latter Proposition and Lemma 5.4. The appropriate projective reparametrization φ is uniquely determined by the initial condition $\varphi(0) = 0$, $\varphi'(0) = a$, and $\varphi''(0) = b$. □

5.7. Examples. Let us recall that in the case of a $|1|$–grading all elements from $P_+ = \exp \mathfrak{g}_1$ act trivially on $T_o(G/P) = \mathfrak{g}/\mathfrak{p}$, so the P–action on this space factorizes over G_0. In particular, any G_0–invariant subset in $\mathfrak{g}_{-1}$ is actually P–invariant, hence the various types of distinguished geodesics are determined by the P–orbits of tangent vectors in $T_o(G/P)$. In other words, all geodesics tangent to vectors in a common P–orbit belong always to the same class (and thus behave the same).

Example 1. Conformal Riemannian structures. The principal group of the geometry corresponds to $G = O(p+1, q+1)$ and the parabolic subgroup P is the Poincaré conformal group, which can be indicated by a block upper triangular matrices with blocks of ranks 1, $p+q$, and 1, see e.g. [**26**, 4.4]. In an appropriate matrix representation, given by the inner product on $\mathbb{R}^{p+q+2}$ with the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & J & 0 \\ 1 & 0 & 0 \end{pmatrix}$ in the standard basis, the grading of the Lie algebra $\mathfrak{g}$ has the form

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^tJ & 0 \end{pmatrix} : X \in \mathbb{R}^{p+q} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a \end{pmatrix} : A \in \mathfrak{o}(p,q), a \in \mathbb{R}, \right\}, \ \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & Z & 0 \\ 0 & 0 & -JZ^t \\ 0 & 0 & 0 \end{pmatrix} : Z \in \mathbb{R}^{p+q*} \right\}.$$

Here J is the matrix defining the standard pseudo–metric of signature (p,q) on $\mathbb{R}^{p+q} = \mathfrak{g}_{-1}$.

Any element $\begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \in G_0$ acts on $\mathfrak{g}_{-1}$ by the transcription $X \mapsto a^{-1}AX$, where $A \in O(p,q)$. Hence the space $\mathfrak{g}_{-1}$ splits into three different orbits of the action of G_0 according to the sign of $\|X\|^2 = X^t JX$, so we get three distinct types of generalized geodesics which emanate in directions of positive, negative, and zero length, respectively. A direct calculation shows that $[X[XZ]] = -2Z(X)X - \|X\|^2 JZ^t$, where $Z(X) = ZX$ is a real number. The orbit of null vectors is of a particular interest since $[X[XZ]] = -2Z(X)X$ in that case. This just means that the family $\mathcal{C}_A^\xi$ of geodesics tangent to a null vector ξ is 1–dimensional and, in particular, all curves from $\mathcal{C}_A^\xi$ differ by a reparametrization. Of course, curves of this type have tangent vectors null in all their points and they are called null–geodesics. For remaining two cases, where tangent vectors are not null, the bracket $[X[XZ]]$ takes all values in $\mathfrak{g}_{-1}$, i.e. the second derivative may be chosen arbitrarily and the dimension of the appropriate spaces $\mathcal{C}_A^\xi$ equals to $\dim(\mathfrak{g}_{-1})$. More explicitly, for any $Y \in \mathfrak{g}_{-1}$ such that $\|Y\|^2 \neq 0$, one can choose $X = Y$ and $Z = aY^t J$, with an appropriate $a \in \mathbb{R}$, to get $[X[XZ]] = Y$.

In particular, for any $X \in \mathfrak{g}_{-1}$ there always exists an element $Z \in \mathfrak{g}_1$ such that $[X[XZ]] = X$, so all geodesics carry a natural projective structure.

Example 2. Almost Grassmannian structures. In this case, $G = SL(n+m, \mathbb{R})$ and the parabolic subgroup P is the stabilizer of $\mathbb{R}^n \subset \mathbb{R}^{n+m}$, so it consists of block upper triangular matrices with the blocks of sizes n and m on the diagonal. On the infinitesimal level,

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} : X \in \mathbb{R}^{mn} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : \operatorname{tr}(A) + \operatorname{tr}(B) = 0 \right\}, \ \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} : Z \in \mathbb{R}^{nm} \right\}.$$

First, it is easy to see that the subgroup G_0 consists of block diagonal matrices, and its action on $\mathfrak{g}_{-1}$ is given by $X \mapsto TXS^{-1}$ for $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \in G_0$. Two elements of $\mathfrak{g}_{-1}$ lie in the same G_0–orbit if and only if they have the same rank, so we get $\min(n, m)$ distinct types of generalized geodesics on manifolds with almost Grassmannian structures of type (n, m). Further, the computation of the iterated bracket yields $[X[XZ]] = -2XZX$. In particular, the choice of the pseudoinverse matrix $Z = X^\dagger$ provides always a multiple of X and so all generalized geodesics enjoy the distinguished projective structure.

If the rank of X is one, then we may choose X to be the matrix with the left upper element $x_{11} = 1$ and all other 0. Then $[X[XZ]]$ equals to $z_{11}X$ for all Z and this behavior is shared by all matrices of rank one. Hence geodesics corresponding to rank one matrices behave like null–geodesics in conformal geometries of indefinite signature. The other extreme is that X has maximal rank, where one gets a lot of freedom in the available second derivatives of the curves. The case that all elements of $\mathfrak{g}_{-1}$ have the form $[X[XZ]]$ with a fixed $X \in \mathfrak{g}_{-1}$ occurs only if $m = n$ and X has rank n.

Example 3. Projective structures. Projective structures are the special case $n = 1$ of Example 2 above, in particular, the rank of $X \neq 0$ is always one. More explicitly, the product ZX is a real number, so the bracket $[X[XZ]]$ is always a multiple of X, which in particular implies that all generalized geodesics are determined, as unparametrized curves, by the direction in one point. This agrees with the classical definition of a projective structure as a class of affine connections sharing the same unparametrized geodesics. All such connections are parametrized by smooth one–forms on the base manifold and they correspond to the Weyl connections underlying the Cartan connection in the sense of Section 3. See Example 3.8 for an elementary and very clear discussion on generalized geodesics in the homogeneous projective space.

6. More refinements

In the whole section, let M be a manifold equipped with a parabolic geometry of some fixed type (G,P) with a $|k|$–graded Lie algebra $\mathfrak{g}$. We will focus on geodesics of types $\mathcal{C}_{\mathfrak{g}_{-j}}$, with $0<j\le k$, in order to improve the estimate on jets deduced in 4.4. The most general result is in Theorem 6.4, but since the proof is a bit technical, we prefer to discuss two simpler special cases first. Let us begin with some useful observations.

6.1. Technicalities. From 4.1 we know that for each $X\in A$ we have to compare $c^{e,X}$ to all curves of the form $c^{\exp Z,Y}$ with $Z\in\mathfrak{p}_+=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_k$ and $Y\in A$. Now, according to [**26**, 3.11], we can get a nicer presentation of $\exp Z$, namely, there are unique elements $Z_i\in\mathfrak{g}_i$ for $i=1,\dots,k$ such that $\exp Z=\exp Z_1\cdots\exp Z_k$. Since $\mathrm{Ad}\circ\exp=\exp\circ\,\mathrm{ad}$, we get $\mathrm{Ad}_{\exp Z}X=\sum_{i=0}^{\infty}\frac{1}{i!}\mathrm{ad}_Z^i(X)$. Further for $\mathrm{Ad}_{\exp Z}=\mathrm{Ad}_{\exp Z_1}\circ\cdots\circ\mathrm{Ad}_{\exp Z_k}$ and the linearity of $\mathrm{Ad}_b:\mathfrak{g}\to\mathfrak{g}$ we may write

$$\mathrm{Ad}_{\exp Z}X=\sum_{i_1,\dots,i_k}\tfrac{1}{i_1!\cdots i_k!}(\mathrm{ad}_{Z_1}^{i_1}\circ\cdots\circ\mathrm{ad}_{Z_k}^{i_k})(X). \tag{1}$$

Moreover, if $X\in\mathfrak{g}_{-j}$ then a summand in the right hand side lies in $\mathfrak{g}_\ell$ if and only if $i_1+2i_2+\cdots+ki_k-j=\ell$. Hence the only nonzero terms of the above sum correspond to all k–tuples of nonnegative integers $(i_1,\dots,i_k)$ such that $0\le i_1+2i_2+\cdots+ki_k\le k+j$.

In all cases discussed below, i.e. for all $A=\mathfrak{g}_{-j}$ with $1\le j\le k$, the curves $c^{e,X}$ and $c^{\exp Z,Y}$ share the same tangent vector only if $Y=X$. Then the curve u associated to these geodesics by equation 4.2(1) yields $\delta u(0)=X-\mathrm{Ad}_{\exp Z}X$ and from (1) we get

$$\delta u(0)=-\sum_{i_1,\dots,i_k}\tfrac{1}{i_1!\cdots i_k!}(\mathrm{ad}_{Z_1}^{i_1}\circ\cdots\circ\mathrm{ad}_{Z_k}^{i_k})(X), \tag{2}$$

where the sum runs over all k–tuples $(i_1,\dots,i_k)$ such that $0<i_1+2i_2+\cdots+ki_k\le k+j$. Denoting $\delta u(0)$ by W, Lemma 4.3 implies that $(\delta u)^{(i)}(0)=\mathrm{ad}_{-X}^i(W)$ and consequently, by Lemma 4.2, in order to prove that geodesics of a type in question are determined by its r–jets we have to show that $\mathrm{ad}_X^i(W)\in\mathfrak{p}$ for all $i\le r-1$ implies $\mathrm{ad}_X^i(W)\in\mathfrak{p}$ for all i.

Finally, for each $\ell\ge1$ define W'_ℓ to be the sum of those terms in the expression (2) of W for which all exponents i_j with $j>\ell$ are zero, and put $W''_\ell=W-W'_\ell$. In particular, we have got $W''_h=0$, i.e. $W'_h=W$, for all $h\ge k$.

Further we need another observation for the proofs. For any $X,Y\in\mathfrak{g}$, the Jacobi identity reads as $\mathrm{ad}_X\circ\mathrm{ad}_Y=\mathrm{ad}_{[X,Y]}+\mathrm{ad}_Y\circ\mathrm{ad}_X$. We shall see in a moment that this generalizes to

$$\mathrm{ad}_X^n\circ\mathrm{ad}_Y=\sum_{j=0}^{n}\binom{n}{j}\mathrm{ad}_{\mathrm{ad}_X^j(Y)}\circ\mathrm{ad}_X^{n-j}\,. \tag{3}$$

In particular, if $\mathrm{ad}_X^{\ell+1}(Y) = 0$ for some $\ell \geq 0$ then for each $n > \ell$ the above sum actually runs over all j such that $0 \leq j \leq \ell$, so we can write $\mathrm{ad}_X^n \circ \mathrm{ad}_Y = \left(\sum_{j=0}^{\ell} \binom{n}{j} \mathrm{ad}_{\mathrm{ad}_X^j(Y)} \circ \mathrm{ad}_X^{\ell-j}\right) \circ \mathrm{ad}_X^{n-\ell}$. This means that the highest order iteration of ad_X which might be brought to the right hand side of the expression is $\mathrm{ad}_X^{n-\ell}$. Let us denote the sum in brackets by φ, so we have concluded that the assumption $\mathrm{ad}_X^{\ell+1}(Y) = 0$ implies the existence of a linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that $\mathrm{ad}_X^n \circ \mathrm{ad}_Y = \varphi \circ \mathrm{ad}_X^{n-\ell}$, provided $n > \ell$. Now we are going to prove a slightly more general result.

Lemma. *For any $n, i \in \mathbb{N}$ and $X, Y \in \mathfrak{g}$, the following equality holds,*

$$\mathrm{ad}_X^n \circ \mathrm{ad}_Y^i = \sum_{j_1,\dots,j_i} c_J \cdot \mathrm{ad}_{\mathrm{ad}_X^{j_1}(Y)} \circ \dots \circ \mathrm{ad}_{\mathrm{ad}_X^{j_i}(Y)} \circ \mathrm{ad}_X^{n-|J|}, \tag{4}$$

where the sum runs over all i–tuples $J = (j_1, \dots, j_i)$ such that $0 \leq |J| \leq n$, $|J| = j_1 + \dots + j_i$, and the coefficients are $c_J = \frac{n!}{j_1! \dots j_i! (n-|J|)!}$.

