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# REMARKS ON DEVELOPMENT OF CURVES

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ABSTRACT. The classical notion of development of curves easily generalizes to the wide class of Cartan geometries. This construction provides certain correspondence between smooth curves on the base manifold and smooth curves in the homogeneous space of the geometry in question. In this vein, there are specific curves on the base manifold distinguished in terms of their developments. Explicit relationship between the initial curve and its development is worked out.

## 1. INTRODUCTION

Given a smooth manifold $M$ endowed with some geometric structure, certain specific curves on $M$ become useful in number of problems, since they are very simple objects which can inherit a considerable amount of the entire geometric information. Typically, geodesics on Riemannian manifolds are of this feature. This is why distinguished types of curves appear in other geometries, in some of which they are very well-studied, e.g., conformal geodesics and null-geodesics on conformal manifolds. In the framework of Cartan geometries, these types of curves can be studied in an unifying manner using the development into the homogeneous model. This is the motivation for the best understanding of the notion of development in that general setting.

However, let us start with the classical definition of development into the tangent bundle on a smooth manifold with affine connection. Let $c$ be a smooth curve on $M$ with a fixed point $x = c(t_0)$. Let $X$ be the curve in the tangent space $T_xM$ which arises by moving the velocity vectors $\dot{c}(t_0 + t)$ back to $T_xM$ parallelly with respect to the given affine connection. A choice of frame in $T_xM$ provides an identification $T_xM \cong \mathbb{R}^m$ and we consider the development of $c$ at $x$ to be the integral (primitive) curve of $X$ in $\mathbb{R}^m$ passing through the origin, i.e., it is the unique curve $Y$ such that $Y' = X$ and $Y(0) = 0$. Obviously, this is a local construction and the development at $x = c(t_0)$ is defined only for such $t$ where the parallel transport is available. This is the classical definition of the development, as described in, e.g., [4, ch. III, § 5].

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**Proposition 1.1** (well-known). *The development of $c$ at $x = c(t_0)$ is a line segment if and only if the vector field $\dot{c}(t_0 + t)$ along $c$ is parallel, i.e., the curve $c$ is geodesic.*

Following [5, 6, 7], we generalize the construction of development to all manifolds endowed with a structure of Cartan geometry. At the same time, the notion of geodesics generalizes as well so that the Proposition above is satisfied in that more general setting. Development of curves gives rise to a correspondence between curves on the base manifold of the Cartan geometry and curves in the homogeneous space. Under some natural restriction we get a bijective correspondence which respects the order of contact. Main aims of this work are to describe this correspondence precisely and explicitly in the most general setting (Section 3) and to reformulate the general result in the case of split Cartan geometries in order to distinguish generalized geodesics on $M$ in terms of their developments (Section 4). Theorem 3.2 is essential here, the rest are consequences.

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## 2. Notations and preliminaries

Following [6], for a pair $(G, H)$ of a Lie group $G$ and a closed subgroup $H \subset G$, we define *Cartan geometry* of type $(G, H)$ on a smooth manifold $M$ to be certain curved analogy of the homogeneous bundle $G \to G/H$ with the Maurer–Cartan form on $G$. More precisely, denoting by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to given Lie groups, Cartan geometry of type $(G, H)$ on $M$ consists of a principal $H$-bundle $\mathcal{G} \to M$ with a $\mathfrak{g}$-valued one-form $\omega$ on $\mathcal{G}$ which satisfies the following conditions:

(a) $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for each $u \in \mathcal{G}$,
(b) $(r^h)^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega$ for all $h \in H$,
(c) $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{h}$.

Symbol Ad above means the adjoint representation of $G$ on $\mathfrak{g}$ and $\zeta_X$ is the fundamental vector field of the principal right action of $H$ on $\mathcal{G}$, generated by an element $X \in \mathfrak{h}$.

The *curvature* of the Cartan geometry is defined as the 2-form $K = d\omega + \frac{1}{2}[\omega, \omega]$ on $\mathcal{G}$ with values in $\mathfrak{g}$. Cartan geometry is called *split* if there is a fixed subalgebra $\mathfrak{n} \subset \mathfrak{g}$ complementary to $\mathfrak{h}$, i.e., $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ on the level of vector spaces. Cartan geometry is called *reductive* if there is a fixed complementary subspace $\mathfrak{n} \subset \mathfrak{g}$ which is $H$-invariant, i.e., invariant according to the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ restricted to $H$.

For any $X \in \mathfrak{g}$, the *constant vector field* $\omega^{-1}(X)$ is the unique vector field on $\mathcal{G}$ which satisfies the condition $\omega(\omega^{-1}(X)(u)) = X$ for all $u \in \mathcal{G}$. Next, we can identify the tangent bundle $TM$ with the associated bundle $\mathcal{G} \times_H (\mathfrak{g}/\mathfrak{h})$ where the representation of the structure group $H$ on $\mathfrak{g}/\mathfrak{h}$ is induced by the adjoint action on $\mathfrak{g}$. The identification is provided by the prescription $\{u, X + \mathfrak{h}\} \mapsto Tp{\cdot}\omega^{-1}(X + \mathfrak{h})(u)$, where $p$ denotes the projection $\mathcal{G} \to M$.

