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Equations and symmetries of generalized geodesics

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Abstract. We are interested in equations of distinguished curves in general Cartan geometries. In this paper we present a way to construct equations for non-parameterized distinguished curves via symmetry algebras of model curves and Cartan's method of moving frame. We also discuss the correspondence, well-known in particular geometries, between maps preserving generalized geodesics of specific type and morphisms of the geometric structure. As examples we compute equations together with their symmetries for generalized geodesics in projective, projective contact and Lagrangean contact geometries.

1. Introduction

For a Lie group G and a closed subgroup $H \subset G$, the Cartan geometry of type (G, H) on a smooth manifold M consists of the following data [**13**]:

- a principal fiber bundle $\mathcal{G} \to M$ with the structure group H;
- Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ with $\mathfrak{g}$ to be the Lie algebra of G.

Cartan geometry is called split if there is a (fixed) subalgebra $\mathfrak{n} \subset \mathfrak{g}$ complementary to $\mathfrak{h} \subset \mathfrak{g}$, the Lie algebra of H. The principal H-bundle $G \to G/H$ with the Maurer–Cartan form $\omega_G \in \Omega^1(G, \mathfrak{g})$ is the flat (or homogeneous) model of Cartan geometries of type (G, H). Cartan geometries appeared first in the pioneer works of E. Cartan [**2, 3**] under the name of generalized spaces. One of his ideas was to generalize his moving frame method to submanifolds in Cartan geometries [**4**]. This works especially smoothly for curves, where the structure equations are automatically satisfied, and leads to the notion of distinguished curves.

First, there are special types of curves in the homogeneous space G/H, namely, the orbits of one-parameter subgroups of G, known as homogeneous curves. This determines special classes of curves on all manifolds endowed with the structure of Cartan geometry of the same type via the notion of development [**11, 15**]. Explicitly, the curve on M is a *distinguished curve* if and only if it develops (at any point) into a curve of the form

$$h \exp(tX)o = \exp(t \operatorname{Ad}_h(X))o \tag{1.1}$$

for some $h \in H$ and $X \in \mathfrak{g}$. In fact, any H-invariant set $\mathcal{C}$ of curves in G/H, mapping 0 to the origin $o = eH$, leads to a well-defined set of curves on M which we call the *distinguished curves of type $\mathcal{C}$*. This is the way how to distinct curves of different

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properties. Particular examples of such curves are geodesics in Euclidean, affine and projective geometries, conformal circles and null-geodesics in conformal geometries, chains in hypersurface CR geometries, and others. All these Cartan geometries are split and the mentioned types of distinguished curves can be specified, according to the notation above, by the condition $X \in A$ where A is a subset in $\mathfrak{n}$. In those cases one speaks about *generalized geodesics of type A*.

One of the questions of this paper is whether the way from a given Cartan geometry to the family of distinguished curves of a specified type can be reversed, i.e., whether one can recover the whole geometry having just the family of distinguished curves. There are three major examples with affirmative and negative answers we have in mind. First, consider an affine geometry with geodesics as distinguished curves. Then it is well-known that there are many non-equivalent affine geometries having the same sets of geodesics considered as non-parameterized curves. So, we see that the affine geometry can not be recovered from its geodesics. On the other hand, any projective geometry is uniquely determined by its geodesics [**2**]. In particular, smooth map keeps the set of non-parameterized geodesics invariant if and only if it is a projective motion. Besides the projective geometries, the second well-known instance of the feature above to be satisfied is the conformal geometry; see [**14**] for the infinitesimal version of the later statement in the case of definite-signature conformal metric concerning conformal circles with distinguished parameters. However, using the techniques presented below, one can prove the same considering just non-parameterized conformal circles. Note that the same question in the case of indefinite conformal metric with null-geodesics as the specified type of distinguished curves is completely different and more or less trivial to answer.

Let us recall here the converse statement, namely, that any morphism of Cartan geometry $(\mathcal{G}, \omega)$ respects the distinguished curves of any specified type, is trivially satisfied due to the equivalent definition of distinguished curves as projections of flow lines of constant vector fields in $\mathfrak{X}(\mathcal{G})$, cf. [**13**].

The main aim of this paper is to develop a method leading up to the system of differential equations, say $\mathcal{E}$, which describe any specified class of distinguished curves. In fact, this is a modern presentation of Cartan's familiar ideas involved in [**2, 3, 4**]. The rest of this paper is devoted to making an intuition concerning the questions above. Concretely, we discuss the infinitesimal symmetries of $\mathcal{E}$ which are essentially useful, however, one has to be a bit careful in this field. At any rate, there is no hope to get result of general character in this way but it serves just as a test of the conjecture above should be satisfied or not. We wish to come back to this topic elsewhere.

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2. Symmetry algebras and distinguished curves

Here we summarize all basics, skipping those details which can be found in the referred literature. In this section we consider a fixed Cartan geometry of type (G, H)

and distinguished curves specified by $\mathcal{C}$, an H-invariant set of homogeneous curves in G/H mapping 0 to the origin.

One of the convenient tools describing basic properties of homogeneous curves is the symmetry algebra of a curve defined in [**6**]. In particular, it encodes the order of initial condition, which determines the homogeneous curve of the current type uniquely, as well as it helps to decide whether the admissible distinguished reparameterizations are projective or affine [**5, 7**]. This approach also leads to a handy criterion for a curve to be a distinguished curve of specified type, Corollary 1. Hence, with the help of the Cartan's moving frame method [**4**], one obtains the system $\mathcal{E}$ of ordinary differential equations describing the curves in question. Such systems can be thought as a deformation of special invariant differential equations on a model space, which were well-studied for low-dimensional geometries, see [**12**].

Having equations of distinguished curves in hand, we use the progress of [**12**] in order to look for the infinitesimal symmetries of $\mathcal{E}$. The main output is the system of partial differential equations, called determining equations, whose solutions are the infinitesimal symmetries. Occasionally, it happens that the infinitesimal symmetries of $\mathcal{E}$ are in fact infinitesimal transformations of the geometric structure we began with. Of course, this must be encoded somehow in the shape of determining equations, which we demonstrate in Sections 3 and 4 in the cases of projective and Lagrangean contact geometries.