Proof. We prove the claim by induction on i. For $i = 1$ we obtain the formula (3) which is visible by induction on n as follows. So $n = 1$ gives the Jacobi identity and let us assume inductively that $n > 1$ and the formula holds for $1, \dots, n-1$. Then from $\mathrm{ad}_X^n \circ \mathrm{ad}_Y = \mathrm{ad}_X \circ (\mathrm{ad}_X^{n-1} \circ \mathrm{ad}_Y)$ we get

$$\begin{aligned}
\mathrm{ad}_X^n \circ \mathrm{ad}_Y &= \sum_{j=0}^{n-1} \tbinom{n-1}{j} \left(\mathrm{ad}_X \circ \mathrm{ad}_{\mathrm{ad}_X^j(Y)} \right) \circ \mathrm{ad}_X^{n-1-j} = \\
&= \sum_{j=0}^{n-1} \tbinom{n-1}{j} \left(\mathrm{ad}_{\mathrm{ad}_X^{j+1}(Y)} \circ \mathrm{ad}_X^{n-1-j} + \mathrm{ad}_{\mathrm{ad}_X^j(Y)} \circ \mathrm{ad}_X^{n-j} \right) = \\
&= \mathrm{ad}_{\mathrm{ad}_X^n(Y)} + \mathrm{ad}_X^n + \sum_{j=1}^{n-1} \left(\tbinom{n-1}{j-1} + \tbinom{n-1}{j} \right) \mathrm{ad}_{\mathrm{ad}_X^j(Y)} \circ \mathrm{ad}_X^{n-j},
\end{aligned}$$

which indeed yields the formula (3) since $\binom{n-1}{j-1} + \binom{n-1}{j} = \binom{n}{j}$ and terms ad_X^n and $\mathrm{ad}_{\mathrm{ad}_X^n(Y)}$ correspond to $j = 0$ and $j = n$, respectively. Thus we have proved the case $i = 1$.

Now let us assume that $i > 1$ and formula (4) holds for $i - 1$. Obviously, $\mathrm{ad}_X^n \circ \mathrm{ad}_Y^i = (\mathrm{ad}_X^n \circ \mathrm{ad}_Y^{i-1}) \circ \mathrm{ad}_Y$ and by induction we get

$$\mathrm{ad}_X^n \circ \mathrm{ad}_Y^i = \sum_{0 \leq |J| \leq n} c_J \cdot \mathrm{ad}_{\mathrm{ad}_X^{j_1}(Y)} \circ \dots \circ \mathrm{ad}_{\mathrm{ad}_X^{j_{i-1}}(Y)} \circ \left(\mathrm{ad}_X^{n-|J|} \circ \mathrm{ad}_Y \right).$$

Applying formula (3) to the expression in brackets we obtain

$$\sum_{0 \leq |J| \leq n} \sum_{j=0}^{n-|J|} c_{j,J} \cdot \mathrm{ad}_{\mathrm{ad}_X^{j_1}(Y)} \circ \dots \circ \mathrm{ad}_{\mathrm{ad}_X^{j_{i-1}}(Y)} \circ \mathrm{ad}_{\mathrm{ad}_X^j(Y)} \circ \mathrm{ad}_X^{n-|J|-j},$$

where the coefficients are $c_{j,J} = c_J \cdot \binom{n-|J|}{j} = \frac{n!}{j_1! \dots j_{i-1}! j! (n-|J|-j)!}$, so we may write $c_{j,J} = c_{\bar{J}}$ for the i–tuple $\bar{J} = (j_1, \dots, j_{i-1}, j)$ and the result follows. $\square$

An easy consequence of (4) is that if $\mathrm{ad}_X^{\ell+1}(Y) = 0$ for some $\ell \geq 0$ then for any i–tuple J all its elements may be supposed to be less or equal to ℓ, hence the sum actually runs over all J such that $0 \leq |J| \leq i\ell$. Altogether, the presumption $\mathrm{ad}_X^{\ell+1}(Y) = 0$ yields the existence of a linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that

$$\mathrm{ad}_X^n \circ \mathrm{ad}_Y^i = \varphi \circ \mathrm{ad}_X^{n-i\ell},$$

provided that the condition $n > i\ell$ is satisfied.

6.2. Geodesics of type $\mathcal{C}_{\mathfrak{g}_{-k}}$. For a $|k|$–graded $\mathfrak{g}$, let us begin with an extreme choice $A = \mathfrak{g}_{-k}$ defining the geodesics which emanate in directions from $TM \setminus T^{-k+1}M$ but not necessarily in all of them. In general, tangent vectors of these geodesics usually form a smaller subset in each tangent space. In other words, with respect to the truncated adjoint action of P on $\mathfrak{g}_-$, the P–orbit of $A = \mathfrak{g}_{-k}$ is always contained in $\mathfrak{g}_- \setminus \mathfrak{g}_{-k+1}$ but the two sets are not equal in general, see Example 7.6.

Nevertheless, geodesics of type $\mathcal{C}_{\mathfrak{g}_{-k}}$ in $|k|$–graded parabolic geometries embody a lot of properties which have been described in the previous section for irreducible parabolic geometries.

Theorem. *Each generalized geodesic of type $\mathcal{C}_{\mathfrak{g}_{-k}}$ in a $|k|$–graded parabolic geometry is uniquely determined by its 2–jet in a single point.*

Moreover, if two of such curves coincide up to parametrization, then this reparametrization is projective. Conversely, given a generalized geodesic of type $\mathcal{C}_{\mathfrak{g}_{-k}}$ corresponding to $(u, X) \in \mathcal{G} \times \mathfrak{g}_{-k}$, every projective change of parametrization defines a geodesic of the same type if and only if there exists a $Z \in \mathfrak{g}_k$ such that $[X[XZ]] = X$.

Proof. First, we have to prove that geodesics of type $\mathcal{C}_{\mathfrak{g}_{-k}}$ are determined by 2–jets. We shall rely on notation and results from 6.1. With respect to the above observations, this is equivalent to prove that conditions $\delta u(0) = W \in \mathfrak{p}$ and $\mathrm{ad}_X(W) \in \mathfrak{p}$ imply $\mathrm{ad}_X^i(W) \in \mathfrak{p}$ for all i. From $W \in \mathfrak{p}$ we conclude that vanishing of the component in $\mathfrak{g}_{-k+1}$ is equivalent to $[Z_1\, X] = 0$. Hence we may omit all terms in the expansion for which i_1 is the only nonzero exponent. Similarly, vanishing of the component in $\mathfrak{g}_{-k+2}$ implies $[Z_2\, X] = 0$, so we omit terms in which only i_1 and i_2 are nonzero. Inductively, we get $[Z_\ell\, X] = 0$ for all $\ell = 1, \dots, k-1$ and so $W = -[Z_k\, X] - \frac{1}{2}[[Z_k[Z_k\, X]]$. Now the condition $[XW] \in \mathfrak{p}$ implies $[X[Z_k\, X]] = 0$ which yields $[XW] = 0$ and so $\mathrm{ad}_X^i(W) = 0$ for all i, exactly as in 5.1.

Concerning reparametrizations, we may adapt the proofs of Lemma 5.5 and Proposition 5.6 along the same lines. Using the notation from there, the condition $\delta u(0) \in \mathfrak{p}$ implies $Y = \varphi'(0)X$ and moreover, inductively as above, we get $[Z_\ell\, Y] = 0$ for all $\ell \leq k-1$, which is the only difference to the $|1|$–graded case. Further, $(\delta u)'(0) \in \mathfrak{p}$ if and only if $\varphi''(0)X = \varphi'(0)^2[Y[Y\, Z_k]]$ and we finish the proof exactly as in the $|1|$–graded case. □

6.3. Geodesics of type $\mathcal{C}_{\mathfrak{g}_{-1}}$. The other extreme class of geodesics is provided by the choice $A = \mathfrak{g}_{-1}$ which is a P–invariant linear subspace in $\mathfrak{g}_-$, so geodesics of type $\mathcal{C}_{\mathfrak{g}_{-1}}$ emanate from any fixed point in M in all directions of $T^{-1}M$. The following Proposition is a special case of Theorem 6.4 and the same observation holds for the proofs, so the following paragraphs may be skipped over.

Proposition. *Each generalized geodesic of type $\mathcal{C}_{\mathfrak{g}_{-1}}$ in a $|k|$–graded parabolic geometry is uniquely determined by its $(k+1)$–jet in a single point.*

Let us start the proof by a claim which concludes the Proposition in a very easy way. Recall the sums W'_ℓ and W''_ℓ defined in 6.1.

Claim. *Let $X \in \mathfrak{g}_{-1}$ and let $\operatorname{ad}^i_X(W) \in \mathfrak{p}$ for all $i \le \ell$. Then for each $m \le \ell$ we have $\operatorname{ad}^{m+1}_X(Z_m) = 0$ and, in particular, for each n we get $\operatorname{ad}^n_X(W'_\ell) \in \mathfrak{p}$.*

Proof. We prove this claim by induction on ℓ. If $\ell = 1$, we only suppose that $\operatorname{ad}_X(W) \in \mathfrak{p}$. Looking at formula 6.1(2) for W and taking into account that $X \in \mathfrak{g}_{-1}$, we see that $\operatorname{ad}_X(W) \in \mathfrak{p}$ implies (and is actually equivalent to) $[X[Z_1\, X]] = 0$ and thus $\operatorname{ad}^2_X(Z_1) = 0$. Hence it remains to show that $\operatorname{ad}^n_X(W'_1) \in \mathfrak{p}$ for all n. By definition, $W'_1 = \sum_{i=1}^{k+1} \frac{1}{i!} \operatorname{ad}^i_{Z_1} X$, thus $\operatorname{ad}^n_X(W'_1) \in \mathfrak{p}$ is equivalent to vanishing of $(\operatorname{ad}^n_X \circ \operatorname{ad}^i_{Z_1})(X)$ for all $i \le n$. From the consequence of Lemma 6.1 we know that $\operatorname{ad}^2_X(Z_1) = 0$ implies that $\operatorname{ad}^n_X \circ \operatorname{ad}^i_{Z_1} = \varphi \circ \operatorname{ad}^{n-i+1}_X \circ \operatorname{ad}_{Z_1}$ for some linear map φ, where by assumption $n - i + 1 \ge 1$. Hence applying this element to X we get $(\operatorname{ad}^n_X \circ \operatorname{ad}^i_{Z_1})(X) = \varphi \circ \operatorname{ad}^{n-i+2}_X(Z_1)$ which vanishes since $n - i + 2 \ge 2$. This completes the proof of the case $\ell = 1$.

Assume inductively that $\ell > 1$ and we have proved the result for $\ell-1$. Given that $\operatorname{ad}^i_X(W) \in \mathfrak{p}$ for all $i \le \ell$, we by induction conclude that $\operatorname{ad}^{m+1}_X(Z_m) = 0$ for $m < \ell$ and $\operatorname{ad}^n_X(W'_{\ell-1}) \in \mathfrak{p}$ for all n. In particular, the condition $\operatorname{ad}^\ell_X(W) \in \mathfrak{p}$ implies $\operatorname{ad}^\ell_X(W''_{\ell-1}) \in \mathfrak{p}$. By definition, $W''_{\ell-1} \in \mathfrak{g}_{\ell-1} \oplus \cdots \oplus \mathfrak{g}_k$ and the only summand of $W''_{\ell-1}$ which belongs to $\mathfrak{g}_{\ell-1}$ is $[Z_\ell\, X]$. Hence $\operatorname{ad}^\ell_X([Z_\ell\, X]) \in \mathfrak{g}_{-1}$ is the only term in $\operatorname{ad}^\ell_X(W''_{\ell-1})$ which does not lie in $\mathfrak{p}$, so the condition $\operatorname{ad}^\ell_X(W''_{\ell-1}) \in \mathfrak{p}$ implies $\operatorname{ad}^{\ell+1}_X(Z_\ell) = 0$. Now it remains to show that $\operatorname{ad}^n_X(W'_\ell) \in \mathfrak{p}$ for all n. Since we know by induction that $\operatorname{ad}^n_X(W'_{\ell-1}) \in \mathfrak{p}$, it suffices to consider $\operatorname{ad}^n_X(W'_\ell - W'_{\ell-1})$. From the expression for W we conclude that

$$W'_\ell - W'_{\ell-1} = \sum_{i_1,\dots,i_\ell} \tfrac{1}{i_1!\cdots i_\ell!} (\operatorname{ad}^{i_1}_{Z_1} \circ \cdots \circ \operatorname{ad}^{i_\ell}_{Z_\ell})(X),$$

where each i_ℓ is always positive. Obviously, the condition $\operatorname{ad}^n_X(W'_\ell - W'_{\ell-1}) \in \mathfrak{p}$ is equivalent to vanishing of $(\operatorname{ad}^n_X \circ \operatorname{ad}^{i_1}_{Z_1} \circ \ldots \circ \operatorname{ad}^{i_\ell}_{Z_\ell})(X)$ for all multi–indices $(i_1, \dots, i_\ell)$ such that $i_1 + 2i_2 + \cdots + \ell i_\ell \le n$. Since $\operatorname{ad}^{m+1}_X(Z_m) = 0$, we see from Lemma 6.1 that $\operatorname{ad}^n_X \circ \operatorname{ad}^{i_m}_{Z_m} = \varphi \circ \operatorname{ad}^{n-mi_m}_X$ for some linear map φ provided that $n > mi_m$. Thus we conclude

$$\operatorname{ad}^n_X \circ \operatorname{ad}^{i_1}_{Z_1} \circ \ldots \circ \operatorname{ad}^{i_\ell}_{Z_\ell} = \psi \circ \operatorname{ad}^{n-i_1-2i_2-\cdots-\ell(i_\ell-1)}_X \circ \operatorname{ad}_{Z_\ell},$$

and by assumption $n - i_1 - 2i_2 - \cdots - \ell(i_\ell - 1) \ge \ell$. Applying this element to X we obtain $\psi \circ \operatorname{ad}^r_X(Z_\ell)$, where by construction $r \ge \ell + 1$, so this vanishes and the statement follows. □

Proof of Proposition. Considering the claim in the case $\ell = k$, we see that $\operatorname{ad}^i_X(W) \in \mathfrak{p}$ for all $i \le k$ implies that $\operatorname{ad}^n_X(W'_k) \in \mathfrak{p}$ for all n. Since we have observed above that $W'_k = W$, this completes the proof. □

6.4. Geodesics of type $\mathcal{C}_{\mathfrak{g}_{-j}}$. The most general case discussed here is provided by the geodesics of type $\mathcal{C}_{\mathfrak{g}_{-j}}$ with arbitrary $j = 1, \dots, k$. Geodesics of this type are always tangent to $T^{-j}M$ in all points. More precisely, they emanate from a given point in certain directions of $T^{-j}M \setminus T^{-j+1}M$ which correspond just to the P–orbit of $\mathfrak{g}_{-j}$ in $\mathfrak{g}_-$ with respect to the truncated adjoint action.