In the case of split Cartan geometries, with the splitting $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, we identify $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{n}$ and via this identification the subalgebra $\mathfrak{n}$ becomes an $H$-module. The action of the group $H$ on $\mathfrak{n}$ is called the truncated adjoint representation and it will be denoted by $\underline{\mathrm{Ad}}$. Obviously, this action is determined by the condition $\underline{\mathrm{Ad}}(h) = \pi \circ \mathrm{Ad}(h)$ for all $h \in H$, where $\pi$ is the projection $\mathfrak{g} \to \mathfrak{n}$ in the direction of $\mathfrak{h}$. Now we can write $TM \cong \mathcal{G} \times_H \mathfrak{n}$ with respect to this action and the identification above turns into the mapping $\{u, X\} \mapsto Tp \cdot \omega^{-1}(X)(u)$ with the inverse $\xi \mapsto \{u, \pi(\omega(\hat{\xi}))\}$, where $\hat{\xi} \in T_u\mathcal{G}$ is an arbitrary lift of the vector $\xi \in T_{p(u)}M$. Further, any fixed subalgebra $\mathfrak{n}$ gives rise to a horizontal distribution $\omega^{-1}(\mathfrak{n})$ on $\mathcal{G}$, hence each splitting enters a general connection on $\mathcal{G}$. The equivariancy of $\omega$ yields that this connection is principal if and only if $\underline{\mathrm{Ad}} = \mathrm{Ad}$, i.e., the geometry in question is reductive.

**Example 2.1.** As we suggested before, the most important example of Cartan geometry of type $(G, H)$ is the principal bundle $G \to G/H$ together with the Maurer–Cartan form on $G$. The structure equation says the curvature vanishes, so this is called flat, or homogeneous, model of Cartan geometries of type $(G, H)$. Much of the geometry of the homogeneous model carries over to all Cartan geometries of the same type and this is the main reason of its importance.

**Example 2.2.** Each manifold with an affine connection is a very obvious example of Cartan geometry of type $(G, H)$ where $G = A(m, \mathbb{R})$ is the group of affine motions on $\mathbb{R}^m$ and $H = GL(m, \mathbb{R})$ is the stabilizer of origin. The bundle of linear frames $P^1M$ plays the role of $\mathcal{G}$ and the Cartan connection $\omega \in \Omega^1(P^1M, \mathfrak{a}(m, \mathbb{R}))$ is given by $\theta + \gamma$, where $\theta \in \Omega^1(P^1M, \mathbb{R}^m)$ is the soldering form and $\gamma \in \Omega^1(P^1M, \mathfrak{gl}(m, \mathbb{R}))$ is the principal connection corresponding to the linear connection on the tangent bundle. This is an example of split and reductive Cartan geometry.

The homogeneous space $G/H$ of affine geometry is $\mathbb{R}^m$ and the left multiplication in $G$ restricted to $H \subseteq GL(m, \mathbb{R})$ descends to the standard action on $\mathbb{R}^m = G/H$. Further, the reductivity of the geometry means that the Ad-representation of $G$ restricted to $H$ keeps the complement $\mathfrak{n} = \mathbb{R}^m$ to $\mathfrak{h}$ invariant. It is easy to see this restricted action coincides with the standard representation of $H$ on $\mathbb{R}^m = \mathfrak{n}$, too.

## 3. Cartan's space and developments

For a Cartan geometry of type $(G, H)$ with the principal bundle $p : \mathcal{G} \to M$, the *Cartan's space* of $M$ is the associated bundle $SM = \mathcal{G} \times_H (G/H)$ with respect to the action of $H$ on $G/H$ given by the left multiplication. The standard construction of the so called extension $\tilde{\mathcal{G}}$ of the principal bundle $\mathcal{G}$ to the structure group $G$ extends uniquely the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ to the principal connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ and this induces general connections on all bundles associated to $\mathcal{G}$ with respect to the action of $H$ on the standard fiber given by the restriction of an action of the group $G$. Hence we get a general connection on the Cartan's space $SM = \mathcal{G} \times_H (G/H) = \tilde{\mathcal{G}} \times_G (G/H)$ which will be essential for the construction of the development of curves.