2.1. As above, let $\mathcal{C}$ be a set of homogeneous curves in G/H. Without any loss of generality, we assume $\mathcal{C}$ to be an orbit of the structure group H, i.e., any two curves of $\mathcal{C}$ are conjugated by an element of H. In fact, the study of distinguished curves goes in this direction in order to discern classes of curves of different behavior.

2.2. Fix a homogeneous curve $L(t) = \exp(tX)o$, the representative of $\mathcal{C} = \mathcal{C}_L$. Following [**6**], compute the symmetry algebra $\operatorname{sym} L \subset \mathfrak{g}$ as follows. For the sequence

$$\mathfrak{h} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \dots \tag{2.1}$$

of Lie subalgebras, defined recursively as $\mathfrak{a}_{i+1} = \{Y \in \mathfrak{a}_i : [Y, X] \subset \langle X \rangle + \mathfrak{a}_i\}$, let r be the order that the sequence stabilizes from. Now we put the symmetry algebra of L to be the subalgebra $\operatorname{sym} L = \langle X \rangle + \mathfrak{a}_r$ of $\mathfrak{g}$.

Equivalently, for any (non-parameterized) curve $L \subset G/H$, the symmetry algebra $\operatorname{sym} L \subset \mathfrak{g}$ is defined as

$$\operatorname{sym} L = \{X \in \mathfrak{g} : \underline{R}_X(p) \in T_pL \text{ for all } p \in L\} \tag{2.2}$$

where $\underline{R}_X$ denotes the vector field on G/H generated by $X \in \mathfrak{g}$ so that $\underline{R}_X(p) = \frac{d}{dt}\big|_0 \exp(tX)p$. In fact, $\underline{R}\colon \mathfrak{g} \to \mathfrak{X}(G/H)$ is an (anti-)homomorphism of Lie algebras and any subalgebra of $\mathfrak{g}$ gives rise to an integrable distribution on G/H. Obviously, for any $p \in L$, $\underline{R}(\operatorname{sym} L)(p) \subseteq T_pL$ and the curve L is homogeneous if and only if $\underline{R}(\operatorname{sym} L)(p) = T_pL$. For later use, let us mention that under the usual identification $T(G/H) \cong G \times_H (\mathfrak{g}/\mathfrak{h})$, via the Maurer–Cartan form ω_G on G, the vector field $\underline{R}_X \in \mathfrak{X}(G/H)$ is written as

$$\underline{R}_X(gH) = [\![g, \operatorname{Ad}_{g^{-1}}(X) + \mathfrak{h}]\!], \tag{2.3}$$

for any $gH \in G/H$. Easily, homogeneous curves L_1 and L_2 coincide up to conjugation by an element of H (say $h \in H$) if and only if the symmetry algebras $\operatorname{sym}(L_1)$ and $\operatorname{sym}(L_2)$ are conjugated (via $\operatorname{Ad}_h \in GL(\mathfrak{g})$).

2.3. Let $(p\colon \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, H), and let c be a (parameterized) smooth curve on M. Denote by $C \subset M$ the non-parameterized image of c. Then, by definition, c is a curve of type $\mathcal{C}_L$ if and only if it develops (at any point) into a curve which is conjugated to L by an element of H. Fixing a point $x \in C$ and following the notation of [**15**], this just means that there is an element $u \in p^{-1}(x) \subset \mathcal{G}$ such that $(\operatorname{dev}_x c)(t) = [\![u, L(t)]\!] \subset S_xM \cong G/H$, where $SM = \mathcal{G} \times_H (G/H)$ is the Cartan's space of M.

Here we use the following definition of development. For $x = c(t_0)$, let $\hat{c}\colon I \to \mathcal{G}$ be any curve over C such that $p(\hat{c}(t)) = c(t_0+t)$ and $\hat{c}(0) = u$. Let further $Y : I \to \mathfrak{g}$ be given as $Y(t) = \omega\big(\frac{d}{dt}\hat{c}(t)\big)$. By the existence and uniqueness theorem for ordinary differential equations on Lie groups, there is a unique curve $a\colon I \to G$, such that $a^*\omega_G = Y$ and $a(0) = e$, and then the development of c is given as $(\operatorname{dev}_x c)(t) = [\![u, a(t)o]\!] \subset S_xM$. Since ω_G is the Maurer–Cartan form on G, then $a^*\omega_G = \delta a$ is just the Darboux (or left logarithmic) derivative of a. Hence we say, that c is a curve of type $\mathcal{C}_L$, for $L(t) = \exp(tX)o$, if and only if there is a lift $\hat{c}$ of c in $\mathcal{G}$ as above such that $\omega\big(\frac{d}{dt}\hat{c}(t)\big) = X$, especially, $X = \delta\exp(tX)$ is constant. Concerning non-parameterized curves, the following Proposition appears.

Proposition 1. *Let $C \subset M$ be an immersed 1-dimensional submanifold. Then the following two conditions are equivalent:*

(1) C admits a (local) parameterization that turns C into a distinguished curve of type $\mathcal{C}_L$,

(2) there is a (local) smooth section $s\colon C \to \mathcal{G}$ of the projection $\mathcal{G} \to M$ such that $s^*\omega \in \Omega^1(C, \mathfrak{g})$ takes values in $\operatorname{sym} L \subset \mathfrak{g}$.

Proof. Consider $c\colon I \to C$, $x = c(t_0) \in C$, and $\hat{c}\colon I \to \mathcal{G}$ over c as above such that $\operatorname{dev}_x c = [\![u, L]\!]$ with $u = \hat{c}(0) \in p^{-1}(x)$ and $L(t) = \exp(tX)o$. Then, defining the section $s\colon C \to \mathcal{G}$ by the prescription $c(t) \mapsto \hat{c}(t - t_0)$, we have got $\operatorname{Im}(s^*\omega) = X$ which belongs to $\operatorname{sym} L$ by definition.