Theorem. *Each generalized geodesic of type $\mathcal{C}_{\mathfrak{g}_{-j}}$ in a $|k|$–graded parabolic geometry, $1 \leq j \leq k$, is uniquely determined by its r–jet in a single point provided that $rj \geq k+1$.*

Proof of this Theorem generalizes the proof of Proposition 6.3 including some ideas of the proof of Theorem 6.2. Let us begin with a more general version of Claim 6.3, then the result follows immediately.

Claim. *Let $X \in \mathfrak{g}_{-j}$ and let $\operatorname{ad}_X^i(W) \in \mathfrak{p}$ for all $i \leq \ell$. Then for each $s \leq \ell$ and m satisfying $sj \leq m < (s+1)j$ we have $\operatorname{ad}_X^{s+1}(Z_m) = 0$ and, in particular, for each $m < (\ell+1)j$ we get $\operatorname{ad}_X^n(W'_m) \in \mathfrak{p}$ for all n.*

Proof. As in the special case of $j = 1$, we prove this claim by induction on ℓ. For $\ell = 1$ we suppose $W \in \mathfrak{p}$ and $\operatorname{ad}_X(W) \in \mathfrak{p}$. The condition $W \in \mathfrak{p}$ yields that vanishing of the component in $\mathfrak{g}_{-j+1}$ is equivalent to $[Z_1 X] = 0$ and by the same induction as in the proof of 6.2 we obtain that $[Z_\ell X] = 0$ for all $\ell < j$, which proves the first statement for $s = 0$. From $\operatorname{ad}_X(W) \in \mathfrak{p}$ we conclude that vanishing of the component in $\mathfrak{g}_{-j}$ is equivalent to $[X[Z_j X]] = 0$. Similarly, the component in $\mathfrak{g}_{-j+1}$ is $[X[Z_{j+1}X]] + [X[Z_1[Z_j X]]]$ and its vanishing is equivalent to $[X[Z_{j+1}X]] = 0$ since the second summand vanishes due to relations which have been achieved so far. Inductively, one can conclude that $\operatorname{ad}_X^2(Z_m) = 0$ for each m satisfying $j \leq m < 2j$, which is the first statement for $s = 1 = \ell$.

Now, for each n, it remains to prove that $\operatorname{ad}_X^n(W'_{2j-1}) \in \mathfrak{p}$, which yields $\operatorname{ad}_X^n(W'_m) \in \mathfrak{p}$ for all $m < 2j$. The above condition is equivalent to vanishing of all terms $(\operatorname{ad}_X^n \circ \operatorname{ad}_{Z_1}^{i_1} \circ \dots \circ \operatorname{ad}_{Z_{2j-1}}^{i_{2j-1}})(X)$ with $i_1 + 2i_2 + \dots + (2j-1)i_{2j-1} < j(n+1)$. Let i_h be the last nonzero exponent in this expression and let us suppose $h \geq j$ at first. Lemma 6.1 with relations $\operatorname{ad}_X(Z_m) = 0$ for $m < j$ and $\operatorname{ad}_X^2(Z_m) = 0$ for $m < 2j$ yields the above term equals to

$$(\varphi \circ \operatorname{ad}_X^{n-(i_j+\dots+i_h-1)} \circ \operatorname{ad}_{Z_h})(X) = \varphi \circ \operatorname{ad}_X^{n-(i_j+\dots+i_h)+2}(Z_h)$$

for some linear map φ. The term in question vanishes if the latter exponent is grater or equal 2, i.e. $n+1 > i_j + \dots + i_h$, which is obviously satisfied by the assumption $i_1 + 2i_2 + \dots + hi_h < j(n+1)$. If $h < j$ then we have to deal with a term $\varphi \circ \operatorname{ad}_X^{n+1}(Z_h)$ which vanishes trivially. So we have proved the case $\ell = 1$.

Let us assume that $\ell > 1$ and the claim holds for $1, \dots, \ell - 1$. Given that $\operatorname{ad}_X^n(W) \in \mathfrak{p}$ for all $i \leq \ell$ we obtain by induction that $\operatorname{ad}_X^{s+1}(Z_m) = 0$ for each s and m satisfying $s \leq \ell - 1$ and $sj \leq m < (s+1)j$ and, in particular, $\operatorname{ad}_X^n(W'_{j(\ell-1)}) \in \mathfrak{p}$ for all n. We have to prove that $\operatorname{ad}_X^{\ell+1}(Z_m) = 0$ and $\operatorname{ad}_X^n(W'_m) \in \mathfrak{p}$ for all n and m satisfying $\ell j \leq m < (\ell+1)j$. In particular, the condition $\operatorname{ad}_X^\ell(W) \in \mathfrak{p}$ implies $\operatorname{ad}_X^\ell(W''_{j(\ell-1)}) \in \mathfrak{p}$, where $W''_{j(\ell-1)}$ belongs to $\mathfrak{g}_{j(\ell-2)+1} \oplus \dots \oplus \mathfrak{g}_k$ by definition. The component of $W''_{j(\ell-1)}$ which lies in $\mathfrak{g}_{j(\ell-1)}$ is the sum of elements of the form $(\operatorname{ad}_{Z_1}^{i_1} \circ \dots \circ \operatorname{ad}_{Z_{j\ell}}^{i_{j\ell}})(X)$ where $i_1 + 2i_2 + \dots + j\ell\, i_{j\ell} = j\ell$. In particular, the only element from the sum with $i_{j\ell} = 1$ corresponds to $[Z_{j\ell}, X]$ and the others have $i_{j\ell} = 0$. Now, all terms $(\mathrm{ad}_X^\ell \circ \mathrm{ad}_{Z_1}^{i_1} \circ \cdots \circ \mathrm{ad}_{Z_{j\ell-1}}^{i_{j\ell-1}})(X)$ with $\sum_{m=1}^{j\ell-1} m\, i_m = j\ell$ vanishes due to the same arguments as above, hence the condition $\mathrm{ad}_X^\ell(W''_{j(\ell-1)}) \in \mathfrak{p}$ implies $\mathrm{ad}_X^\ell([Z_{j\ell}, X]) = 0$. More precisely, by the inductive assumptions for any $m < \ell j$ and $s = \lfloor \frac{m}{j} \rfloor$ (the greatest integer $\leq \frac{m}{j}$) we have $\mathrm{ad}_X^{s+1}(Z_m) = 0$, so there is a linear map φ such that

$$(\mathrm{ad}_X^\ell \circ \mathrm{ad}_{Z_1}^{i_1} \circ \cdots \circ \mathrm{ad}_{Z_h}^{i_h})(X) = (\varphi \circ \mathrm{ad}_X^r \circ \mathrm{ad}_{Z_h})(X) = \varphi \circ \mathrm{ad}_X^{r+1}(Z_h)$$

where i_h is the last nonzero number from the sequence $i_1, \ldots, i_{j\ell-1}$ and

$$r = \ell - \sum_{m=1}^{h} \left\lfloor \frac{m}{j} \right\rfloor i_m + \left\lfloor \frac{h}{j} \right\rfloor.$$

Now the condition $r + 1 \geq \lfloor \frac{h}{j} \rfloor$ is equivalent to $\ell + 1 \geq \sum_{m=1}^{h} \lfloor \frac{m}{j} \rfloor i_m$ which follows immediately from $\ell = \sum_{m=1}^{j\ell-1} \frac{m}{j}\, i_m = \sum_{m=1}^{h} \frac{m}{j}\, i_m$. So we have really proved that $\mathrm{ad}_X^\ell([Z_{j\ell}, X]) = 0$. Inductively, one can conclude that $\mathrm{ad}_X^{\ell+1}(Z_m) = 0$ for each m satisfying $\ell j \leq m < (\ell + 1)j$.

Finally, we have to prove $\mathrm{ad}_X^n(W'_{j(\ell+1)-1}) \in \mathfrak{p}$ for each n, which is equivalent to vanishing of all terms $(\mathrm{ad}_{Z_1}^{i_1} \circ \cdots \circ \mathrm{ad}_{Z_{j(\ell+1)-1}}^{i_{j(\ell+1)-1}})(X)$ with $\sum_{m=1}^{j(\ell+1)-1} m\, i_m < j(n+1)$. The same ideas as above leads to the conclusion that any such term vanishes if $n + 1 \geq \sum_{m=1}^{h} \lfloor \frac{m}{j} \rfloor i_m$, where h is an index of the last nonzero exponent i_h in the term in question. Now the latter inequality is obviously satisfied due to the assumption $n + 1 > \sum_{m=1}^{h} \frac{m}{j}\, i_m$, so the proof is complete. □

Proof of Theorem. With respect to the above Claim, let ℓ be an integer such that $j(\ell + 1) - 1 \geq k$. Then $W'_{j(\ell+1)-1} = W$ and the assumption $\mathrm{ad}_X^i(W) \in \mathfrak{p}$ for $i \leq \ell$ implies $\mathrm{ad}_X^i(W) \in \mathfrak{p}$ for all i. Altogether, geodesics of type $\mathcal{C}_{\mathfrak{g}_{-j}}$ are determined by $(\ell+1)$–jets if the number ℓ satisfies the inequality $j(\ell+1) - 1 \geq k$, i.e. $j(\ell + 1) \geq k + 1$. Hence the result follows. □

7. More examples

In this section we present a complete classification of generalized geodesics for several specific more–graded parabolic geometries. First of all, we introduce a recipe according to which all computations are conducted. The process will be very natural and rather similar to that of $|1|$–graded parabolic geometries applied in 5.7.

Examples 7.2, 7.3, and 7.4 represent some special cases of parabolic contact structures. According to the standard identification $T(G/P) = G \times_P \mathfrak{g}_-$, the underlying contact structure corresponds to the P–invariant subspace $\mathfrak{g}_{-1} \subset \mathfrak{g}_-$ and we are dealing with $|2|$–gradings where $\mathfrak{g}_{-2}$ is one–dimensional and the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is nondegenerate. In these cases geodesics of type $\mathcal{C}_{\mathfrak{g}_{-2}}$ always emanate in all directions from $TM \setminus T^{-1}M$, in other words, the P–orbit of $\mathfrak{g}_{-2}$ is the entire complement of $\mathfrak{g}_{-1}$ in $\mathfrak{g}_-$. A very well known instance of this type of generalized geodesics is provided by the Chern–Moser chains on CR–structures of hypersurface type, Example 7.4. In general, from Theorem 6.2 we know that curves of this type are determined by their 2–jet in a point as parametrized curves, and it follows that they are uniquely determined by their direction in one point up to parametrization, by dimension reasons. Moreover, each such geodesic carries a natural projective structure of distinguished parametrizations.

A slightly more general example of this type was studied for 6–dimensional CR–structures of codimension 2 in [**20**]. The whole discussion of those cases is included in Example 7.5. Finally, Example 7.6 contains a classification of generalized geodesics in the so called x—x—dot geometries which are $|2|$–graded too. In the latter cases, the subalgebra $\mathfrak{g}_{-2}$ is not one–dimensional and, in contrast to the contact geometries, geodesics of type $\mathcal{C}_{\mathfrak{g}_{-2}}$ does not exhaust all directions from $TM \setminus T^{-1}M$, so the discussion is a bit richer.