More concretely, the extension of $\mathcal{G}$ is the principal $G$-bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_H G$, the bundle $\mathcal{G}$ is seen as a subbundle of $\tilde{\mathcal{G}}$ due to the canonical reduction $u \mapsto \{u, e\}$, and the connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ is determined by the conditions to be principal and to coincide with $\omega$ on $T\mathcal{G} \subset T\tilde{\mathcal{G}}$. Altogether, the principal connection $\tilde{\omega}$ satisfies:

(a) $\tilde{\omega}(u)|_{T_u\mathcal{G}} = \omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ for each $u \in \mathcal{G}$,

(b) $\tilde{\omega}(\zeta_X(u)) = X$ for all $X \in \mathfrak{g}$, $u \in \tilde{\mathcal{G}}$,

(c) the horizontal lift of vector $\xi = \{u, X + \mathfrak{h}\} \in T_{p(u)}M$ is

$$\tilde{\xi}(u) = \omega^{-1}(X)(u) - \zeta_X(u) \in T_u\tilde{\mathcal{G}}\,.$$

First two conditions are obvious from definition. The third one is a consequence, in particular, the lift $\tilde{\xi}$ is well-defined, projects onto $\xi$, and $\tilde{\omega}(\tilde{\xi}) = X - X = 0$. See, e.g., [7] for details.

Another ingredient for the construction of developments is the canonical global section of the bundle projection $SM \to M$, given by the prescription $x \mapsto \{u, o = eH\}$ for any $u \in \mathcal{G}$ staying over $x \in M$. Indeed, the definition does not depend on a choice of $u \in \mathcal{G}$ and the section is denoted by $O$. Hence, any fiber of the vertical tangent bundle $VSM = \mathcal{G} \times_H T(G/H)$ restricted to $O(M)$ can be thought as an analogy of the tangent space, since the bundles $VSM|_{O(M)}$ and $TM$ are canonically isomorphic via the isomorphism $T_o(G/H) \cong \mathfrak{g}/\mathfrak{h}$. In this vein, the base manifold $M$ seems to be 'osculated' in each point by the homogeneous space $G/H$ on the level of tangent spaces.

**Remarks.** In the case of affine geometry, the Cartan's space $SM$ coincides with the tangent bundle $TM$ since the standard fibers $G/H$ and $\mathfrak{g}/\mathfrak{h}$ are the same $\mathbb{R}^m$ and the two actions of the structure group equal, as shown at the end of Example 2.2. Next, the construction above reminds the osculation of non-degenerated real submanifolds in $\mathbb{C}^n$ by quadrics which represent the modeling spaces of the induced CR-structure. This notion goes really back to Cartan, see [5] for more details and comments.

**Definition 3.1.** For $c$ being a parametrized curve on $M$ with a fixed point $x = c(t_0)$, we define the *development* of $c$ at $x$ to be a curve $\operatorname{dev}_x c$ in the fiber $S_xM \cong G/H$ as follows. First, the curve $c$ can be seen as a parametrized curve in the Cartan's space $SM$ due to the composition with the canonical section $O$. Then, locally, the value $(\operatorname{dev}_x c)(t)$ is obtained by moving the point $O(c(t_0 + t))$ back to the fiber $S_xM$ using the parallel transport induced by $\tilde{\omega}$. More precisely, if $\tilde{c}(s)$ is the parallel curve in $SM$ determined by the initial condition $\tilde{c}(0) = O(c(t_0 + t))$ and covering the path $s \mapsto c(t_0 + t + s)$ in $M$ then $(\operatorname{dev}_x c)(t) = \tilde{c}(-t)$.

The whole construction works only on a neighbourhood $I$ of $t_0 \in \mathbb{R}$ where the parallel transport in $SM$ along $c$ is well-defined. Then, for $J = I - t_0$, the development $\operatorname{dev}_x c$ is a curve $J \to S_xM$ which maps $0 \in J$ to $O(x) \in S_xM$. For each $u \in p^{-1}(x)$, the development of $c$ at $x$ can be represented by a unique curve $\bar{c} : J \to G/H$ mapping $0 \in J$ to the origin $o \in G/H$ so that $(\operatorname{dev}_x c)(t) = \{u, \bar{c}(t)\}$. Another frame $uh \in p^{-1}(x)$ changes the curve to $\ell_{h^{-1}} \circ \bar{c}$. Fixing the frame $u \in p^{-1}(x)$, we often consider the development at $x$ as a map which assigns to each curve $c$ passing through $x \in M$ the unique curve $\bar{c}$ in $G/H$ as before.