Conversely, let $c\colon I \to C$ be any parameterization of C such that $c(0) = x$. Let $s\colon C \to \mathcal{G}$ be a section such that the assumption $\operatorname{Im}(s^*\omega) \subset \operatorname{sym} L$ is satisfied, i.e., for $\hat{c} = s \circ c$ and $Y(t) = \omega\big(\frac{d}{dt}\hat{c}(t)\big)$ we have $\operatorname{Im}(Y) \subset \operatorname{sym} L$. Let further $\operatorname{Sym}(L)$ be a virtual subgroup in G (not necessarily closed) corresponding to the Lie algebra $\operatorname{sym}(L)$. Then from [**6**, Theorem 2] it follows that L, considered as unparameterized curve, (locally) coincides with the orbit of $\operatorname{Sym}(L)$ through o. Since $Y(t) \in \operatorname{sym}(L)$ for all $t \in I$, we see that $a(t) \in \operatorname{Sym}(L)$, and thus $a(t)o$ belongs to L.

Let $\underline{a}\colon t \mapsto a(t)o$. Show that $\underline{a}'(0) \neq 0$, i.e., $\underline{a}$ defines a parameterization of L in a neighborhood of the origin. Indeed, under the identification $T(G/H) \cong G \times_H (\mathfrak{g}/\mathfrak{h})$ as in 2.2, the tangent vector field of $\underline{a}$ is written as $\underline{a}'(t) = [\![a(t), Y(t) + \mathfrak{h}]\!]$. Hence, evaluated in 0, it is equal to $\underline{a}'(0) = [\![e, Y(0) + \mathfrak{h}]\!]$. But $Y(0) \notin \mathfrak{h}$, by definition, since $\hat{c}$ is transversal to the fibers of $p\colon \mathcal{G} \to M$. □

2.4. On a coordinate neighborhood U of a point $x \in M$, any section $s : U \to \mathcal{G}$ of the projection $\mathcal{G} \to M$ defines the Cartan gauge $\theta = s^*\omega \in \Omega^1(U, \mathfrak{g})$ which can only changes, under the change of section by a map $h : U \to H$, according to the formula

$$\tilde{\theta} = \operatorname{Ad}_{h^{-1}} \theta + \delta h. \tag{2.4}$$

Any two Cartan gauges satisfying the condition above, on the intersection of domains, are called compatible and the relation 'to be compatible' is an equivalence relation on the set of local 1-forms $\{\theta_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\}$, see [**13**, Ch. 5] for details. We will work in this framework below and, in these terms, Proposition 1 can be formulated as follows.

COROLLARY 1. *Let C be a non-parameterized curve in M. Then the following two conditions are equivalent:*

(1) C is a curve of type $\mathcal{C}_L$,
(2) for any $x \in C$, there is a neighborhood $U \ni x$ and Cartan gauge $\theta \in \Omega^1(U, \mathfrak{g})$ such that $\mathrm{Im}(\theta|_C) \subseteq \mathrm{sym}(L)$.

Start with a Cartan gauge $\theta_0 = \theta|_C \in \Omega^1(C, \mathfrak{g})$ along C (correctly, along $C \cap U$). The first necessary condition for the curve C to be a curve of type $\mathcal{C}_L$ is on the tangent space level. Namely, for some parameterization $c : I \to C$, the tangent vectors $\dot{c}$ has to be contained in the subset of TM which corresponds to the H-invariant set $\mathrm{Ad}_H(X) + \mathfrak{h} \subseteq \mathfrak{g}/\mathfrak{h}$, provided the curve L representing the class $\mathcal{C}_L$ is generated by X. If this is the case, one can surely find a calibration $h : U \to H$ such that $\theta_1 = \mathrm{Ad}_{h^{-1}} \theta_0 + \delta h$, restricted to C, takes values in $\langle X \rangle + \mathfrak{h}$. Considering c to be a curve of type $\mathcal{C}_L$ (up to reparameterization), we repeat this idea to build up a sequence of Cartan gauges θ_i whose restrictions to C take values in $\langle X \rangle + \mathfrak{a}_{i-1}$, $i \in \mathbb{N}$, where the subalgebras $\mathfrak{a}_j \subseteq \mathfrak{h}$ are as in (2.1). Basically, this is the idea of moving frame, cf. [**6**].

Conversely, considering c to be a general curve on M, the question on $\mathrm{Im}(\theta_i|_C) \subseteq \langle X \rangle + \mathfrak{a}_{i-1}$ in each step yields some differential conditions on c which must be satisfied in order c to be a curve of type $\mathcal{C}_L$ up to ith order and up to reparameterization. Finally, the last step, corresponding to $\mathrm{sym}(L) = \langle X \rangle + \mathfrak{a}_r$, gives the system of differential equations $\mathcal{E}_L \subset J^{r+1}(\mathbb{R}, M)$ we are interested in. In general, no all of the above constraints are differential equalities but often also inequalities. The typical instance of that case are the first-order conditions on chains in contact parabolic geometries, see Sections 4 and 5. However, we can always deal just with the final system of differential equations of order $r + 1$ keeping in mind that the initial conditions on the solution to be the right curve have to satisfy all the constraints up to order r.

2.5. Now, in the half-time, write r instead of $r + 1$ and consider the system $\mathcal{E}_L \subset J^r(\mathbb{R}, M)$ of ordinary differential equations to be of the form $F_\nu(t, x^{(r)}) = 0$, $\nu = 1, \ldots, N$, where any $F_\nu : J^r(\mathbb{R}, M) \to \mathbb{R}$ is a smooth function and $x^{(r)}$ represents the derivatives of $x = (x^1, \ldots, x^m)$, $m = \dim M$, up to order r. Let further $\xi \in \mathfrak{X}(\mathbb{R} \times M)$ be a vector field on the space of independent and dependent variables, written as

$$\xi(t, x) = \psi(t, x)\frac{\partial}{\partial t} + \varphi^i(t, x)\frac{\partial}{\partial x^i}, \tag{2.5}$$

and let $\xi^{(r)} \in \mathfrak{X}(J^r(\mathbb{R}, M))$ be its rth prolongation, explicitly described in [**12**, Theorem 4.16]. Note that ξ generates fiber-wise point transformations of $\mathbb{R} \times M$ if and only if the function ψ depends only on t, which is the case we are focused on.