7.1. Recipe. The recipe presented here is based only on general results from Section 4. The developed process will serve us to solve the following problems in some specific parabolic geometries with nontrivial filtration of tangent bundle. We have to find

- ⋆ the sufficient order of jet which determines geodesics of type $\mathcal{C}_A$ uniquely,
- ⋆ the family of generalized geodesics which emanate with a given tangent vector from one point,
- ⋆ the class of preferred parametrizations on geodesics of the type in question.

1. First of all, we start with the description of distinguished types of tangent vectors. According to 2.4, they correspond to P–invariant subsets in $\mathfrak{g}_-$ with respect to the truncated adjoint action $\underline{\mathrm{Ad}}$. To any such subset we will look for its G_0–invariant subsets which define generalized geodesics emanating in the actual directions. Sometimes two distinct G_0–invariant subsets define the same class of curves so one of them will be omitted in further discussions. More concretely, for G_0–invariant subsets $A, B \subset \mathfrak{g}_-$ the classes of curves $\mathcal{C}_A$ and $\mathcal{C}_B$ obviously coincide if for any $X \in A$ there is an element $b \in P$ such that $\mathrm{Ad}_b(X) \in B$, and conversely. Although this condition is very restrictive, it happens in some interesting cases, see e.g. 7.3, but it has not to be necessary as the computations in 7.2 or 7.4 show. Further, curves of type $\mathcal{C}_A$ are contained in a class of curves $\mathcal{C}_B$ only if the P–orbit of A belongs to the P–orbit of B, hence we may only control G_0–invariant subsets in the common P–orbit in order to omit the superfluous cases.

2. From 4.1 we know that it suffices to compare geodesics $c^{e,X}$ and $c^{\exp Z,Y}$ (with $X, Y \in A$ and $Z \in \mathfrak{p}_+$) in the homogeneous space G/P in order to answer all above questions. Let A be a G_0–invariant subset in $\mathfrak{g}_-$ and $c^{e,X}$ be a generalized geodesic of type $\mathcal{C}_A$. Step by step, following Lemmas 4.2 and 4.3, we will search $Z \in \mathfrak{p}_+$ and $Y \in A$ such that the curves $c^{e,X}$ and $c^{\exp Z,Y}$ coincide. At the same time we get the order of jet which decides the two curves are equal.

The first condition $\delta u(0) \in \mathfrak{p}$ restricts $Z \in \mathfrak{p}_+$ to fulfill the condition $Y = \underline{\mathrm{Ad}}^{-1}_{\exp Z}(X) \in A$. The other conditions $(\delta u)^{(i)}(0) \in \mathfrak{p}$ further reduce possible $Z \in \mathfrak{p}_+$ in order the two curves share a common $(i+1)$–jet. All such elements form a subspace in $\mathfrak{p}_+$ denoted by the symbol B^{i+1}. More precisely, for any $r \geq 1$ we put $B^r = \{Z \in \mathfrak{p}_+ : j_0^r c^{e,X} = j_0^r c^{\exp Z,Y} \text{where } Y = \underline{\mathrm{Ad}}^{-1}_{\exp Z}(X) \in A\}$. In particular, $B^1 = \{Z \in \mathfrak{p}_+ : \underline{\mathrm{Ad}}^{-1}_{\exp Z}(X) \in A\}$. Generalized geodesics of a given type are obviously determined by a jet of order r if the condition $(\delta u)^{(r-1)}(0) \in \mathfrak{p}$ implies $(\delta u)^{(r)}(0) = 0$ or, more generally, if the sequence of B^i stabilizes at $i = r$. In that case, the assumption $(\delta u)^{(i)}(0) \in \mathfrak{p}$ implies $(\delta u)^{(i+1)}(0) \in \mathfrak{p}$ for all $i \geq r-1$.

3. Now we are interested in the dimension of the set of geodesics of type $\mathcal{C}_A$ sharing the same tangent vector ξ. This set is denoted by $\mathcal{C}_A^\xi$ as in 5.3 and its dimension does not depend on a particular vector but only on its type. In view of the above arguments, let $\xi = \{e, X\}$ be the fixed vector and further let r be an order of jet which determines generalized geodesics with the given tangent vector uniquely. Obviously, for any $Z \in B^r$ the curves $c^{e,X}$ and $c^{\exp Z, \underline{\mathrm{Ad}}^{-1}_{\exp Z}(X)}$ coincide. If another representative of the vector $\{e, X\}$ is chosen then the analogously defined subsets $B^i \subseteq \mathfrak{p}_+$ are naturally identified with the initial ones, so the set $\mathcal{C}_A^\xi$ is parametrized by the quotient B^1/B^r, which is easy to describe.

4. Finally, we are interested in distinguished parametrizations of generalized geodesics of a given type. Lemma 4.5 helps us to find all reparametrizations which appear when two generalized geodesics parametrize the same curve. In all examples, the function φ is estimated according to its behavior in 0, so we must always check if the estimation is right, i.e. equation 4.5(4) holds true for all t. Reparametrizations which appear in this way will be either projective or affine (which is not a coincidence, rather the general rule, cf. [**9**]).

On the other hand, for any generalized geodesic $c^{g,X}$ and any reparametrization φ of the admissible type the curve $c^{g,X} \circ \varphi$ is a generalized geodesic too. This is trivially satisfied if φ is affine. Otherwise, the statement is true if and only if $\varphi''(0)$ can get any nonzero (and so all) value. This can be easily verified in all cases discussed below. Further, the value of $\varphi''(0)$ is expressed during the computation in variables $\varphi'(0)$, X, and Z so that $\varphi''(0) = -2\varphi'(0)^2 Z(X)$, where $Z(X)$ corresponds to the central part of the bracket $[Z, X]$. All details are left to the reader, however, this heavily remind the formula (5) in 3.8.

5. Most of necessary computations has been done with the help of the computational system Maple, so all technical details are presented in the format of Maple worksheets [**29**] and we will only summarize the achieved results of any particular example into a table of the form:

type	X	B^1 B^2 B^3	ord	dim	R	$\varphi''(0)$

The first column denotes the type of generalized geodesics in question, i.e. a G_0–invariant subset $A \subset \mathfrak{g}_-$. The second column presents a general element $X \in A$, in particular, all indicated entries are supposed to be nonzero. Columns B^i describe the sets defined in the point 2 above and if the sequence B^i stabilizes we use the symbol † instead of rewriting the previous result. The next column is the minimal order of jet which determines geodesics of type $\mathcal{C}_A$ uniquely. The column dim contains the dimension of the set $\mathcal{C}^\xi_A$ from the point 3 above with respect to the vector $\xi = \{e, X\}$. The number is simply the difference $\dim(B^1) - \dim(B^3)$ since the sufficient order is never grater than 3 in all examples discussed below. For all geodesics with the common tangent vector $\xi = \{e, X\}$ we have to consider all subsets $A \subset \mathfrak{g}_-$ with a common P–orbit containing X. These are indicated in the table by a one common row. A star $*$ in an expression always means any entry of the parameter in question.

The rest of the table is devoted to reparametrizations. The column R describes the subset in B^1 such that any curve $c^{\exp Z, Y}$ with $Z \in R$ and $Y = \underline{\mathrm{Ad}}^{-1}_{\exp Z}(X)$ coincide with $c^{e,X}$ up to a reparametrization φ in the sense of 4.5(4). The last column then expresses the value of $\varphi''(0)$ with respect to the chosen $Z \in R$, in particular, for $Z \in B^1$ we have $\varphi'(0) = 1$. The geodesics in question allow only affine class of reparametrizations if and only if $R/B^3 = 0$, otherwise, the quotient $R/B^3 \subseteq B^1/B^3$ is 1–dimensional and any nonzero element of R/B^3 determines a nonzero value of $\varphi''(0)$. Together with conditions $\varphi'(0) = 1$ and $\varphi(0) = 0$ one can get an explicit expression of the corresponding projective reparametrization φ.

7.2. Projective contact structures. Projective contact structures are contact structures with a projective structure on the contact distribution. The standard homogeneous model in an odd dimension $2n-1$ is the real projective space $\mathbb{R}P^{2n-1}$ with a transitive action of the semisimple Lie group $Sp(2n, \mathbb{R}) \subseteq SL(2n, \mathbb{R})$ which is obtained as a restriction of the usual action of the group $SL(2n, \mathbb{R})$ on $\mathbb{R}P^{2n-1}$. The stabilizer of any ray in $\mathbb{R}^{2n}$ is isomorphic to the parabolic subgroup in $G = Sp(2n, \mathbb{R})$ which is indicated by the Satake diagram x—o$\cdots$o=<=o. In an appropriate matrix representation, the parabolic subgroup P is formed by upper triangular matrices with blocks of sizes 1, $2n-2$, and 1 on the diagonal. See [**26**, 4.4] for details.

The lowest dimensional nontrivial case corresponds to $n = 2$, so we suppose $G = Sp(4, \mathbb{R})$ given as a group of linear automorphisms on $\mathbb{R}^4$ which preserve the symplectic form determined by the matrix $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ in the standard basis. The Lie algebra $\mathfrak{sp}(4, \mathbb{R})$ consists of block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A is an arbitrary matrix of size 2, $B = B^T$, $C = C^T$, and $D = -A^T$. The symbol ^{T} means the transposition with respect to the antidiagonal. Hence the grading of the Lie algebra $\mathfrak{g}$ has the form

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\}, \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X_1 & 0 & 0 \\ 0 & X_1^t J & 0 \end{pmatrix} : X_1 \in \mathbb{R}^2 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R}, A \in \mathfrak{sp}(2, \mathbb{R}) \right\},$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & Z_1 & 0 \\ 0 & 0 & JZ_1^t \\ 0 & 0 & 0 \end{pmatrix} : Z_1 \in \mathbb{R}^{2*} \right\}, \mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : z \in \mathbb{R} \right\},$$

where J denotes the submatrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The truncated adjoint action of the structure group P on $\mathfrak{g}_-$ is described as follows. First of all, let us identify an element

$\begin{pmatrix} 0 & 0 & 0 \\ X_1 & 0 & 0 \\ x & X_1^t J & 0 \end{pmatrix} \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ with the pair $(x, X_1) \in \mathbb{R} \times \mathbb{R}^2$ and, similarly, elements from $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ are written as $(Z_1, z) \in \mathbb{R}^{2*} \times \mathbb{R}$. With this notation, any element $\begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \in G_0 = \mathbb{R}_+ \times Sp(2, \mathbb{R})$ maps $(x, X_1) \mapsto (a^{-2}x, a^{-1}AX)$ and $\exp(Z_1, z) \in P_+$ acts according to the transcription $(x, X_1) \mapsto (x, X_1 - xJZ_1^t)$. Now it is obvious that there are two complementary P–invariant subsets in $\mathfrak{g}_-$ and the corresponding tangent directions are those inside and outside of the contact subbundle, respectively. Further there are three distinct G_0–invariant subsets in $\mathfrak{g}_-$:

$$\begin{aligned} A_1 &= \mathfrak{g}_{-1} \setminus \{0\}, \\ A_2 &= \mathfrak{g}_{-2} \setminus \{0\}, \\ A_3 &= \mathfrak{g}_- \setminus (\mathfrak{g}_{-1} \cup \mathfrak{g}_{-2}). \end{aligned}$$

Clearly, the subset A_1 is P–invariant, so only the geodesics of type $\mathcal{C}_{A_1}$ emanate in directions from the contact distribution. The P–orbit of A_2, denoted by $P(A_2)$, equals to $P(A_3) = \mathfrak{g}_- \setminus \mathfrak{g}_{-1}$, so for any vector which is transversal to the contact subbundle there may be geodesics of types $\mathcal{C}_{A_2}$ and $\mathcal{C}_{A_3}$ tangent to the given vector. But, for any $X = (x, X_1) \in A_3$ there are $Y = (x, 0) \in A_2$ and $Z = (-c^{-1}X_1^t J, 0) \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that the curve $\exp tX$ equals to $\exp Z \exp tY$ modulo P. Hence any geodesic of the generic type is a “chain” and so we consider only geodesics of types $\mathcal{C}_{\mathfrak{g}_{-1}}$ and $\mathcal{C}_{\mathfrak{g}_{-2}}$ in the sequel. Altogether, for any type of tangent vectors there are geodesics of a unique type which emanate in those directions.

Resume. The expression $(Z_1(X_1) = 0, *)$ below denotes the set $\{(Z_1, z) \in \mathfrak{g}_1 \oplus \mathfrak{g}_2 : Z_1(X_1) = 0\}$. Similar abbreviations are often used hereafter.

type	X	B^1	B^2	B^3	ord	dim	R	$\varphi''(0)$
A_1	$(0, X_1)$	$\mathfrak{p}_+$	$(Z_1(X_1) = 0, *)$	†	2	1	$(Z_1, *)$	$-2\, Z_1(X_1)$
A_2	$(x, 0)$	$\mathfrak{g}_2$	0	†	2	1	$(0, z)$	$-2\, zx$

Remarks. In both particular cases, the geodesics are determined by a 2–jet in one point and for a fixed vector there is a 1–dimensional family of curves of an appropriate type tangent to the given vector. Moreover, geodesics of any mentioned type carry the projective class of distinguished parametrizations, so each generalized geodesic is uniquely given by a tangent direction as an unparametrized curve. This mimics the behavior of geodesics in classical projective geometries where the filtration of the tangent bundle is trivial and the whole classification allows only geodesics of a unique type with just the same properties, cf. Example 3 in 5.7.