**Theorem 3.2.** *Let $p : \mathcal{G} \to M$ be the principal bundle of Cartan geometry of type $(G, H)$. For a smooth curve $c$ on $M$, fix a point $x = c(t_0)$ and arbitrary $u \in p^{-1}(x)$. Let $\hat{c}$ be any lift of $c$ in $\mathcal{G}$ starting at $u$ and let $X$ be the curve in $\mathfrak{g}$ defined as $X(t) = \omega\big(\frac{d}{dt}\hat{c}(t)\big)$. Then the development of $c$ at $x = p(u)$ is the curve* $(\operatorname{dev}_x c)(t) = \{u, a(t)H\} \subset S_xM$, *where $a$ is the unique curve in $G$ satisfying*

$$\delta a = X \text{ and } a(0) = e\,. \tag{1}$$

Symbol $\delta$ means the left logarithmic, or Darboux, derivative determined by the Maurer–Cartan form $\omega_G$ on $G$. In general, for a smooth map $f : N \to G$, the left logarithmic derivative $\delta f = f^*\omega_G$ is a $\mathfrak{g}$-valued one-form on $N$, however, in the case of $\dim N = 1$, it is usually understood as a curve in $\mathfrak{g}$ due to the identification with the image of $1 \in T_s\mathbb{R}$ for all $s$. Conversely, for a given one-form $\omega : TN \to \mathfrak{g}$, one can ask whether there is a smooth map $f : N \to G$ such that $\omega = \delta f$. The classical result of Cartan shows that the only sufficient (and also necessary) condition for the existence of local solutions is the structure equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$ to be satisfied. Further, if $f_1, f_2 : N \to G$ are two maps such that $\delta f_1 = \delta f_2$ then there is a unique element $c \in G$ such that $f_2 = \ell_c \circ f_1$. Altogether, this is a non-abelian generalization of the existence and uniqueness of the primitive function in elementary calculus—all proofs and details can be found in [6, ch. 3, §§5–6].

**Theorem 3.3** (Cartan). *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $g \in G$ be a fixed element. Let $\omega$ be a $\mathfrak{g}$-valued one-form on the smooth manifold $N$ satisfying the structure equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$. Then, for each point $x \in N$, there is a neighborhood $U$ of $x$ and a unique smooth map $f : U \to G$ such that $\omega|_U = \delta f$ and $f(x) = g$.*

Next, for any smooth maps $f_1, f_2 : N \to G$, an analogy of the Leibniz rule for the left logarithmic derivative has the form

$$\delta(f_1 \cdot f_2)(x) = (\delta f_2)(x) + \mathrm{Ad}_{f_2(x)^{-1}} \circ (\delta f_1)(x) \tag{2}$$

as shown in e.g. [3, p. 39]. Especially, for $f_2 = f_1^{-1}$, the left hand side is the derivative of the constant map to $e \in G$, so we conclude that the following holds for any $f : N \to G$ and all $x \in N$,

$$(\delta f^{-1})(x) = -\mathrm{Ad}_{f(x)} \circ (\delta f)(x)\,. \tag{3}$$

**Proof of Theorem 3.2.** We have to prove that the parallel curve over $c$ in $SM = \tilde{\mathcal{G}} \times_G (G/H)$ starting at $\{u, a(t)H\}$ attains the point $O(c(t_0 + t)) = \{\hat{c}(t), eH\}$ just in the time $t$, where clearly $c(t_0) = x = p(u)$. The essence of the proof is to describe the parallel curve $\tilde{c}$ in $\tilde{\mathcal{G}}$, covering $c$ and starting at $u = \hat{c}(0) = \tilde{c}(0)$, with respect to the principal connection $\tilde{\omega}$ as described at the very beginning of this section. Then the condition above reads as $\{\tilde{c}(t), a(t)H\} = \{\hat{c}(t), eH\}$, so we write $\tilde{c}(t) = \hat{c}(t) \cdot a^{-1}(t)$ for later use. Now we are going to determine $a : J \to G$ so that $\tilde{c}$ is the appropriate parallel curve in $\tilde{\mathcal{G}}$, in particular, $a(0) = e$ must be satisfied.

The definition of $X$ gives $\frac{d}{dt}\hat{c}(t) = \omega^{-1}(X(t))(\hat{c}(t))$ and the vector field $\frac{d}{dt}a^{-1}(t)$ is obviously written as $T_o\ell_{a^{-1}(t)} \cdot (\delta a^{-1})(t)$. Hence the derivative of $\tilde{c}(t) = \hat{c}(t) \cdot a^{-1}(t)$ yields

$$\tfrac{d}{dt}\tilde{c}(t) = Tr^{a^{-1}(t)} \cdot \omega^{-1}(X(t))(\hat{c}(t)) + \zeta_{(\delta a^{-1})(t)}(\hat{c}(t) \cdot a^{-1}(t))\,. \tag{4}$$