If the system of differential equations $\mathcal{E}_L$ is regular then ξ is an infinitesimal symmetry of $\mathcal{E}_L$ if and only if $\xi^{(r)} \cdot F_\nu\big|_{\mathcal{E}_L} = 0$ for all $\nu = 1, \ldots, N$, see [**12**, Ch. 6]. In fact, these conditions form an over-determined linear system of partial differential equations, known as determining equations, for the functions ψ and φ^i from the coordinate expression of ξ in (2.5). In general, that is an elementary but rather tedious task to find the determining equations of the system in question but, fortunately, there is number of softwares which help to bridge over this part of computation, see [**1**] for an instance.

Further, at least in the case of locally flat Cartan geometries, the resulting system of determining equations can be solved explicitly, hence one completely describes the Lie algebra of infinitesimal symmetries of $\mathcal{E}_L$. Here we just refer to the next sections for the concrete presentation of mentioned techniques.

2.6. We devote this paragraph to the promised connection between the automorphisms of Cartan geometry and symmetries of distinguished curves of specified type. Obviously, map $f : M \to M$ is a morphism of Cartan geometry if and only if, for any Cartan gauge θ, the pullback $f^*\theta$ is compatible with θ, i.e., there is a smooth map $h : U \to H$ such that $f^*\theta = \mathrm{Ad}_{h^{-1}}\theta + \delta h$, see 2.4. Note that this condition is satisfied for any Cartan gauge if and only if it is satisfied for one of them, by transitivity of the relation 'to be compatible'. For our purposes, we have to find the infinitesimal analogy of the above compatibility condition. An easy computations yields that vector field $\xi \in \mathfrak{X}(M)$ is an infinitesimal transformation of the Cartan geometry if and only if, for one (or, equivalently, any) Cartan gauge θ, there is a smooth map $Y : U \to \mathfrak{h}$ such that

$$\mathcal{L}_\xi\theta = -\operatorname{ad}_Y\theta + dY \tag{2.6}$$

holds.

Altogether, the question on whether has an infinitesimal symmetry ξ of $\mathcal{E}_L$ to be an infinitesimal transformation of the Cartan geometry is equivalent to the question whether, for general ξ satisfying the system of determining equations from 2.5, is there a smooth map $Y : U \to \mathfrak{h}$ such that the condition (2.6) is satisfied for a Cartan gauge $\theta \in \Omega^1(U, \mathfrak{g})$. In order to resolve the later question, one has first to look for such a $Y : U \to \mathfrak{h}$ that $\mathcal{L}_\xi\theta \equiv -\operatorname{ad}_Y\theta \mod \mathfrak{h}$, because of dY contributes only to $\mathfrak{h}$. The rest should be concluded (if the conjecture is true) by a game with coefficients of ξ and their partial derivatives involving the identities which follow from assumption, i.e., that the system of determining equations is satisfied by ξ.

Note that in the homogeneous model one also solves this problem by explicit solution of determining equations, as suggested in 2.5, which leads to the very visible description of the Lie algebra $\inf(\mathcal{E}_L)$ of infinitesimal symmetries of $\mathcal{E}_L$. Then one concludes by comparing the dimensions of Lie algebras in question, due to the inclusion $\mathfrak{g} \subseteq \inf(\mathcal{E}_L)$ which is here by definition.

Thus, computation of the symmetry algebra of $\mathcal{E}_L$ in the flat case can be considered as a test on whether the class of distinguished curves of type $\mathcal{C}_L$ does determine the Cartan geometry. If this test fails, i.e., the dimension of $\inf(\mathcal{E}_L)$ is bigger than $\dim\mathfrak{g}$, then the answer should be negative (the typical instance is the case of affine connections). However, even in the case when the test fails it is still possible (usually, due to some global arguments) that a map keeping the set of curves of type $\mathcal{C}_L$ stable is a transformation of the geometric structure. See Section 5 for example. Anyway, there are still more arguments required to establish correctly an answer to our question, so we wish to visit this problem elsewhere in a more conceptual way. Especially, in order to make the test and the hypothesis precise, two essential things are needed to clarify: first, which classes of distinguished curves should be considered as models and, second, which kind of global arguments can arise…

In the next, we demonstrate the just presented techniques in three particular cases. In Section 3 we go carefully through the example of projective plane geometry in order to make the reader familiar with all the general notions above. Sections 4 and 5 represent essential points of the process in the case of 3-dimensional Lagrangean contact and projective contact geometry, respectively. Computation for all these geometries in higher dimensions is completely analogous, only a bit more longer.

3. Projective geometry

3.1. Projective 2-dimensional geometry is a split Cartan geometry of type (G, H) modeled by the projective plane $\mathbb{RP}^2$ with the principal group $G = SL(3,\mathbb{R})$ of projective transformations and $H \subset G$, the stabilizer of some fixed point in $\mathbb{RP}^2$. On the infinitesimal level, we schematically write

$$\mathfrak{h} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} : \text{trace} = 0 \right\} \tag{3.1}$$

with the natural choice of complementary subalgebra

$$\mathfrak{n} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \right\}. \tag{3.2}$$

There are no distinguished directions and no distinguished types of generalized geodesics in projective geometries. In other words, any element of $\mathfrak{n}$ lies in the H-orbit of the vector

$$X_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{n}, \tag{3.3}$$

and any generalized geodesic in G/H is a shift gL_0 of the curve $L_0 = \exp(tX_0)o$, for some $g \in G$. Hence we set $\mathcal{C} = \{gL_0 : g \in H\}$.

3.2. The symmetry algebra of the curve L_0 is computed in two steps such that $\operatorname{sym}(L_0) = \langle X_0 \rangle + \mathfrak{a}_1$, explicitly,

$$\operatorname{sym}(L_0) = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} : \text{trace} = 0 \right\}. \tag{3.4}$$

The shape of the symmetry algebra involves the basic properties of generalized geodesic as suggested in introduction. In particular, in our case, non-parameterized geodesics are uniquely given by a direction in one point, which is a consequence of $\operatorname{sym}(L_0) = \langle X_0 \rangle + \mathfrak{a}_1$, i.e., $r = 1$, and any such curve admits the projective class of distinguished parameters, which one concludes from the pair of Lie algebras $(\langle X_0 \rangle + \mathfrak{a}_r, \mathfrak{a}_r)$, following [**5**].