Moreover, the above classification goes through unchanged for small dimensions and so it seems to be valid for a general n, although the above case $n = 2$ is rather special because $G_0 = \mathbb{R}_+ \times Sp(2n-2, \mathbb{R})$ and $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$. In particular, from the description of the truncated action it is obvious that there are neither new distinguished G_0–invariant nor P–invariant subsets in $\mathfrak{g}_-$ which would define new types of curves and new tangent directions, respectively.

7.3. Lagrangean contact structures. Lagrangean contact structures are contact structures endowed with a fixed decomposition of the contact subbundle into a direct sum of two distinguished subbundles such that the differential of the one–form defining the contact distribution restricted to one of these subbundles vanishes. Complementary subbundles with this property are called a Lagrangean pair. The standard homogeneous model is provided by the projectivized cotangent bundle to the real projective space $\mathbb{R}P^n$ with a transitive action of the Lie group $G = SL(n+1, \mathbb{R})$. The stabilizer of any point in $P(T^*(\mathbb{R}P^n))$ is a parabolic subgroup of G which can be indicated in a block–wise form as subgroup of upper triangular matrices with blocks of ranks 1, $n-1$, and 1. The corresponding Satake diagram is x—o $\cdots$ o—x, where the number of uncrossed nodes is n. See [**25**] for details and precise definitions.

Let us consider the lowest dimensional case which corresponds to the principal group $G = SL(3, \mathbb{R})$ and P being the Borel subgroup. On the infinitesimal level, the grading of the Lie algebra $\mathfrak{g}$ has the form

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ *&0&0 \end{pmatrix} \right\}, \ \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0&0&0 \\ *&0&0 \\ 0&*&0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} *&0&0 \\ 0&*&0 \\ 0&0&* \end{pmatrix} : \mathrm{trace} = 0 \right\}, \ \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0&*&0 \\ 0&0&* \\ 0&0&0 \end{pmatrix} \right\}, \ \mathfrak{g}_2 = \left\{ \begin{pmatrix} 0&0&* \\ 0&0&0 \\ 0&0&0 \end{pmatrix} \right\},$$

where the stars represent arbitrary real entries.

Geometrically, there are four different classes of tangent vectors on manifolds endowed with a Lagrangean contact structure. They correspond to P–invariant subsets in $\mathfrak{g}_-$ as follows. Vectors tangent to one of the two Lagrangean subbundles correspond to linear subspaces $\mathfrak{g}^L_{-1} = \left\{ \begin{pmatrix} 0&0&0 \\ *&0&0 \\ 0&0&0 \end{pmatrix} \right\}$ and $\mathfrak{g}^R_{-1} = \left\{ \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&*&0 \end{pmatrix} \right\}$ in $\mathfrak{g}_{-1}$, respectively. Remaining vectors in the contact distribution are given by $\mathfrak{g}_{-1} \setminus (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1})$ and, finally, those outside of the contact subbundle correspond to $\mathfrak{g}_- \setminus \mathfrak{g}_{-1}$. All the mentioned subsets are really P–invariant which is easily visible by the explicit description of the truncated adjoint action of P on $\mathfrak{g}_-$. More precisely, any element $\exp \begin{pmatrix} 0&z_L&w \\ 0&0&z_R \\ 0&0&0 \end{pmatrix} \in P_+$ acts as $\begin{pmatrix} 0&0&0 \\ x_L&0&0 \\ y&x_R&0 \end{pmatrix} \mapsto \begin{pmatrix} 0&0&0 \\ x_L+yz_R&0&0 \\ y&x_R-yz_L&0 \end{pmatrix}$ and the action of the subgroup G_0 rescales each entry of a matrix in $\mathfrak{g}_-$ by a nonzero factor. Hence the G_0–orbits in $\mathfrak{g}_-$ are determined simply by the nonzero entries of a matrix and so we get seven G_0–invariant subsets in $\mathfrak{g}_-$. Let us focus on the subsets $\left\{ \begin{pmatrix} 0&0&0 \\ *&0&0 \\ *&0&0 \end{pmatrix} \right\} \subset \mathfrak{g}_{-2} \times \mathfrak{g}^L_{-1}$ and $\left\{ \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ *&*&0 \end{pmatrix} \right\} \subset \mathfrak{g}_{-2} \times \mathfrak{g}^R_{-1}$ at first. Any element of either of these subsets can be mapped to $\mathfrak{g}_{-2}$ by some Ad_b with $b \in P$, and vice versa. From the point 1 in 7.1 we know that these subsets lead to the same curves as $A = \mathfrak{g}_{-2}$, so geodesics of these two types may be omitted in further discussion. More explicitly, for any $X = \begin{pmatrix} 0&0&0 \\ x_L&0&0 \\ y&0&0 \end{pmatrix} \in \mathfrak{g}^L_{-1} \times \mathfrak{g}_{-2}$ and $Z = \begin{pmatrix} 0&0&0 \\ 0&0&z_R \\ 0&0&0 \end{pmatrix} \in \mathfrak{p}$ one can compute $\mathrm{Ad}_{\exp Z} X = \begin{pmatrix} 0&0&0 \\ x_L+yz_R&0&0 \\ y&0&0 \end{pmatrix}$, which provides the above mentioned correspondence if $y \neq 0$; the second case is similar. Altogether we get five types of generalized geodesics given by the following G_0–invariant subsets in $\mathfrak{g}_-$:

$$
\begin{aligned}
A_1 &= \mathfrak{g}^L_{-1} \setminus \{0\},\\
A_2 &= \mathfrak{g}^R_{-1} \setminus \{0\},\\
A_3 &= \mathfrak{g}_{-1} \setminus (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1}),\\
A_4 &= \mathfrak{g}_{-2} \setminus \{0\},\\
A_5 &= \mathfrak{g}_- \setminus (\mathfrak{g}_{-1} \cup \mathfrak{g}_{-2} \times (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1})).
\end{aligned}
$$

Obviously, the first three subsets are P–invariant, so only geodesics either of type $\mathcal{C}_{A_1}$ or $\mathcal{C}_{A_2}$ emanate in a direction from the Lagrangean pair and for remaining directions in the contact distribution there are geodesics only of type $\mathcal{C}_{A_3}$. Further, $P(A_4) = P(A_5) = \mathfrak{g}_- \setminus \mathfrak{g}_{-1}$ and for any vector out of the contact subbundle there are geodesics of both types $\mathcal{C}_{A_4}$ and $\mathcal{C}_{A_5}$ tangent to the given vector.

Resume. Direct computations lead to the following results.

type	X	B^1	B^2	B^3	ord	dim	R	$\varphi''(0)$
A_1	$\begin{pmatrix} 0 & 0 & 0 \\ x_L & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathfrak{p}_+$	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$	$\dagger$	2	1	$\begin{pmatrix} 0 & s & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$	$-2sx_L$
A_2	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & x_R & 0 \end{pmatrix}$	$\mathfrak{p}_+$	$\begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\dagger$	2	1	$\begin{pmatrix} 0 & * & * \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}$	$-2sx_R$
A_3	$\begin{pmatrix} 0 & 0 & 0 \\ x_L & 0 & 0 \\ 0 & x_R & 0 \end{pmatrix}$	$\mathfrak{p}_+$	$\mathfrak{g}_2$	0	3	$3 = 2n + 1$	$s\begin{pmatrix} 0 & x_R & 0 \\ 0 & 0 & x_L \\ 0 & 0 & 0 \end{pmatrix}$	$-sx_Lx_R$
A_4	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$	$\mathfrak{g}_2$	0	$\dagger$	2	1	$\begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$-2sy$
A_5	$\begin{pmatrix} 0 & 0 & 0 \\ x_L & 0 & 0 \\ y & x_R & 0 \end{pmatrix}$	$\mathfrak{p}_+$	0	$\dagger$	2	$3 = 2n + 1$	0	0

Remarks. Of course, geodesics of types $\mathcal{C}_{A_1}$ and $\mathcal{C}_{A_2}$ have got exactly the same properties which are similar to those of null–geodesics in conformal geometries, i.e. in a given direction there is just one unparametrized geodesic emanating in this direction and any $Z \in \mathfrak{p}_+$ provides at most a new (projective) reparametrization of the curve. These properties survive even in a more dimensional cases, where, in addition, a new specific class of the tangent vectors in the kernel of the algebraic bracket on the contact subbundle appears. Of course, this leads to a new type of generalized geodesics.

Geodesics of type $\mathcal{C}_{A_4}$ represent an exact analogy to CR–chains discussed in 7.4. In the most generic case given by the subset A_5 there are no two geodesics with a common tangent vector, which would be the same up to a reparametrization, so here only affine reparametrizations appear. Altogether, for any vector out of the contact distribution there is just one unparametrized "chain" with a canonical projective structure and a 3–dimensional family of uniquely parametrized geodesics of the generic type $\mathcal{C}_{A_5}$.

7.4. CR–structures of hypersurface type. Hypersurface CR–structures are contact structures with an almost complex structure on the contact distribution. The standard homogeneous model of dimension $2n+1$ is provided by real hyperquadrics in $\mathbb{C}^{n+1}$ where the whole structure is induced by the complex structure of

$\mathbb{C}^{n+1}$. Usually we consider quadrics given by the equation $\mathrm{Im}(w) = h(z, \bar{z})$ where $(z, w) = (z_1, \ldots, z_n, w)$ are coordinates in $\mathbb{C}^{n+1}$ and h is a nondegenerated Hermitean form on $\mathbb{C}^n$. The CR–subbundle is established in the tangent bundle of the quadric by the condition $\mathrm{Re}(w) = 0$ and we say the quadric has CR–dimension n and real codimension 1. Let us assume the signature of h is (p, q), $p + q = n$. Similarly to conformal geometries, the Lie group $G = SU(p+1, q+1)$ acts transitively on the CR–quadric due to the identification with the complex projectivization of a light cone in $\mathbb{C}^{n+2}$. The stabilizer of the origin is a parabolic subgroup in semisimple G. In an appropriate representation of G, the parabolic subgroup P can be viewed as a subgroup of block upper triangular matrices with the middle blocks belonging to $SU(p, q)$. These structures provide other real forms of the Lagrangean contact structures presented in 7.3. All details can be found in [**12**], [**15**], [**20**], and others.

The classification of generalized geodesics will depend on the signature of the modeling quadric, so we discuss two lowest dimensional cases corresponding to strictly definite and indefinite case, respectively.

A. Definite case. The lowest possible case is provided by the CR–dimension 1. The modeling quadric is 3–dimensional CR–sphere defined by the equation $\mathrm{Im}(w) = |z|^2$ with respect to coordinates (z, w) in $\mathbb{C}^2$. The group of CR–automorphisms of the CR–sphere is a quotient of the semisimple Lie group $SU(2, 1)$ by the noneffective kernel which is isomorphic to $\mathbb{Z}_3$. Hence the principal group of the geometry is $G = SU(2, 1)/\mathbb{Z}_3$. Let $SU(2, 1)$ be given as a group of linear automorphisms of $\mathbb{C}^3$ preserving the Hermitean inner product with the matrix $\begin{pmatrix} 0 & 0 & -\frac{i}{2} \\ 0 & 1 & 0 \\ \frac{i}{2} & 0 & 0 \end{pmatrix}$ in the standard basis. The stabilizer of the origin is a parabolic subgroup $B/\mathbb{Z}_3$ where B is the Borel subgroup in $SU(2, 1)$. On the infinitesimal level, the grading of the Lie algebra $\mathfrak{g} = \mathfrak{su}(2, 1)$ has the form

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix} : u \in \mathbb{R} \right\}, \ \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -2i\bar{x} & 0 \end{pmatrix} : x \in \mathbb{C} \right\},$$
$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} w & 0 & 0 \\ 0 & ir & 0 \\ 0 & 0 & -\bar{w} \end{pmatrix} : w \in \mathbb{C}, r \in \mathbb{R}, \ r + 2\mathrm{Im}(w) = 0 \right\},$$
$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & 2iz & 0 \\ 0 & 0 & \bar{z} \\ 0 & 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\}, \ \mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : v \in \mathbb{R} \right\}.$$

Geometrically, there are two classes of tangent vectors on CR–manifolds of codimension 1 with the definite Levi form. The vectors from the CR–subbundle correspond to vectors in the P–invariant subset $\mathfrak{g}_{-1} \subset \mathfrak{g}_-$ and the vectors transversal to the CR–distribution are described by the complement $\mathfrak{g}_- \setminus \mathfrak{g}_{-1}$ in $\mathfrak{g}_-$. More explicitly, the truncated adjoint action of P on $\mathfrak{g}_-$ is given as follows. Any element $\exp \begin{pmatrix} 0 & 2iz & v \\ 0 & 0 & \bar{z} \\ 0 & 0 & 0 \end{pmatrix} \in P_+$ maps $\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ u & -2i\bar{x} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ x+u\bar{z} & 0 & 0 \\ u & -2i(\bar{x}+uz) & 0 \end{pmatrix}$ and the action of the subgroup G_0 multiplies by nonzero complex numbers on $\mathfrak{g}_{-1}$ and by nonzero real numbers on $\mathfrak{g}_{-2}$. Hence the nontrivial G_0–orbits in $\mathfrak{g}_-$ are given just by nonzero entries of the matrix, so we get the following G_0–invariant subsets in $\mathfrak{g}_-$:

$$A_1 = \mathfrak{g}_{-1} \setminus \{0\},$$
$$A_2 = \mathfrak{g}_{-2} \setminus \{0\},$$
$$A_3 = \mathfrak{g}_- \setminus (\mathfrak{g}_{-1} \cup \mathfrak{g}_{-2}).$$

Obviously, $P(A_1) = A_1$ and $P(A_2) = P(A_3) = \mathfrak{g}_- \setminus \mathfrak{g}_{-1}$, so for any direction in the CR–distribution only geodesics of type $\mathcal{C}_{A_1}$ can emanate in this direction and for any vector which is transversal, there are geodesics of both types $\mathcal{C}_{A_2}$ and $\mathcal{C}_{A_3}$ tangent to this vector.