The request for vectors $\frac{d}{dt}\tilde{c}(t)$ to be horizontal means that $\tilde{\omega}\big(\frac{d}{dt}\tilde{c}(t)\big) = 0$ for all $t$. From the right equivariancy of the principal connection $\tilde{\omega}$ we conclude that the latter condition is equivalent to

$$\mathrm{Ad}_{a(t)} X(t) + (\delta a^{-1})(t) = 0\,, \tag{5}$$

since $\tilde{\omega}(\zeta_X) = X$ for any $X \in \mathfrak{g}$ and $\omega^{-1}(X(t))(\hat{c}(t))$ belongs to $T_{\hat{c}(t)}\mathcal{G}$ where $\tilde{\omega}$ and $\omega$ coincide. According to (3), equation (5) just means

$$X(t) = (\delta a)(t) \tag{6}$$

since the map $\mathrm{Ad}(a(t)) : \mathfrak{g} \to \mathfrak{g}$ is invertible for all $t$. Now, Theorem 3.3 guarantees the (local) existence and uniqueness of $a$ with the initial condition $a(0) = e$. This is due to the fact that $N$ is one-dimensional in our case, so all two-forms on $N$ are trivial and so the structure equation is trivially satisfied. □

**Remarks.** (a) Theorem 3.2 provides an alternative view on development of curves, which can also serve as a definition, cf. [6, p. 209]. Although it is implicitly involved in the statement, we now directly show, as an exercise, that the development of $c$ at $x$ constructed in this way really does not depend on the choice of lift $\hat{c}$. In the current setting, let $\hat{c}_2 = \hat{c}\cdot h$ be another lift of $c$ starting at $u \in \mathcal{G}$, i.e., $h$ is a curve in $H$ such that $h(0) = e$. Then, considering the connection $\omega$ on the principal $H$-bundle $\mathcal{G}$ instead of $\tilde{\omega}$ on the $G$-bundle $\tilde{\mathcal{G}}$, the same arguments as in the proof above—namely, the equivariancy of $\omega$ applied to the derivative of $\hat{c}_2(t) = \hat{c}(t)\cdot h(t)$—shows that the corresponding curve $X_2$ in $\mathfrak{g}$ is

$$\omega\big(\tfrac{d}{dt}\hat{c}_2(t)\big) = \mathrm{Ad}_{h^{-1}(t)} X(t) + (\delta h)(t) \tag{7}$$

where $X = \delta a$. The resulting development is then represented by the unique curve $a_2$ in $G$ such that $\delta a_2 = X_2$ and $a_2(0) = e$. But, $\delta(a\cdot h) = X_2$ and $a(0)\cdot h(0) = e$, hence $a_2 = a\cdot h$ by the uniqueness of the solution. Curves $a$ and $a_2$ share the same projection to $G/H$, so the development is well-defined in this way.

(b) Next, Theorem 3.2 clearly demonstrates the property which everybody expects developments in the homogeneous model to have, namely, that developments of curves in $M = G/H$ into the fiber of the Cartan's space $SM = G \times_H (G/H)$ are just the initial curves but shifted to the origin. Indeed, for a smooth curve $c(t) = b(t)H$ in $M = G/H$ and a point $x = c(t_0)$, let $\hat{c}(t) = b(t + t_0)$ be the lift of $c$ in $G$ starting at $u = b(t_0)$. By the definition of $X$, we get $X = \delta\hat{c}$ and, according to $u \in G$, the development of $c$ is represented by the unique solution of $\delta a = X$ with the initial data $a(0) = e$. But, $\hat{c} = \ell_u \circ a$ since $\hat{c}(0) = u = b(t_0)$, hence the resulting curve in $G/H$ is a shift of $c$ according to

$$a(t) = b(t_0)^{-1}\cdot b(t + t_0)\,. \tag{8}$$

Another representative $b_2$ of $c$ leads to the same curve in $G/H$ but conjugated by $h = b(t_0)^{-1}b_2(t_0) \in H$, as in the general case. In particular, for curves which map 0 to the origin $o = eH$, the developments at $o$ with respect to $e \in p^{-1}(o)$ are absolutely unchanged.

Back to the general picture, the construction of developments establishes a correspondence between smooth curves on $M$ which map 0 to a fixed point $x \in M$ and smooth curves in the fiber $S_xM$ of the Cartan's space mapping 0 to $O(x)$. Now we are going to show this correspondence is bijective, Proposition 3.5.

First of all, we formulate a slight modification of Theorem 3.3 where we consider a general Cartan geometry with the total space $\mathcal{G}$ and Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, instead of the Lie group $G$ with the Maurer–Cartan form $\omega_G \in \Omega^1(G, \mathfrak{g})$, and the pullback $f^*\omega$, instead of the left logarithmic derivative $\delta f = f^*\omega_G$. We further consider

$\dim N = 1$, which is also the only possibility allowing an analogy to Theorem 3.3 in the case of general Cartan geometry with non-vanishing curvature. In that case, $\mathfrak{g}$-valued one-forms are seen as curves in $\mathfrak{g}$, the structure equation is trivially satisfied, hence we get the following formulation.