3.3. For a coordinate system on $U \subset M$, any Cartan gauge $\theta \in \Omega^1(U, \mathfrak{g})$ can be calibrated by $h : U \to H$ such that the $\mathfrak{n}$-part of θ coincides with dx (due to surjectivity of the map $H \to GL(\mathfrak{n})$ induced by Ad). Hence starting on this level, we write

$$\theta_0 = \begin{bmatrix} a_k & z_{1k} & z_{2k} \\ \delta^1_k & a^1_{1k} & a^1_{2k} \\ \delta^2_k & a^2_{1k} & a^2_{2k} \end{bmatrix} dx^k \tag{3.5}$$

where of course we sum over $k = 1, 2$ and the trace has to vanish. Consider a general curve $c : I \to M$ and $C \subset M$ to be the non-parameterized image of c. Then the pullback of θ_0 to the curve $c(t) = (x^1(t), x^2(t))$ is

$$c^*\theta_0 = \begin{bmatrix} a_k & z_{1k} & z_{2k} \\ \delta^1_k & a^1_{1k} & a^1_{2k} \\ \delta^2_k & a^2_{1k} & a^2_{2k} \end{bmatrix} \dot{x}^k dt. \tag{3.6}$$

An appropriate calibration leads to a compatible Cartan gauge θ_1 along C with values in $\langle X_0 \rangle + \mathfrak{a}_0$, i.e., with zero in the (3,1)-entry. In particular, for

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{\dot{x}^2}{\dot{x}^1} & 1 \end{bmatrix} \tag{3.7}$$

we really get, according to (2.4) and summing over $k, \ell = 1, 2$,

$$c^*\theta_1 = \begin{bmatrix} a_k\dot{x}^k & z_{\ell k}\frac{\dot{x}^\ell}{\dot{x}^1}\dot{x}^k & z_{2k}\dot{x}^k \\ \dot{x}^1 & a^1_{\ell k}\frac{\dot{x}^\ell}{\dot{x}^1}\dot{x}^k & a^1_{2k}\dot{x}^k \\ 0 & (a^2_{\ell k}\frac{\dot{x}^\ell}{\dot{x}^1} - a^1_{\ell k}\frac{\dot{x}^2\dot{x}^\ell}{(\dot{x}^1)^2})\dot{x}^k & (a^2_{2k} - a^1_{2k}\frac{\dot{x}^2}{\dot{x}^1})\dot{x}^k \end{bmatrix} dt + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\ddot{x}^2\dot{x}^1 - \dot{x}^2\ddot{x}^1}{(\dot{x}^1)^2} & 0 \end{bmatrix} dt. \tag{3.8}$$

Of course, we have assumed $\dot{x}^1 \neq 0$, without any loss of generality.

3.4. At this moment, the equations for geodesics are read in the (3,2)-entry of $c^*\theta_1$ in order to take values in the subalgebra $\mathrm{sym}(L_0) = \langle X_0 \rangle + \mathfrak{a}_1 \subset \mathfrak{g}$. Altogether, the system of ordinary differential equations describing (non-parameterized) geodesics is just one equation of second order, namely,

$$\ddot{x}^2\dot{x}^1 - \dot{x}^2\ddot{x}^1 = \sum_{k,\ell=1}^{2} (a^1_{\ell k}\dot{x}^2\dot{x}^\ell - a^2_{\ell k}\dot{x}^\ell\dot{x}^1)\dot{x}^k. \tag{3.9}$$

Visibly, there is no contribution of functions z_{1k} and z_{2k} into the equations above and, as an exercise, one can verify that each affine geodesic (with an arbitrary parameter) of any linear connection from the projective class is really solution of this system.

3.5. Consider a vector field $\xi \in \mathfrak{X}(\mathbb{R} \times M)$, as in 2.5, with $\psi = \psi(t)$. Then the determining differential equations for ξ to be an infinitesimal symmetry of the system (3.9) are found to be

$$\varphi^i_{,t} = 0, \tag{3.10}$$

$$\varphi^i_{,jk} + \sum_{\ell=1}^{2}(a^i_{jk,\ell}\varphi^\ell + a^i_{j\ell}\varphi^\ell_{,k}) = (-1)^i \left(\varphi^i_{,j}(a^1_{1k} - a^2_{2k}) - a^i_{jk}(\varphi^1_{,1} - \varphi^2_{,2})\right), \tag{3.11}$$

$$\begin{aligned}\varphi^i_{,ii} + \sum_{\ell=1}^{2}(a^i_{ii,\ell}\varphi^\ell + a^i_{i\ell}\varphi^\ell_{,i}) &= 2\left(\varphi^j_{,ij} + (a^i_{ji}\varphi^j_{,i} - a^j_{ii}\varphi^i_{,j})\right)\\ &+ (-1)^j\left(\varphi^j_{,i}(a^1_{1j} - a^2_{2j}) - a^j_{ij}(\varphi^1_{,1} - \varphi^2_{,2})\right)\\ &+ \sum_{\ell=1}^{2}\left((a^j_{ji,\ell}\varphi^\ell + a^j_{j\ell}\varphi^\ell_{,i}) + (a^j_{ij,\ell}\varphi^\ell + a^j_{i\ell}\varphi^\ell_{,j})\right),\end{aligned}$$

for all $i, j, k \in \{1, 2\}$ such that $j \neq i$ and $k \neq i$. Lower indices after the comma denote the partial derivatives with respect to $x = (x^1, x^2)$. In particular, the first set of equations reads as $\frac{\partial\varphi^i}{\partial t} = 0$, i.e., the functions φ^1 and φ^2 depend only on x. The second clear consequence of the system above is that $\psi = \psi(t)$ may be arbitrary, i.e., any reparameterization of a solution of (3.9) is again solution; just as one could anticipate.