Resume.

type	X	B^1 B^2 B^3	ord	dim	R	$\varphi''(0)$
A_1	$\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -2i\bar{x} & 0 \end{pmatrix}$	$\mathfrak{p}_+$ $\mathfrak{g}_2$ 0	3	$3 = 2n+1$	$s\begin{pmatrix} 0 & 2\bar{x} & 0 \\ 0 & 0 & ix \\ 0 & 0 & 0 \end{pmatrix}$	$-2s\lvert x\rvert^2$
A_2	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}$	$\mathfrak{g}_2$ 0 $\dagger$	2	1	$\begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$-2vu$
A_3	$\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ u & -2i\bar{x} & 0 \end{pmatrix}$	$\mathfrak{p}_+$ 0 $\dagger$	2	$3 = 2n+1$	0	0

Remarks. The case of real hypersurfaces in $\mathbb{C}^2$ was carefully studied by Cartan who involves the chains (in our setting, the geodesics of type $\mathcal{C}_{\mathfrak{g}_{-2}}$) among the system of invariants, which resolve two real hypersurfaces in $\mathbb{C}^2$ are equivalent under a biholomorphic transformation. Furthermore, the standard homogeneous models of embedded CR–manifolds, the CR–quadrics, are obtained by osculating the initial CR–manifold up to the second order in any its point, and this is probably the most initial example of the construction of the Cartan's space; see 2.7 for the abstract definition and [**16**] for other comments and details.

B. Indefinite case. Let us consider the case with CR–dimension 2 and Hermitean form of signature $(1, 1)$. The modeling quadric is the 5-dimensional hypersurface $\mathrm{Im}(w) = |z_1|^2 - |z_2|^2$ in $\mathbb{C}^3$. The group of CR–automorphisms of this quadric is isomorphic to $G = SU(2, 2)/K$, where K is the discrete noneffective kernel. Let us suppose the group $SU(2, 2)$ is given by the Hermitean inner product with the block matrix $\begin{pmatrix} 0 & 0 & -\frac{i}{2} \\ 0 & J & 0 \\ \frac{i}{2} & 0 & 0 \end{pmatrix}$ where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The stabilizer of the origin is the block upper triangular matrix as above with the middle block of size 2×2. On the infinitesimal level we get

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix} : u \in \mathbb{R} \right\}, \ \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X_1 & 0 & 0 \\ 0 & -2i\bar{X}_1^t J & 0 \end{pmatrix} : X_1 \in \mathbb{C}^2 \right\},$$
$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} w & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & -\bar{w} \end{pmatrix} : w \in \mathbb{C}, W \in \mathfrak{u}(1,1), \ \mathrm{tr}(W) + 2\,\mathrm{Im}(w) = 0 \right\}$$
$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & 2iZ_1 & 0 \\ 0 & 0 & J\bar{Z}_1^t \\ 0 & 0 & 0 \end{pmatrix} : Z_1 \in \mathbb{C}^{2*} \right\}, \ \mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : v \in \mathbb{R} \right\}.$$

In contrast to the definite case, there are new types of tangent vectors in the CR–subbundle given by the sign of their length. More precisely, the property of positive, negative, and zero length of a vector is invariant with respect to the truncated adjoint action of the structure group P which factors over $G_0 = \mathbb{R} \times U(1,1)$ on the CR–subbundle. Explicitly, the action of an element $\begin{pmatrix} w & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \bar{w}^{-1} \end{pmatrix} \in G_0$ on $\mathfrak{g}_-$ is given by the transcription $(u, X_1) \mapsto (|w|^{-2}u, w^{-1}AX_1)$ if we identify each element $\begin{pmatrix} 0 & 0 & 0 \\ X_1 & 0 & 0 \\ u & -2i\bar{X}_1^t J \end{pmatrix} \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ with the pair $(u, X_1) \in \mathbb{R} \times \mathbb{C}^2$. Similarly, each element $\begin{pmatrix} 0 & 2iZ_1 & v \\ 0 & 0 & J\bar{Z}_1^t \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is determined by a pair $(Z_1, v) \in \mathbb{C}^{2*} \times \mathbb{R}$ and $\exp(Z_1, v) \in P_+$ acts as $(u, X_1) \mapsto (u, X_1 + uJ\bar{Z}_1^t)$. Altogether, there are three distinguished classes of vectors in the CR–distribution and one class of vectors which are transversal. For the transversal directions there are four G_0–invariant subsets with the common P–orbit $\mathfrak{g}_- \setminus \mathfrak{g}_{-1}$, established by the conditions $X_1 = 0$, $\|X_1\|^2 = \bar{X}_1^t J X_1 = 0$, and $\|X_1\|^2 \neq 0$, respectively. First of all, let us focus on the subset $A = \{(u, X_1) : u \neq 0, \|X_1\| = 0\}$ in order to show that all geodesics of type $\mathcal{C}_A$ are chains, i.e. curves of type $\mathcal{C}_{\mathfrak{g}_{-2}}$. This is satisfied since for any $X = (u, X_1) \in A$ there is $Y = (u, 0) \in \mathfrak{g}_{-2}$ and $Z = (u^{-1}\bar{X}_1^t J, 0) \in \mathfrak{p}_+$ such that $\exp tX = \exp Z \exp tY$ modulo P. Hence, there are six G_0–invariant subsets left to be discussed, which are indicated as follows. Let us denote by the symbol $\mathfrak{g}_{-1}^0$ the set of all nonzero vectors in $\mathfrak{g}_{-1}$ which have the zero length, analogously for $\mathfrak{g}_{-1}^+$ and $\mathfrak{g}_{-1}^-$. Now, all distinct classes of generalized geodesics correspond to the following G_0–invariant subsets in $\mathfrak{g}_-$:

$$
\begin{aligned}
A_1 &= \mathfrak{g}_{-1}^0, \\
A_2 &= \mathfrak{g}_{-1}^+, \\
A_3 &= \mathfrak{g}_{-1}^-, \\
A_4 &= \mathfrak{g}_{-2} \setminus \{0\}, \\
A_5 &= \mathfrak{g}_{-1}^+ \times (\mathfrak{g}_{-2} \setminus \{0\}), \\
A_6 &= \mathfrak{g}_{-1}^- \times (\mathfrak{g}_{-2} \setminus \{0\}).
\end{aligned}
$$

The first three subsets are P–invariant and the remaining three lie in the common P–orbit $\mathfrak{g}_- \setminus \mathfrak{g}_{-1}$. For any vector out of the CR–subbundle there are geodesics of the latter three types tangent to that vector.

Resume. In the table below we only refer to the properties which characterize the subsets in question instead of the explicit description.

type	X	B^1	B^2	B^3	ord	dim	R	$\varphi''(0)$
A_1	$(0, \|X_1\|^2 = 0)$	$\mathfrak{p}_+$	$(Z_1(X_1) = 0, *)$	$\dagger$	2	2	$(Z_1(X_1) \in \mathbb{R}, *)$	$-2Z_1(X_1)$
$A_{2,3}$	$(0, \|X_1\|^2 \neq 0)$	$\mathfrak{p}_+$	$\mathfrak{g}_2$	0	3	$5 = 2n+1$	$(-is\bar{X}_1^t J, 0)$	$-2s\|X_1\|^2$
A_4	$(u, 0)$	$\mathfrak{g}_2$	0	$\dagger$	2	1	$(0, v)$	$-2vu$
$A_{5,6}$	$(u, \|X_1\|^2 \neq 0)$	$\mathfrak{p}_+$	0	$\dagger$	2	$5 = 2n+1$	0	0

Remarks. This classification generalizes that in the definite case and it seems to be valid for hypersurface CR–manifolds of a general CR–dimension n. In particular, for $n = 1$ the geodesics of type $\mathcal{C}_{A_1}$ can not appear and the couples A_2, A_3 and A_5, A_6 reduce either to the positive or the negative case, so we get just the table in the case of the definite signature above.

For general n, the notion of chains (the geodesics of type $\mathcal{C}_{A_4}$) is well understood due to Chern and Moser, who extend the ideas of Cartan for $n = 1$ to all nondegenerated CR–manifolds of hypersurface type. Through any point in each direction transversal to the CR–subbundle there emanates a unique unparametrized chain which carries a distinguished class of projective reparametrizations. Next, there is another invariant class of curves on nondegenerated CR–manifolds of indefinite types which completes the system of chains in some specific sense. These was introduced by Koch as null–chains and they correspond to geodesics of type $\mathcal{C}_{A_1}$ in our setting, see [**15**] for details. Through any point in each null–direction in the CR–distribution, there is a 2–dimensional family of (parametrized) null–chains, each of them endowed with a canonical projective structure; the dimension 2 is computed as $2 = 2n + 1 - 2(n - 1) - 1$.

Further we discuss geodesics of generic types corresponding to the subsets $A_{2,3}$ and $A_{5,6}$, which are tangent and transversal to the CR–subbundle, respectively. Their properties are visible from the table above, the dimensions of $\mathcal{C}^{\xi}_{A_i}$ are obvious. In particular, the geodesics of types $\mathcal{C}_{A_5}$ and $\mathcal{C}_{A_6}$, which complete the system of distinguished curves transversal to the CR–subbundle, carry only affine class of distinguished reparametrizations.

7.5. CR–structures of codimension 2. While CR–geometries of codimension 1 are always parabolic, there are only a few cases among CR–geometries of higher codimension where the CR–manifolds in question allow to be endowed with a structure of parabolic geometry. One of the exceptional cases corresponds to the CR–structures of CR–dimension 2 and real codimension 2. The modeling quadrics in $\mathbb{C}^4$ are given by the system of equations $\mathrm{Im}(w_j) = h_j(z, \bar{z})$, $j = 1, 2$, where (z_1, z_2, w_1, w_2) are coordinates in $\mathbb{C}^4$ and h_j are Hermitean forms on $\mathbb{C}^2$. Hence the list of modeling quadrics depends just on the classification of nondegenerated Hermitean forms on $\mathbb{C}^2$ with values in $\mathbb{C}^2$. Any such form takes in suitable coordinates one of the following forms

$$h(z, \bar{z}) = (|z_1|^2, |z_2|^2),$$
$$h(z, \bar{z}) = (|z_1|^2, \mathrm{Re}(z_1\bar{z}_2)),$$
$$h(z, \bar{z}) = (\mathrm{Re}(z_1\bar{z}_2), \mathrm{Im}(z_1\bar{z}_2)),$$

which are called hyperbolic, parabolic, and elliptic, respectively. From these possibilities only the hyperbolic and elliptic quadric carries the structure of a $|2|$–graded parabolic geometry, which is due to the fact that in both cases the automorphism group of the corresponding quadric is a semisimple Lie group and the stabilizer of any its point is a parabolic subgroup. All details can be found in [**20**] and references therein. Generalized geodesics on hyperbolic and elliptic CR–manifolds have been discussed in [**28**] and we only summarize the achieved results here.