**Lemma 3.4.** *Let $\mathcal{G} \to M$ be the principal bundle of the Cartan geometry with the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ and let $u \in \mathcal{G}$ be a fixed frame. Let $X$ be a smooth curve in $\mathfrak{g}$. Then, for each $t_0 \in \mathbb{R}$, there is a neighborhood $I$ of $t_0$ and a unique smooth curve $c : I \to \mathcal{G}$ such that $X|_I = \omega(\dot{c})$ and $c(t_0) = u$.*

**Proof.** In order to prove this statement, it suffices to follow the proof of Theorem 3.3 as written in, e.g., [6, pp. 116–118]. The essence of that proof is a construction of the graph of $f$ in $N \times G$. Locally, this graph is found as the leaf through the point $(x, g)$ of some integrable distribution of rank $= \dim N$ on $N \times G$. The distribution is determined by the one-form $\omega$ on $N$ and the Maurer–Cartan form $\omega_G$, and the integrability is guaranteed by the fact that the structure equation is satisfied both for $\omega_G$ and $\omega$ by the assumption.

In the curved case, all constructions match as well but the integrability of the distribution in question is no longer satisfied in general, except the case with $\dim N = 1$. □

Now we see that Lemma 3.4 provides the inverse to the map assigning to any smooth curve $c$ on $M$ with $c(0) = x$ the representative of its development $\operatorname{dev}_x c$ with respect to a fixed frame $u \in p^{-1}(x)$. The restriction to curves mapping 0 to $x$ is clear since, for any reparametrization of the form $\varphi(t) = t + t_0$, the curves $c$ and $c \circ \varphi$ have got the same development. Altogether, we obtain the following identification.

**Proposition 3.5.** *For a fixed $u \in \mathcal{G}$, the development of curves on $M$ into the homogeneous space $G/H$ establishes a bijective correspondence between the sets*

$$\{\text{curves on } M \text{ mapping } 0 \text{ to } x = p(u)\} \rightleftarrows \{\text{curves on } G/H \text{ mapping } 0 \text{ to } o = eH\}\ .$$

**Proof.** The right arrow is the development with respect to $u \in \mathcal{G}$ as described above. We have only to describe the left arrow in more details so that it is the inverse to the map before.

Let $c$ be a curve in $G/H$ which maps 0 to the origin, and let $a : I \to G$ be any lift of $c$ such that $a(0) = e$. For the fixed $u \in \mathcal{G}$, there is a unique curve $c_1$ in $\mathcal{G}$ such that $c_1(0) = u$ and $\omega(\dot{c}_1) = \delta a$, due to Lemma 3.4. Thus, we get the projection $p \circ c_1$ to $M$ as the image of $c$. Of course, the result does not depend on the choice of $a$, provided that $a(0) = e$. The rest is clear, namely, we have really constructed the inverse to the map in question. □

The identification above is not canonical at all but, by the definition of development, it depends on fixed $u \in \mathcal{G}$ over $x$ so that another frame $uh \in p^{-1}(x)$ changes the image of the right arrow by the conjugation of $h \in H$. As an exercise, one can identify the set of curves on $M$ mapping 0 to a fixed point $x \in M$ with the set of conjugacy classes of curves on $G/H$ mapping 0 to the origin. Anyway, the bijection above is compatible with taking jets in 0 as follows.

**Proposition 3.6.** *Let $c_1$ and $c_2$ be smooth curves in $M$ such that $c_1(0) = x = c_2(0)$. Then, for any positive integer $r$, $j^r_0 c_1 = j^r_0 c_2$ if and only if $j^r_0(\operatorname{dev}_x c_1) = j^r_0(\operatorname{dev}_x c_2)$.*

**Proof.** Fix a frame $u \in \mathcal{G}$ over $x = c_1(0) = c_2(0)$. On a neighborhood $U$ of $x \in M$, let us consider a local section $\sigma : U \to \mathcal{G}$ mapping $x$ to $u$. For $i = 1, 2$ and small $t$, the curve $\hat{c}_i = \sigma \circ c_i$ is a lift of $c_i$ with $\hat{c}_i(0) = u$. Since $\sigma$ is a diffeomorphism onto the image, the following equivalence is clear: $j^r_0 c_1 = j^r_0 c_2$ if and only if $j^r_0 \hat{c}_1 = j^r_0 \hat{c}_2$.

Let $X_i(t) = \omega\big(\frac{d}{dt}\hat{c}_i(t)\big)$ be the curves in $\mathfrak{g}$, as in Theorem 3.2. The form $\omega : T\mathcal{G} \to \mathfrak{g}$ is an absolute parallelism, and $\ell$-jet of $X_i$ is expressed in terms of $\ell$-jet of $\omega$ and $(\ell+1)$-jet of $\hat{c}_i$, hence, it is obvious that if $j^r_0\hat{c}_1 = j^r_0\hat{c}_2$ then $j^{r-1}_0 X_1 = j^{r-1}_0 X_2$. For the converse implication, let as assume the equality $j^{r-1}_0 X_1 = j^{r-1}_0 X_2$ to be satisfied. Clearly, if $X_1(0) = X_2(0)$ then $j^1_0\hat{c}_1 = j^1_0\hat{c}_2$ and, inductively, the equality $j^\ell_0\hat{c}_1 = j^\ell_0\hat{c}_2$ implies $j^{\ell+1}_0\hat{c}_1 = j^{\ell+1}_0\hat{c}_2$ for all $\ell = 1,\dots,r-1$. Altogether: $j^r_0\hat{c}_1 = j^r_0\hat{c}_2$ if and only if $j^{r-1}_0 X_1 = j^{r-1}_0 X_2$.

