

LIE CONTACT STRUCTURES AND CHAINS

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ABSTRACT. Lie contact structures generalize the classical Lie sphere geometry of oriented hyperspheres in the standard sphere. They can be equivalently described as parabolic geometries corresponding to the contact grading of orthogonal real Lie algebra. It follows the underlying geometric structure can be interpreted in several equivalent ways. In particular, we show this is given by a split-quaternionic structure on the contact distribution, which is compatible with the Levi bracket.

In this vein, we study the geometry of chains, a distinguished family of curves appearing in any parabolic contact geometry. Also to the system of chains there is associated a canonical parabolic geometry of specific type. Up to some exceptions in low dimensions, it turns out this can be obtained by an extension of the parabolic geometry associated to the Lie contact structure if and only if the latter is locally flat. In that case we can show that chains are never geodesics of an affine connection, hence, in particular, the path geometry of chains is always non-trivial. Using appropriately this fact, we conclude that the path geometry of chains allows to recover the Lie contact structure, hence, in particular, transformations preserving chains must preserve the Lie contact structure.

1. Introduction

The Lie sphere geometry is the geometry of oriented hyperspheres in the standard sphere established by S. Lie. A generalization of the corresponding geometric structure to general smooth manifold is provided by [7] and further studied by other authors. Here we briefly present the basic ideas and outline the purposes of this paper.

1.1. Classics. Let us consider the vector space $\mathbb{R}^{n+4}$ with an inner product of signature $(n+2, 2)$. The projectivization of the cone of non-zero null-vectors in $\mathbb{R}^{n+4}$ is a hyperquadric in the projective space $\mathbb{RP}^{n+3}$, which is called the *Lie quadric* and denoted by Q^{n+2} . The standard sphere S^{n+1} is then realized as the intersection of Q^{n+2} with a hyperplane in $\mathbb{RP}^{n+3}$. There is a bijective correspondence between the Lie quadric Q^{n+2} and the set of the so called kugels of S^{n+1} . The kugel of S^{n+1} is an oriented hypersphere or a point (called a point sphere) in S^{n+1} . The point is that kugels corresponding to the same projective line are in oriented contact, with the point sphere as the common contact point. Hence each projective line in Q^{n+2} is uniquely represented by the common contact point and the common unit normal of the family of kugels in contact. This establishes a bijective correspondence between the set of projective lines in Q^{n+2} (i.e. isotropic planes in $\mathbb{R}^{n+4}$) and the unit tangent sphere bundle of S^{n+1} , denoted by $T_1(S^{n+1})$. Either of the two is then understood as the model Lie contact structure in dimension $2n+1$.

By definition, the Lie transformation group G is the group of projective transformations of $\mathbb{RP}^{n+3}$ preserving the Lie quadric. Hence G is isomorphic to $PO(n+2, 2)$, the quotient of $O(n+2, 2)$ by its center which is $\{\pm \text{id}\}$. The group G acts transitively (and effectively) on the set of projective lines in Q^{n+2} , and hence on

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$T_1(S^{n+1})$, and preserves the canonical contact structure. As a homogeneous space, $T_1(S^{n+1}) \cong G/P$ where $P \subset G$ is the stabilizer of some element. It turns out P is a parabolic subgroup of G .

The generalization of the concepts above to general contact manifold yields the notion of the Lie contact structure, which is defined in section 3 in [7] as a reduction of the adapted frame bundle to an appropriate subgroup of the structure group. Applying the Tanaka's theory, the equivalence problem for Lie contact manifolds is solved in principle by Theorem 4.3 in [7] and rather explicitly in [6]. Anyway, there is established an equivalence between Lie contact structures and normal Cartan geometries of type (G, P) . Important examples of Lie contact structures are observed on the unit tangent sphere bundles of Riemannian manifolds.