A. Hyperbolic quadric. According to the above ideas, the hyperbolic quadric in $\mathbb{C}^4$ is expressed by the equations

$$\operatorname{Im}(w_1) = |z_1|^2,\ \operatorname{Im}(w_2) = |z_2|^2.$$

Obviously, this is a direct product of two CR–spheres from the definite part of 7.4. The product of the two actions of $SU(2,1)/\mathbb{Z}_3$ on each CR–sphere defines a transitive action of the Lie group $SU(2,1)/\mathbb{Z}_3 \times SU(2,1)/\mathbb{Z}_3$ on the hyperbolic quadric. Any element of the latter group acts by a CR–automorphism but there is another (involutive) automorphism which interchanges the two spheres in the product. Hence the group of automorphisms of the hyperbolic quadric is isomorphic to $G = (SU(2,1)/\mathbb{Z}_3 \times SU(2,1)/\mathbb{Z}_3) \rtimes \mathbb{Z}_2$. The stabilizer of the origin is $P = (B/\mathbb{Z}_3 \times B/\mathbb{Z}_3) \rtimes \mathbb{Z}_2$ where B is the Borel subgroup in $SU(2,1)$. Obviously, G is semisimple and P parabolic.

The following discussion is based only on results achieved for the CR–sphere in 7.4 due to the product structure of the hyperbolic quadric. In particular, the tangent bundle is a direct product of two subbundles which correspond to vanishing left and right part of $\mathfrak{g}_- = \mathfrak{g}_-^L \times \mathfrak{g}_-^R$, respectively. This distinguishes a class of vectors in the tangent bundle which we call singular. All P–invariant and G_0–invariant subsets in $\mathfrak{g}_- = \mathfrak{g}_-^L \times \mathfrak{g}_-^R$ are obtained as products of P and G_0–invariant subsets in each slot up to their interchanging. Especially, the singular directions correspond to the P–invariant subset $\mathfrak{g}_-^L \cup \mathfrak{g}_-^R$ which is just the P–orbit of $\mathfrak{g}_-^L$. Similarly, all P and G_0–invariant subsets in $\mathfrak{g}_-$ discussed below are written in the brief form, i.e. $A \times B \subset \mathfrak{g}_-^L \times \mathfrak{g}_-^R$ means its P and G_0–orbit $A \times B \cup B \times A$, respectively. Altogether, we shall see that there are five types of tangent vectors on the hyperbolic quadric. First, let us consider the subsets

$$\begin{aligned}
A_1 &= \{0\} \times (\mathfrak{g}_{-1}^R \setminus \{0\}),\\
A_2 &= \{0\} \times (\mathfrak{g}_{-2}^R \setminus \{0\}),\\
A_3 &= \{0\} \times (\mathfrak{g}_-^R \setminus (\mathfrak{g}_{-1}^R \cup \mathfrak{g}_{-2}^R)).
\end{aligned}$$

Obviously, the subset $P(A_1) = A_1$ corresponds to singular vectors in the CR–subbundle and $P(A_2) = P(A_3) = \{0\} \times (\mathfrak{g}_-^R \setminus \mathfrak{g}_{-1}^R)$ gives singular vectors which are transversal. Curves of these types have been fully classified in 7.4.

Now, by the symbol $\mathfrak{g}_{--}^L$ we denote the set of generic elements in $\mathfrak{g}_-^L$, i.e. $\mathfrak{g}_{--}^L = \mathfrak{g}_-^L \setminus (\mathfrak{g}_{-1}^L \cup \mathfrak{g}_{-2}^L)$, and similarly for $\mathfrak{g}_{--}^R$ and $\mathfrak{g}_{--}$. The symbol S represents the set of singular tangent vectors, i.e. $S = \{0\} \times \mathfrak{g}_-^R$ with respect to the conventions above. All remaining G_0–invariant subsets in $\mathfrak{g}_-$ determining geodesics in nonsingular directions are

$$\begin{aligned}
A_4 &= \mathfrak{g}_{-1} \setminus S,\\
A_5 &= \mathfrak{g}^L_{-1} \times \mathfrak{g}^R_{-2} \setminus S,\\
A_6 &= \mathfrak{g}^L_{-1} \times \mathfrak{g}^R_{--} \setminus S,\\
A_7 &= \mathfrak{g}_{-2} \setminus S,\\
A_8 &= \mathfrak{g}^L_{-2} \times \mathfrak{g}^R_{--} \setminus S,\\
A_9 &= \mathfrak{g}_{--} \setminus S.
\end{aligned}$$

The subset A_4 is P–invariant and the corresponding vectors in the tangent bundle of the hyperbolic quadric are the nonsingular vectors in the CR–distribution. Further, the subset $P(A_5) = P(A_6) = \mathfrak{g}^L_{-1} \times (\mathfrak{g}^R_- \setminus \mathfrak{g}^R_{-1})$ determines a specific class of nonsingular vectors not belonging to the CR–subbundle and, finally, the subset $P(A_7) = P(A_8) = P(A_9) = \mathfrak{g}_- \setminus \mathfrak{g}_{-1}$ corresponds to the generic nonsingular directions out of the CR–subbundle.

Resume. The product structure of the hyperbolic quadric with the isolated action of the structure group on each slot leads directly to the following results compiled only from the CR–sphere case. Below we use a natural notation where any element $X \in \mathfrak{g}_- = \mathfrak{g}^L_{-2} \times \mathfrak{g}^L_{-1} \times \mathfrak{g}^R_{-2} \times \mathfrak{g}^R_{-1}$ is indicated by a quadruple $\left(\begin{smallmatrix} u_L & u_R \\ x_L & x_R \end{smallmatrix}\right)$ with $u_i \in \mathbb{R}$ and $x_i \in \mathbb{C}$. Similar conventions are kept for $Z \in \mathfrak{g}^L_1 \times \mathfrak{g}^L_2 \times \mathfrak{g}^R_1 \times \mathfrak{g}^R_2 = \mathfrak{p}_+$.

type	X	B^1	B^2	B^3	ord	dim	R	$\varphi''(0)$
A_4	$\left(\begin{smallmatrix} 0 & 0 \\ x_L & x_R \end{smallmatrix}\right)$	$\mathfrak{p}_+$	$\mathfrak{g}_2$	0	3	6	$s\left(\begin{smallmatrix} \lvert x_R\rvert^2 x_L & \lvert x_L\rvert^2 x_R \\ 0 & 0 \end{smallmatrix}\right)$	$-2s\,\lvert x_L x_R\rvert^2$
A_5	$\left(\begin{smallmatrix} 0 & u_R \\ x_L & 0 \end{smallmatrix}\right)$	$\mathfrak{p}^L_+ \times \mathfrak{g}^R_2$	$\mathfrak{g}^L_2 \times \{0\}$	0	3	4	$s\left(\begin{smallmatrix} u_R x_L & 0 \\ 0 & \lvert x_L\rvert^2 \end{smallmatrix}\right)$	$-2s\,u_R\lvert x_L\rvert^2$
A_6	$\left(\begin{smallmatrix} 0 & u_R \\ x_L & x_R \end{smallmatrix}\right)$	$\mathfrak{p}_+$	$\mathfrak{g}^L_2 \times \{0\}$	0	3	6	0	0
A_7	$\left(\begin{smallmatrix} u_L & u_R \\ 0 & 0 \end{smallmatrix}\right)$	$\mathfrak{g}_2$	0	†	2	2	$s\left(\begin{smallmatrix} 0 & 0 \\ u_R & u_L \end{smallmatrix}\right)$	$-2s\,u_L u_R$
A_8	$\left(\begin{smallmatrix} u_L & u_R \\ 0 & x_R \end{smallmatrix}\right)$	$\mathfrak{g}^L_2 \times \mathfrak{p}^R_+$	0	†	2	4	0	0
A_9	$\left(\begin{smallmatrix} u_L & u_R \\ x_L & x_R \end{smallmatrix}\right)$	$\mathfrak{p}_+$	0	†	2	6	0	0

Remarks. Among all nonsingular types of generalized geodesics there are three of them of a particular interest. The first type is given by the choice $A_4 = \mathfrak{g}_{-1} \setminus A_1$ and geodesics of this type are the only ones which emanate in generic directions of the CR–distribution.

The second distinguished choice is $A_7 = \mathfrak{g}_{-2}$, which generalizes the notion of chains in nonsingular directions as discussed in [**20**], and the last choice $A_9 = \mathfrak{g}_{--}$ defines curves of the generic type. For any nonsingular direction which does not belong to the CR–subbundle there is a 1–dimensional family of unparametrized chains with the projective class of parametrizations, 4–dimensional family of uniquely parametrized geodesics of type $\mathcal{C}_{A_8}$, and 6–dimensional family of uniquely parametrized curves of the generic type $\mathcal{C}_{A_9}$.

B. Elliptic quadric. The elliptic quadric in $\mathbb{C}^4$ is expressed as a graph of

$$\mathrm{Im}(w_1) = \mathrm{Re}(z_1\bar{z}_2),\ \mathrm{Im}(w_2) = \mathrm{Im}(z_1\bar{z}_2).$$

The group of automorphisms of the CR–structure on the elliptic quadric is less visible than in the hyperbolic case, however, the group is isomorphic to

$$G = SL(3,\mathbb{C})/\mathbb{Z}_3 \rtimes \mathbb{Z}_2$$

as shown in [**20**]. Similarly to the hyperbolic case, $\mathbb{Z}_3$ represents a noneffective kernel in $SL(3,\mathbb{C})$ and, on the infinitesimal level, $\mathbb{Z}_2$ provides an interchanging of the two components of the CR–subspace, see below. Isotropy parabolic subgroup P of the origin is the semidirect product $B/\mathbb{Z}_3 \rtimes \mathbb{Z}_2$ where B is the Borel subgroup of $SL(3,\mathbb{C})$ consisting of upper triangular matrices. The Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$, viewed as a real Lie algebra, and the corresponding parabolic subalgebra $\mathfrak{p}$ defines the gradation of $\mathfrak{g}$ according to the five diagonals.

In addition, the subspace $\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_L & 0 & 0 \\ 0 & x_R & 0 \end{pmatrix} : x_L, x_R \in \mathbb{C} \right\} \subset \mathfrak{g}_-$ defining the CR–distribution decomposes into $\mathfrak{g}_{-1} = \mathfrak{g}^L_{-1} \times \mathfrak{g}^R_{-1}$ which induces a product structure on the CR–subbundle. This distinguishes a class of tangent vectors in the CR–subbundle which correspond to the P–invariant subset $P(\mathfrak{g}^L_{-1}) = P(\mathfrak{g}^R_{-1}) = \mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1}$. The classification of remaining P and G_0–invariant subsets in $\mathfrak{g}_-$ is the same as in 7.3 except for all entries are complex now. In particular, there is a G_0–invariant subset $\mathfrak{g}_{-2} \times (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1})$ in $\mathfrak{g}_-$ which determines the same geodesics as the subset $\mathfrak{g}_{-2}$, so we omit this possibility in further discussion. Altogether, all G_0–invariant subsets in $\mathfrak{g}_-$ which define distinct classes of generalized geodesics are

$$\begin{aligned}
A_1 &= (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1}) \setminus \{0\}, \\
A_2 &= \mathfrak{g}_{-1} \setminus (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1}), \\
A_3 &= \mathfrak{g}_{-2} \setminus \{0\}, \\
A_4 &= \mathfrak{g}_- \setminus (\mathfrak{g}_{-1} \cup \mathfrak{g}_{-2} \times (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1})).
\end{aligned}$$

Obviously, subsets A_1 and A_2 are P–invariant and they correspond to distinguished and generic vectors in the CR–subbundle, respectively. Further, the subset $P(A_3) = P(A_4) = \mathfrak{g}_- \setminus \mathfrak{g}_{-1}$ determines vectors out of the CR–distribution.

Resume.

type	X	B^1	B^2	B^3	ord	dim	R	$\varphi''(0)$
A_1	$\begin{pmatrix} 0 & 0 & 0 \\ x_L & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathfrak{p}_+$	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$	$\dagger$	2	2	$s\begin{pmatrix} 0 & \bar{x}_L & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$	$-2s\|x_L\|^2$
A_2	$\begin{pmatrix} 0 & 0 & 0 \\ x_L & 0 & 0 \\ 0 & x_R & 0 \end{pmatrix}$	$\mathfrak{p}_+$	$\mathfrak{g}_2$	0	3	6	$s\begin{pmatrix} 0 & x_L^{-1} & 0 \\ 0 & 0 & x_R^{-1} \\ 0 & 0 & 0 \end{pmatrix}$	$-s$
A_3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$	$\mathfrak{g}_2$	0	$\dagger$	2	2	$s\begin{pmatrix} 0 & 0 & \bar{y} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$-2s\|y\|^2$
A_4	$\begin{pmatrix} 0 & 0 & 0 \\ x_L & 0 & 0 \\ y & x_R & 0 \end{pmatrix}$	$\mathfrak{p}_+$	0	$\dagger$	2	6	0	0

Remarks. The above classification looks like a complex analogy of Example 7.3 with the only difference that the two distinguished subspaces of $T^{-1}M$ in the Lagrangean case are not distinguishable here. Further, geodesics of type $\mathcal{C}_{A_3}$, which generalize chains from hypersurface CR–structures, share just the same properties as in the hyperbolic case.