Next, let $a_1$ and $a_2$ be the unique curves in $G$ such that $\delta a_i = X_i$ and $a_i(0) = e$. By the definition of the left logarithmic derivative, the condition $\delta a_i = X_i$ reads as $\omega_G\big(\frac{d}{dt}a_i(t)\big) = X_i(t)$, for the Maurer–Cartan form $\omega_G : TG \to \mathfrak{g}$, and the same arguments as in the latter paragraph yield: $j^{r-1}_0 X_1 = j^{r-1}_0 X_2$ if and only if $j^r_0 a_1 = j^r_0 a_2$.

Finally, $r$-th jet prolongation of the projection $G \to G/H$ ensures the implication: if $j^r_0 a_1 = j^r_0 a_2$ then $j^r_0(\operatorname{dev}_x c_1) = j^r_0(\operatorname{dev}_x c_2)$. Altogether, we get:

$$j^r_0 c_1 = j^r_0 c_2 \Longrightarrow j^r_0(\operatorname{dev}_x c_1) = j^r_0(\operatorname{dev}_x c_2)\,.$$

The converse implication can be proved just in the same way as before following the inverse to the construction of development. Starting with a local section $\sigma$ of the projection $G \to G/H$ such that $\sigma(o) = e$, we consider $a_i$ to be the lift of $\operatorname{dev}_x c_i$ in the image of $\sigma$, and so forth. The essential point here is the bijective correspondence, established by Lemma 3.4, between curves $a$ in $G$ mapping 0 to $e$ and curves $\hat{c}$ in $\mathcal{G}$ mapping 0 to fixed $u \in \mathcal{G}$ such that $\delta a = \omega\big(\frac{d}{dt}\hat{c}\big)$. In previous paragraphs, we have proved via $X_i = \delta a_i = \omega\big(\frac{d}{dt}\hat{c}_i\big)$ that $j^r_0 a_1 = j^r_0 a_2$ if and only if $j^r_0\hat{c}_1 = j^r_0\hat{c}_2$. The rest is obvious. □

## 4. Split geometries and generalized geodesics

Following Theorem 3.2, the development of $c$ at $x = p(u)$ is represented by a curve $a$ in $G$ which is the solution of $\delta a = X$ with $a(0) = e$. In the case of split Cartan geometries, the image of the complement $\mathfrak{n} \subset \mathfrak{g}$ under the map $\exp : \mathfrak{g} \to G$ forms a subgroup in $G$ complementary to fibers of the projection $G \to G/H$, locally around the origin. In that case, the development above can be described by a unique curve $Y$ in $\mathfrak{n}$ so that $a(t) = \exp(Y(t)) \bmod H$, in other words, this is just expression of $a$ in the normal coordinates through $e \in G$. Further, if we consider the horizontal lift of $c$ instead of a general lift $\hat{c}$ then the image of $X$ belongs to $\mathfrak{n}$ and the entire problem of developments can be solved in $\mathfrak{n}$ as follows.

Considering the curve $a$ of the form $a = \exp\circ Y$, we have to describe $\delta(\exp\circ Y)$ which is $(\delta\exp)\circ TY$ by the definition of $\delta$. So we are going to find the formula for $\delta\exp : T\mathfrak{g} \to \mathfrak{g}$ first. For each linear endomorphism $z \in \mathfrak{gl}(\mathfrak{g})$ let us denote by $g(z) = \frac{e^z-1}{z}$ the formal power series $\sum_{k=0}^{\infty}\frac{1}{(k+1)!}z^k$. Then the formula for the right logarithmic derivative deduced in [3, 4.27] can be easily adapted for the left one so that equality

$$(\delta\exp)(Y) = g(\operatorname{ad}(-Y)) : T_Y\mathfrak{g} \to \mathfrak{g} \tag{9}$$

holds for any $Y \in \mathfrak{g}$. In particular, for $Y : I \to \mathfrak{g}$, the left logarithmic derivative of the composition $\exp\circ Y : I \to G$, understood as a curve in $\mathfrak{g}$, equals to

$g(\operatorname{ad}(-Y(t)))(Y'(t))$. Hence, omitting the variable $t$, equation (1) reads as

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)!}(\operatorname{ad}(-Y))^k(Y') = X\,, \tag{10}$$

which is a first-order ODE with the initial condition $Y(0) = 0$. Considering horizontal lift of $c$, i.e., considering $X \subset \mathfrak{n}$, the latter equation is then put in $\mathfrak{n}$.