3.6. Using the normal coordinates in the case of locally flat projective geometry, one begins with Cartan gauge

$$\theta_0 = \begin{bmatrix} 0 & 0 & 0 \\ dx^1 & 0 & 0 \\ dx^2 & 0 & 0 \end{bmatrix}, \tag{3.12}$$

which leads to an extra easy version of the geodesic equation $\mathcal{E}$,

$$\ddot{x}^2\dot{x}^1 - \dot{x}^2\ddot{x}^1 = 0, \tag{3.13}$$

and the determining equations are

$$\begin{gathered}\varphi^1_{,22} = \varphi^2_{,11} = 0,\\ \varphi^1_{,11} = 2\varphi^2_{,12}, \varphi^2_{,22} = 2\varphi^1_{,21},\end{gathered} \tag{3.14}$$

with φ^1 and φ^2 to be functions only of x.

Now, let $\xi \in \mathfrak{X}(M)$ be an infinitesimal point symmetry of (3.13), i.e., the system of partial differential equations (3.14) is satisfied, provided that $\xi = \varphi^i\frac{\partial}{\partial x^i}$. Write θ instead of θ_0 and try to find a map $Y : U \to \mathfrak{h}$ such that $\mathcal{L}_\xi\theta = -\operatorname{ad}_Y\theta + dY$, following 2.6. Using the abbreviated notation as before, the Lie derivative of θ is

$$\mathcal{L}_\xi\theta = \begin{bmatrix} 0 & 0 & 0 \\ \varphi^1_{,k} & 0 & 0 \\ \varphi^2_{,k} & 0 & 0 \end{bmatrix} dx^k, \tag{3.15}$$

and we have to determine Y such that the $\mathfrak{n}$-parts of $\mathcal{L}_\xi\theta$ and $-\operatorname{ad}_Y\theta$ coincide. Such a Y is deduced to has the form

$$Y = \begin{bmatrix} \frac{1}{3}\varphi^1_{,1} + \frac{1}{3}\varphi^2_{,2} & q_1 & q_2 \\ 0 & -\frac{2}{3}\varphi^1_{,1} + \frac{1}{3}\varphi^2_{,2} & -\varphi^1_{,2} \\ 0 & -\varphi^2_{,1} & \frac{1}{3}\varphi^1_{,1} - \frac{2}{3}\varphi^2_{,2} \end{bmatrix} \tag{3.16}$$

where q_j are arbitrary functions. Then we get the difference $\mathcal{L}_\xi\theta - \operatorname{ad}_Y\theta - dY$ equals to

$$\begin{bmatrix} q_k - \frac{1}{3}\varphi^1_{,1k} - \frac{1}{3}\varphi^2_{,2k} & -q_{1,k} & -q_{2,k} \\ 0 & -q_1\delta^1_k + \frac{2}{3}\varphi^1_{,1k} - \frac{1}{3}\varphi^2_{,2k} & -q_2\delta^1_k + \varphi^1_{,2k} \\ 0 & -q_1\delta^2_k + \varphi^2_{,1k} & -q_2\delta^2_k + \frac{1}{3}\varphi^1_{,1k} + \frac{2}{3}\varphi^2_{,2k} \end{bmatrix} dx^k \tag{3.17}$$

and, using heavily the identities (3.14), this expression reduces to the form

$$\mathcal{L}_\xi\theta - \operatorname{ad}_Y\theta - dY = \begin{bmatrix} q_k dx^k - \frac{1}{2}\varphi^k_{,kk}dx^k & -q_{1,k}dx^k & -q_{2,k}dx^k \\ 0 & -q_1dx^1 + \frac{1}{2}\varphi^1_{,11}dx^1 & -q_2dx^1 + \frac{1}{2}\varphi^2_{,22}dx^1 \\ 0 & -q_1dx^2 + \frac{1}{2}\varphi^1_{,11}dx^2 & -q_2dx^2 + \frac{1}{2}\varphi^2_{,22}dx^2 \end{bmatrix}. \tag{3.18}$$

In order to get zeros in the right-down block, the functions q_1 and q_2 are determined uniquely so that $q_1 = \frac{1}{2}\varphi^1_{,11}$ and $q_2 = \frac{1}{2}\varphi^2_{,22}$, respectively, and the rest of the matrix vanishes either trivially or as a consequence of (3.14). More concretely, the left-up corner vanishes because of vanishing of the whole trace and $q_{j,k} = \frac{1}{2}\varphi^j_{,jjk}$ vanish, for any $j, k \in \{1, 2\}$, as follows: $\varphi^j_{,jjj} = 2\varphi^i_{,jji} = 0$, where $i \neq j$, and, for $j \neq k$, one gets $\varphi^j_{,jjk} = \frac{1}{2}\varphi^k_{,jkk} = \frac{1}{4}\varphi^j_{,jjk}$, hence $\varphi^j_{,jjk} = 0$ as well.

3.7. Alternative way available in the flat case is to solve the system of determining equations (3.14), as suggested in 2.6. Going this way, one can find the general solution of that system looks like

$$\begin{aligned} \varphi^1(x^1, x^2) &= c_1(x^1)^2 + c_2x^1x^2 + c_3x^1 + c_4x^2 + c_5, \\ \varphi^2(x^1, x^2) &= c_1x^1x^2 + c_2(x^2)^2 + c_6x^1 + c_7x^2 + c_8, \end{aligned} \tag{3.19}$$

for arbitrary constants $c_i \in \mathbb{R}$. Hence the dimension of $\inf(\mathcal{E})$ equals to 8 and so $\inf(\mathcal{E}) = \mathfrak{g}$ by dimension reasons. In fact, $G = SL(3, \mathbb{R})$ is the maximal possible symmetry group of second-order ordinary differential equation in two variables which is then necessarily equivalent to that in (3.13), cf. [**8, 12**].

Note that the opposite direction, i.e., the inclusion $\mathfrak{g} \subseteq \inf(\mathcal{E})$ which is trivial in general, can also be verified on this elementary level. In fact, the difference $\mathcal{L}_\xi\theta - \operatorname{ad}_Y\theta - dY$ in (3.17) vanishes if and only if all the relations in (3.14) are satisfied, i.e., ξ is an infinitesimal symmetry of (3.13).