7.6. x—x—dot structures. Let us conclude with the discussion of generalized geodesics in the so called x—x—dot geometries which are established by the Satake diagram x—x—o, i.e. the principal group is $G = SL(4,\mathbb{R})$ with the parabolic subgroup indicated as $P = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}$, see e.g. [**26**, 4.4]. Such structures appear as correspondence spaces in classical twistor theory, and they are related to the geometric theory of ODE's. The following discussion may be also understood as a block–wise generalization of the discussion in 7.3 which corresponds to the x—x case. The final classification will be quite similar to that for the x—x case except for the curves which emanate in directions out of the distribution $T^{-1}M$, where we get two distinct types of tangent vectors.

The grading of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(4,\mathbb{R})$ is described by block matrices of the form

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_2 & 0 & 0 \end{pmatrix} : X_2 \in \mathbb{R}^2 \right\}, \ \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & X_1 & 0 \end{pmatrix} : x \in \mathbb{R}, X_1 \in \mathbb{R}^2 \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & C \end{pmatrix} : a + b + \operatorname{tr}(C) = 0 \right\},$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & Z_1 \\ 0 & 0 & 0 \end{pmatrix} : z \in \mathbb{R}, Z_1 \in \mathbb{R}^{2*} \right\}, \ \mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & 0 & Z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : Z_2 \in \mathbb{R}^{2*} \right\}.$$

The truncated adjoint action of P on $\mathfrak{g}_-$ is described as follows. Any element $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & C \end{pmatrix} \in G_0$ acts as $\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ X_2 & X_1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ bxa^{-1} & 0 & 0 \\ CX_2a^{-1} & CX_1b^{-1} & 0 \end{pmatrix}$ and the action of an element $\exp \begin{pmatrix} 0 & z & Z_2 \\ 0 & 0 & Z_1 \\ 0 & 0 & 0 \end{pmatrix} \in P_+$ is given by the formula $\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ X_2 & X_1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ x+Z_1(X_2) & 0 & 0 \\ X_2 & X_1 - zX_2 & 0 \end{pmatrix}$. In particular, the P–action respects the linear independence of vectors X_1 and X_2 since the condition $a \cdot b \cdot \det(C) = 1$ is still satisfied and vectors X_1, X_2 are independent if and only if the vectors X_2 and $X_1 - zX_2$ are independent too, for any $z \in \mathbb{R}$.

There are two distinct P–invariant subspaces in $\mathfrak{g}_{-1}$ which distinguish two complementary subbundles in $T^{-1}M$. These sets are given by the blocks x and X_1 and we denote them by $\mathfrak{g}^L_{-1}$ and $\mathfrak{g}^R_{-1}$, respectively. Generic vectors in $T^{-1}M$ correspond to the P–invariant complement $\mathfrak{g}_{-1} \setminus (\mathfrak{g}^L_{-1} \cup \mathfrak{g}^R_{-1})$. Further, the P–orbit of $\mathfrak{g}_{-2}$ is $\left\{ \begin{pmatrix} 0 & 0 & 0 \\ Z_1(X_2) & 0 & 0 \\ X_2 & -zX_2 & 0 \end{pmatrix} \right\} \subset \mathfrak{g}_-$, so it never exhausts all vectors in $\mathfrak{g}_- \setminus \mathfrak{g}_{-1}$. Let us denote by C the hyperquadric in $\mathfrak{g}_-$ consisting of all elements such that vectors X_1 and X_2 are linearly dependent. Obviously, $P(\mathfrak{g}_{-2}) = C \setminus \mathfrak{g}_{-1}$ and the complement $\mathfrak{g}_- \setminus C$, formed by all elements in $\mathfrak{g}_-$ where X_1 and X_2 are independent, is P–invariant too. This provides the second class of vectors which are not tangent to $T^{-1}M$. Altogether, we have got five distinct classes of tangent vectors on manifolds with x—x—dot structures.

Now, let us consider the G_0–invariant subsets $\left\{\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ X_2 & 0 & 0 \end{pmatrix}\right\} \subset \mathfrak{g}_{-2} \times \mathfrak{g}_{-1}^L$ and $\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_2 & aX_2 & 0 \end{pmatrix}\right\} \subset \mathfrak{g}_{-2} \times \mathfrak{g}_{-1}^R$. The same arguments as in 7.3 show that generalized geodesics of these two types coincide with geodesics of type $\mathcal{C}_{\mathfrak{g}_{-2}}$, hence we omit these choices in the sequel. Altogether, one can conclude that all distinct types of generalized geodesics correspond to the following G_0–invariant subsets in $\mathfrak{g}_-$:

$$\begin{aligned}
A_1 &= \mathfrak{g}_{-1}^L \setminus \{0\}, \\
A_2 &= \mathfrak{g}_{-1}^R \setminus \{0\}, \\
A_3 &= \mathfrak{g}_{-1} \setminus (\mathfrak{g}_{-1}^L \cup \mathfrak{g}_{-1}^R), \\
A_4 &= \mathfrak{g}_{-2} \setminus \{0\}, \\
A_5 &= C \setminus (\mathfrak{g}_{-1} \cup \mathfrak{g}_{-2} \times (\mathfrak{g}_{-1}^L \cup \mathfrak{g}_{-1}^R)), \\
A_6 &= (\mathfrak{g}_- \setminus C) \cap (\mathfrak{g}_{-2} \times \mathfrak{g}_{-1}^R), \\
A_7 &= (\mathfrak{g}_- \setminus C) \setminus (\mathfrak{g}_{-2} \times \mathfrak{g}_{-1}^R).
\end{aligned}$$

Obviously, $P(A_1) = A_1$, $P(A_2) = A_2$, and $P(A_3) = A_3$, so only geodesics of these types emanate in appropriate directions in $T^{-1}M$. Further, $P(A_4) = P(A_5) = C \setminus \mathfrak{g}_{-1}$ and $P(A_6) = P(A_7) = \mathfrak{g}_- \setminus C$ and so for each vector which is transversal to the distribution $T^{-1}M$, either there are geodesics of types $\mathcal{C}_{A_4}$ and $\mathcal{C}_{A_5}$, or of types $\mathcal{C}_{A_6}$ and $\mathcal{C}_{A_7}$, according to the type of the tangent vector.

Resume. Each element of $\mathfrak{p}_+$ is determined by a triple $(z, Z_1, Z_2) \in \mathbb{R} \times \mathbb{R}^{2*} \times \mathbb{R}^{2*}$ and we use this notation in the table below in order to indicate the properties which characterize the mentioned subsets of $\mathfrak{p}_+$. Moreover, we write the triples as columns for the sake of economy.

type	X	B^1	B^2	B^3	ord	dim	R	$\varphi''(0)$
A_1	$\begin{pmatrix} 0&0&0 \\ x&0&0 \\ 0&0&0 \end{pmatrix}$	$\mathfrak{p}_+$	$\begin{pmatrix} 0 \\ * \\ * \end{pmatrix}$	†	2	1	$\begin{pmatrix} z \\ * \\ * \end{pmatrix}$	$-2xz$
A_2	$\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&X_1&0 \end{pmatrix}$	$\mathfrak{p}_+$	$\begin{pmatrix} * \\ Z_1(X_1)=0 \\ * \end{pmatrix}$	†	2	1	$\begin{pmatrix} * \\ Z_1 \\ * \end{pmatrix}$	$-2Z_1(X_1)$
A_3	$\begin{pmatrix} 0&0&0 \\ x&0&0 \\ 0&X_1&0 \end{pmatrix}$	$\mathfrak{p}_+$	$\begin{pmatrix} 0 \\ Z_1(X_1)=0 \\ * \end{pmatrix}$	$\begin{pmatrix} 0 \\ Z_1(X_1)=0 \\ Z_2(X_1)=0 \end{pmatrix}$	3	3	$\begin{pmatrix} Z_1(X_1) \\ Z_1 \\ Z_2(X_1)=0 \end{pmatrix}$	$-xZ_1(X_1)$
A_4	$\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ X_2&0&0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ Z_1(X_2)=0 \\ * \end{pmatrix}$	$\begin{pmatrix} 0 \\ Z_1(X_2)=0 \\ Z_2(X_2)=0 \end{pmatrix}$	†	2	1	$\begin{pmatrix} 0 \\ Z_1(X_2)=0 \\ Z_2 \end{pmatrix}$	$-2Z_2(X_2)$
A_5	$\begin{pmatrix} 0&0&0 \\ x&0&0 \\ X_2&aX_2&0 \end{pmatrix}$	$\mathfrak{p}_+$	$\begin{pmatrix} 0 \\ Z_1(X_2)=0 \\ Z_2(X_2)=0 \end{pmatrix}$	†	2	3	B^3	0
A_6	$\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ X_2&X_1&0 \end{pmatrix}$	$\begin{pmatrix} * \\ Z_1(X_2)=0 \\ * \end{pmatrix}$	$\begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}$	[illegible]	2	$3 = 2n - 1$	$\begin{pmatrix} Z_1(X_1) \\ Z_1(X_2)=0 \\ Z_2(X_2)=Z_1(X_1) \end{pmatrix}$	$-2Z_2(X_2)$
A_7	$\begin{pmatrix} 0&0&0 \\ x&0&0 \\ X_2&X_1&0 \end{pmatrix}$	$\mathfrak{p}_+$	0	†	2	$5 = 2n + 1$	$\begin{pmatrix} Z_1(X_1) \\ Z_1(X_2)=0 \\ 0 \end{pmatrix}$	$-Z_1(X_1)$

Remarks. The above classification seems to be valid for a general value of n, which corresponds to $n-1$ circles in the Satake diagram x—x—o$\cdots$o—o. For $n = 1$ the first five rows really recover the classification in the x—x case presented in 7.3 (the last two possibilities, where the vectors X_1 and X_2 are independent, could not appear in that case). Further, the dimensions of sets $\mathcal{C}_A^\xi$ for geodesics of first five types does not depend on the value of n. For instance, the dimension of $\mathcal{C}^{\xi}_{A_4}$ is computed as $2n-1-(2n-2)=1$, the others are similar. In particular, geodesics of type $\mathcal{C}_{\mathfrak{g}_{-2}}$ have got just the same properties as the chains in parabolic contact geometries, although the dimension of $\mathfrak{g}_{-2}$ equals n here. Of course, this is due to the bracket $[X[XZ]]$ equals to a multiple of $X \in \mathfrak{g}_{-2}$ for all $Z \in \mathfrak{g}_2$.

List of symbols

Ad	adjoint representation $G \to GL(\mathfrak{g})$
ad	$= T_e \operatorname{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$
$\underline{\operatorname{Ad}}$	truncated adjoint representation, p. 6
$[\ ,\ ]$	Lie bracket
$c^{u,X}$	generalized geodesic corresponding to $(u, X) \in \mathcal{G} \times A$
$\mathcal{C}_A$	geodesics of specific type, 2.4
$\mathcal{C}_A^\xi$	geodesics of type $\mathcal{C}_A$ tangent to ξ, pp. 30, 43
δ	left logarithmic derivative, 2.3
exp	exponential mapping $\mathfrak{g} \to G$
ε	normal coordinates in $e \in G$, p. 28
Fl	flow
γ	principal connection form
$\gamma(\xi)$	lift of $\xi \in TM$ according to γ, pp. 4, 17
G/H	homogeneous space
$\mathcal{G}$	principal bundle of Cartan geometry, 2.1
$\tilde{\mathcal{G}}$	extension of $\mathcal{G}$, p. 7
G_m^r	jet group of order r in dimension m
$GL(m, \mathbb{R})$	$= G_m^1$, general linear group
$j_a^r f$	r–jet of f at a
$J^r(M, N)$	bundle of r–jets of maps $M \to N$
ℓ	left multiplication in Lie group
λ	left action of Lie group
M	smooth manifold
∇	absolute/covariant derivative of general/affine connection, 1.3
∇^ω	invariant derivative of Cartan connection, 2.1
∇^σ	covariant derivative of Weyl connection, 3.4
o	$= eH$, origin in G/H
ω	Cartan connection, 2.1
$\Omega^1(M, V)$	V–valued one–forms on M
P^r	bundle of r–frames
P	rho–tensor, 3.5
$\mathbb{R}$	real numbers
r	principal right action
S	Cartan's space bundle, 2.7
σ	Weyl structure, 3.2
$\rtimes$	semidirect product
T_1^r	bundle of r–velocities
T	$= T_1^1$, tangent bundle
$T_{\mathcal{C}_A}^r$	r–velocities of geodesics of type $\mathcal{C}_A$, 2.6, 5.3
θ	canonical form on $P^r M$
$\mathfrak{X}M$	vector fields on M
ζ_X	fundamental vector field generated by X

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