**Remarks.** Especially, if the subalgebra $\mathfrak{n} \subset \mathfrak{g}$ is abelian then the equation above takes the very nice and easy form

$$Y' = X\,. \tag{11}$$

This happens, for example, in the case of irreducible parabolic geometries whilst, in general parabolic geometries, the complement $\mathfrak{n} \subset \mathfrak{g}$ is only nilpotent. Anyway, the sum in (10) is always finite in these cases.

Another instance of a geometry with $\mathfrak{n}$ abelian is the affine geometry where the formula above just recovers the definition of development at the beginning of Section 1. Indeed, under the identification $TM \cong \mathcal{G} \times_H \mathfrak{n}$ with the notation above, the velocity vector field along $c$ is written as $\dot{c}(t) = \{\hat{c}(t), X(t)\}$, and the definition of parallel displacement in associated bundles provides the equivalence of these two concepts. In the sequel, we conclude the analogy with the affine case by Proposition 4.2 which is a straight generalization of that in 1.1. Following [6, p. 210], we first generalize the notion of geodesics to any split Cartan geometry.

**Definition 4.1.** Let $p : \mathcal{G} \to M$ be a Cartan geometry of type $(G, H)$, split as $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, and let $A \subseteq \mathfrak{n}$ be an arbitrary subset. Smooth curve on the base manifold is called a *generalized geodesic* of type $\mathcal{C}_A$ if it has the shape

$$c^{u,X}(t) = p(\mathrm{Fl}_t^{\omega^{-1}(X)}(u))$$

for some $u \in \mathcal{G}$ and $X \in A$.

Indeed, affine geodesics can be described just in the same way in view of Example 2.2, cf. [4, p. 139]. Further, this definition recovers some well-known types of curves in particular geometries, e.g., the conformal circles in conformal geometries, the Chern–Moser chains in hypersurface CR-geometries, and others.

The generalized geodesic $c^{u,X}$ clearly goes through the point $p(u) = c(0)$ with the tangent vector $\{u, X\} \in T_{p(u)}M$ but, in contrast to the affine (more generally, reductive) case, this initial condition does not determine the generalized geodesic uniquely. In other words, another representative of the tangent vector may define another curve, so we need initial conditions of higher order in order the generalized geodesics to be uniquely specified. Furthermore, according to the choice of $A \subseteq \mathfrak{n}$, there appear curves of various types on the base manifold, which may behave rather different. The very well-known instance of such curves are null-geodesics and null-chains on conformal and CR-manifolds of indefinite signature, respectively.

The notion of development is often used to distinguish curves on all manifolds endowed with a Cartan geometry of fixed type by means of distinguished curves in the homogeneous model. The following Proposition provides an alternative definition of generalized geodesics.

**Proposition 4.2.** *The development of $c$ at $x = c(t_0)$ has the form $(\mathrm{dev}_x c)(t) = \{u, \exp(tX)H\} \subset S_xM$, for some $X \in A$ and $u \in \mathcal{G}$, if and only if $c$ is a geodesic of type $\mathcal{C}_A$.*

**Proof.** According to Theorem 3.2 and conventions from the beginning of this section, we have only to prove that the curve $X(t)$ is constant if and only if $Y(t) = tX$ for some $X \in A$, where $Y$ is the solution of (10) with the initial condition $Y(0) = 0$.

If $X(t) = X$ is the constant curve, we define $Y(t) = tX$ and, conversely, if $Y$ has the form $Y(t) = tX$ then we put $X(t) = X$. Anyway, equation (10) is still satisfied, since the sum $\sum_{k=1}^{\infty} \frac{1}{(k+1)!}(\mathrm{ad}(-tX))^k(X)$ vanishes for all $t$ and so the only nonzero term on the left hand side is $X$. The uniqueness of the development completes the proof. □

**Remarks.** All the observations above are essential for closer study of some basic properties of generalized geodesics, which can be read now on the level of the homogeneous space using their developments. This is presented in [2] in the case of parabolic geometries.

Concerning parabolic geometries, the notion of generalized Weyl structures [1] leads to a class of affine connections on the base manifold, which underlie the given Cartan connection in some sense. Then besides the development of curves into the homogeneous space of the parabolic geometry, we have got the classical development into the tangent space with respect to a fixed affine connection from the class in question. In this vein, it is natural to look for the relationship between these two developments considering a fixed Weyl structure. For conformal and projective geometries, this problem is fully solved by Tanaka in [8, 9] in terms of the so called rho-tensor. Following the process of Theorem 3.2 above, one can easily generalize these results to an arbitrary parabolic geometry. We wish to come back to this topic elsewhere.

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