4. Lagrangean contact geometry

4.1. Lagrangean contact geometry in dimension 3 is a split Cartan geometry of type (G, H) modeled by the projectivization of the tangent space to $\mathbb{RP}^2$. The group $G = SL(3, \mathbb{R})$ consists of all projective transformations naturally prolonged to $PT(\mathbb{RP}^2)$, and H is the stabilizer of a fixed line in the tangent space at some fixed point. On the infinitesimal level, we have

$$\mathfrak{h} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} : \text{trace} = 0 \right\} \tag{4.1}$$

with the complementary subalgebra $\mathfrak{n} = \mathfrak{n}_1^L \oplus \mathfrak{n}_1^R \oplus \mathfrak{n}_2$ such that

$$\mathfrak{n}_1^L = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \mathfrak{n}_1^R = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{bmatrix} \right\}, \text{ and } \mathfrak{n}_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \right\}. \tag{4.2}$$

For simplicity we restrict our attention just to the essential points of the process in the flat case, though most of computations below can be done in general.

There is a natural contact structure on $G/H = PT(\mathbb{RP}^2)$ generated by tangent vectors to curves lifted from $\mathbb{RP}^2$. Via the identification $T(G/H) \cong G \times_H (\mathfrak{g}/\mathfrak{h})$ as in 2.2 and $\mathfrak{n} \cong \mathfrak{g}/\mathfrak{h}$, the contact distribution corresponds to the two-dimensional

H-invariant subspace in $\mathfrak{n}$ defined by $\mathfrak{n}_1 = \mathfrak{n}_1^L \oplus \mathfrak{n}_1^R$. There are several subsets in $\mathfrak{n}$ invariant under the action of H which distinguish tangent vectors in $T(G/H)$. Except those within $\mathfrak{n}_1$, there is just the complement $\mathfrak{n} \setminus \mathfrak{n}_1$ corresponding to vectors lying outside of the contact distribution. There are distinguished curves of the particular interest which emanate in the later directions, namely, the curves represented by $L_0 = \exp(tX_0)o$ where $X_0 \in \mathfrak{n}_2$. These are called *chains*, in analogy with the Chern–Moser chains well-known from hypersurface CR geometries. From the symmetry algebra $\mathrm{sym}(L_0)$ below one can deduce they behave just like the classical chains, i.e., in any direction outside the contact distribution there is a unique unparameterized chain admitting a projective class of distinguished parameters.

4.2. Computing the symmetry algebra of L_0, we get $\mathrm{sym}(L_0) = \langle X_0 \rangle + \mathfrak{a}_1$ so that

$$\mathrm{sym}(L_0) = \left\{ \begin{bmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{bmatrix} : \mathrm{trace} = 0 \right\}. \tag{4.3}$$

4.3. Let us fix the local coordinates (x, y, z) on $G/H = PT(\mathbb{RP}^2)$ so that (x, y) are affine coordinates on $\mathbb{RP}^2$ and $z = \frac{dy}{dx}$. Then the contact structure on G/H is defined by the 1-form $\gamma = dy - z\,dx$. Similarly to 3.3 and 3.6, we start with the Cartan gauge

$$\theta_0 = \begin{bmatrix} 0 & 0 & 0 \\ dx & 0 & 0 \\ dy - z\,dx & dz & 0 \end{bmatrix}, \tag{4.4}$$

and its pullback $c^*\theta_0$ to the curve $c(t) = (x(t), y(t), z(t))$. Calibration

$$h = \begin{bmatrix} 1 & -\frac{\dot z}{\dot y - z\dot x} & 0 \\ 0 & 1 & \frac{\dot x}{\dot y - z\dot x} \\ 0 & 0 & 1 \end{bmatrix} \tag{4.5}$$

leads to a compatible Cartan gauge $c^*\theta_1$ with values in $\langle X_0 \rangle + \mathfrak{h}$. Of course, this is possible if and only if $\dot y - z\dot x = \gamma(\dot c) \neq 0$, i.e., vector $\dot c$ is transverse to the contact distribution.

4.4. Now, the equations on chains are just written in $(1, 2)$ and $(2, 3)$ entries of the matrix $c^*\theta_1$, explicitly, we have got $\mathcal{E}$ consisting of

$$\begin{gathered} \ddot y \dot x - \dot y \ddot x = 0, \\ (\dddot z \dot y - \dot z \dddot y) + z(\dddot x \dot z - \dot x \dddot z) = -2\dot x \dot z^2. \end{gathered} \tag{4.6}$$

4.5. Skipping the explicit description and solution of determining equations, we just conclude that the system (4.6) is really invariant under reparameterizations and the Lie algebra $\mathrm{inf}(\mathcal{E})$ of all infinitesimal symmetries of $\mathcal{E}$ is 8-dimensional and so coincides with $\mathfrak{g}$ by dimension reasons. Altogether, we have got a good reason to believe that the set of chains allows to recover the initial Lagrangean contact geometry.

5. Projective contact geometry

5.1. Three-dimensional projective contact geometry is a split Cartan geometry of type (G,H) modeled by the projective space $\mathbb{RP}^3$ with the principal group $G = Sp(4,\mathbb{R}) \subset SL(4,\mathbb{R})$ acting transitively on $\mathbb{RP}^3$ via the restriction of the standard action of $SL(4,\mathbb{R})$. Let $H \subset G$ be the stabilizer of some fixed point, then $\mathbb{RP}^3 = G/H$. The natural contact structure on $\mathbb{RP}^3$ arises from the symplectic structure on $\mathbb{R}^4$ invariant with respect to the action of $Sp(4,\mathbb{R})$. On the other hand, there is a natural flat projective connection on $\mathbb{RP}^3$ and these two structures are compatible in the following sense. If a straight line (i.e., a geodesic in the flat projective geometry) is tangent to the contact distribution at one point then it is a contact curve. See **[9, 10]** for more details and the compatibility of contact and projective structures in general case.

An appropriate matrix representation leads to the following infinitesimal description,

$$\mathfrak{g} = \left\{ \begin{bmatrix} a & b & c & d \\ x & e & f & c \\ y & g & -e & -b \\ z & y & -x & -a \end{bmatrix} \right\}, \tag{5.1}$$

in particular, $\dim \mathfrak{g} = 10$. Hence, schematically,

$$\mathfrak{h} = \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \in \mathfrak{g} \right\} \tag{5.2}$$

with the complementary subalgebra $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ so that

$$\mathfrak{n}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & * & * & 0 \end{bmatrix} \in \mathfrak{g} \right\} \text{ and } \mathfrak{n}_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix} \right\}. \tag{5.3}$$

As in the Lagrangean contact case, we are focused especially on the flat model hereafter.

There are only two distinct types of tangent vectors in $T(G/H)$, namely, those lying inside and outside of the contact subbundle. Under the familiar identification $T(G/H) \cong G \times_H (\mathfrak{g}/\mathfrak{h})$ and $\mathfrak{n} \cong \mathfrak{g}/\mathfrak{h}$ as before, the former case corresponds to the 2-dimensional subspace $\mathfrak{n}_1 \subset \mathfrak{n}$ whilst the later one to the complement $\mathfrak{n} \setminus \mathfrak{n}_1$. As in the Lagrangean contact case, we have also got the *chains*, the distinguished curves transversal to the contact distribution which are represented by any element of the subset $\mathfrak{n}_2 \subset \mathfrak{n}$ in the same sense as before. Further, there is only one more type of generalized geodesics which can be represented by an arbitrary element of $\mathfrak{n}_1$. Of course, these are the geodesics of the projective structure on the contact distribution and, in particular, they have the same properties as chains up to the initial condition to be directed within the contact subbundle. In general, projective contact structure induces the true projective structure whose geodesics are precisely the just discussed curves.

5.2. Let $L_0 = \exp(tX_0)o$, $X_0 \in \mathfrak{n}_2$, be a chain. Again, symmetry algebra of L_0 has the form $\mathrm{sym}(L_0) = \langle X_0 \rangle + \mathfrak{a}_1$, schematically written as

$$\mathrm{sym}(L_0) = \left\{ \begin{bmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{bmatrix} \in \mathfrak{g} \right\}. \tag{5.4}$$

5.3. Appropriate local coordinates on $G/H = \mathbb{RP}^3$ lead to the Cartan gauge

$$\theta_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ dx & 0 & 0 & 0 \\ dy & 0 & 0 & 0 \\ dz - ydx + xdy & dy & -dx & 0 \end{bmatrix} \tag{5.5}$$

and, in these coordinates, the contact distribution is then given as the kernel of the 1-form $\gamma = dz - ydx + xdy$. Consider the pullback $c^*\theta_0$ to curve $c(t) = (x(t), y(t), z(t))$ and choose the calibration

$$h = \begin{bmatrix} 1 & -\frac{\dot{y}}{\dot{z}-y\dot{x}+x\dot{y}} & \frac{\dot{x}}{\dot{z}-y\dot{x}+x\dot{y}} & 0 \\ 0 & 1 & 0 & \frac{\dot{x}}{\dot{z}-y\dot{x}+x\dot{y}} \\ 0 & 0 & 1 & \frac{\dot{y}}{\dot{z}-y\dot{x}+x\dot{y}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.6}$$

which yields the compatible Cartan gauge $c^*\theta_1$ with values in $\langle X_0 \rangle + \mathfrak{a}_0$. Of course, we have considered $\dot{z} - y\dot{x} + x\dot{y} = \gamma(\dot{c}) \neq 0$.

5.4. The equations on chains are then read from (1,2) and (1,3) or, equivalently, from (2,4) and (3,4) entries of $c^*\theta_1$ in order to take values in $\mathrm{sym}(L_0)$. Hence, the system $\mathcal{E}$ consists of two equations

$$\begin{aligned} (\ddot{y}\dot{z} - \dot{y}\ddot{z}) + y(\ddot{x}\dot{y} - \dot{x}\ddot{y}) &= 0, \\ (\ddot{x}\dot{z} - \dot{x}\ddot{z}) + x(\ddot{x}\dot{y} - \dot{x}\ddot{y}) &= 0. \end{aligned} \tag{5.7}$$

5.5. The determining equations of $\mathcal{E}$ form a 3-variable analogy of those in (3.14), in particular, the system (5.7) is invariant under reparameterizations and the Lie algebra $\mathrm{inf}(\mathcal{E})$ has the maximal possible dimension, i.e., 15. Hence the system of differential equations $\mathcal{E}$ is equivalent to the trivial one, [**8**], and the Lie algebra $\mathrm{inf}(\mathcal{E})$ of infinitesimal symmetries of $\mathcal{E}$ coincides with $\mathfrak{sl}(4, \mathbb{R})$, the Lie algebra of infinitesimal transformations of the true projective structure on $G/H = \mathbb{RP}^3$.

Anyway, we still can claim that chain-preserving infinitesimal transformations coincide with $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{R})$, the Lie algebra of infinitesimal transformations of the projective contact structure. In order to prove this, it just remains to show that any chain-preserving transformation respects the contact distribution, but this is clear more or less by definition.

The right reason of the former result, i.e., $\mathrm{inf}(\mathcal{E}) = \mathfrak{sl}(4, \mathbb{R}) \supset \mathfrak{g}$, is that there is never used the essential constraint $\dot{z} - y\dot{x} + x\dot{y} \neq 0$ in the computation of infinitesimal symmetries of $\mathcal{E}$. Omitting this inequality, one really recovers both the chains and the geodesics of type $\mathfrak{n}_1$ as solutions of $\mathcal{E}$. Of course, this can be nicely presented using the same techniques as yet: choosing a suitable element $X_1 \in \mathfrak{n}_1$, computing the symmetry algebra of $L_1 = \exp(tX_1)o$, and fixing the same coordinates as in 5.3, one ends with the differential equation $\ddot{x}\dot{y} - \dot{x}\ddot{y} = 0$ provided that $\dot{z} - y\dot{x} + x\dot{y} = 0$, i.e., the curve is tangent to the contact distribution. Now, it is an easy exercise to show that any solution of these two equations is also solution of (5.7).

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