1.2. General signature. The ideas above allow a natural generalization considering the inner product on $\mathbb{R}^{n+4}$ to have an arbitrary signature $(p+2, q+2)$, where $p+q=n$. Following the previous approach, we still consider the space of projective lines in the Lie quadric Q^{n+2} , i.e. the space of isotropic planes in $\mathbb{R}^{n+4}$, as the model. As a homogeneous space, this is isomorphic to G/P , where $G = PO(p+2, q+2)$ and $P \subset G$ is the stabilizer of an isotropic plane in $\mathbb{R}^{n+4}$. It is then natural to define the *Lie contact structure of signature (p, q)* as the underlying structure of a parabolic geometry of type (G, P) which is the meaning of the definition in 2.3, which we adopt from section 4.2.5 in [2]. Note that in low dimensions the Lie contact structure may have a specific flavour, which is briefly discussed in remark 2.3.

First of all, the Lie contact structure on M involves a contact structure $H \subset TM$ so that $H \cong L^* \otimes R$, where L and R are auxiliary vector bundles over M of rank 2 and n . Note that the auxiliary bundles has almost no intrinsic geometrical meaning, however, for $H \cong L^* \otimes R$, there is a distinguished subset in each H_x consisting of all the elements of rank one. This is the *Segre cone* which plays a role in the sequel. The maximal linear subspaces contained in the cone have dimension n and it turns out they are isotropic with respect to the Levi bracket. Characterization of the Lie contact structure in these terms is provided by Proposition 2.4.

On the other hand, the tensor product structure of H_x can be naturally rephrased as a *split-quaternionic structure*. The compatibility with the Levi bracket is easy to express and it turns out these data also characterize the Lie contact structure entirely, Proposition 2.5. This is the convenient interpretation of the Lie contact structure we further use below. (Note that in both cases there is minor additional input involved, namely a fixed trivialization of a line bundle over M .)

1.3. Chains. As for any parabolic contact structure, there is a general concept of *chains* which form a distinguished family of curves generalizing the Chern–Moser chains on CR manifolds of hypersurface type. As unparametrized curves (paths), chains are uniquely determined by a tangent direction, transverse to the contact distribution, in one point. A path geometry of chains can be equivalently described as a regular normal parabolic geometry of a specific type over the open subset of PTM consisting of all non-contact directions in TM . It turns out it is possible to relate the parabolic contact geometry and the path geometry of chains on the level of Cartan geometries directly, i.e. without the prolongation. This was done for Lagrangean contact structures and CR structures of hypersurface type in [4].

In the rest of present paper we follow the mentioned construction and the consequences for Lie contact structures. First, dealing with the homogeneous model, one explicitly describes the data allowing the direct relation between the Lie contact geometry and the path geometry of chains, section 3.3. Second, the construction in general is compatible with the normality condition if and only if the parabolic contact structure is torsion free. For Lie contact structures in dimension greater than

or equal to 7, the normality and torsion freeness imply the structure is locally flat, see Theorem 3.5.

Comparing to earlier studies in [4, 5], this brings a rather strong restriction. Under this assumption the tools we use may seem a bit non-proportional, however, to our knowledge there is no elementary argument covering the results below, not even in the homogeneous model. Analyzing the curvature of the induced Cartan geometry, we conclude with some applications for locally flat Lie contact structures: Chains are never geodesics of an affine connection, hence, in particular, the path geometry of chains is never flat, Theorem 4.1. Therefore the harmonic curvature of the path geometry of chains (and an efficient interpretation of the structure) allows one to recover the Lie contact structure. Hence, contact diffeomorphisms preserving chains must preserve the Lie contact structure, Theorem 4.2.

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2. Lie contact structures

In this section we bring the general definition of Lie contact structures with the alternative interpretations of the underlying geometric structure. We start with a necessary background.

2.1. Parabolic contact structures. For a semisimple Lie group G and a parabolic subgroup $P \subset G$, *parabolic geometry* of type (G, P) on a smooth manifold M consists of a principal P -bundle $\mathcal{G} \rightarrow M$ and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G . The Lie algebra $\mathfrak{g}$ is always equipped with the grading of the form $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ such that the Lie algebra $\mathfrak{p}$ of P is $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. By G_0 we denote the subgroup in P , with the Lie algebra $\mathfrak{g}_0$, consisting of all elements in P whose adjoint action preserves the grading of $\mathfrak{g}$. The grading of $\mathfrak{g}$ induces a P -invariant filtration of $\mathfrak{g}$, which gives rise to a filtration of the tangent bundle TM , due to the usual identification $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ via the Cartan connection ω . On the graded vector bundle corresponding to this filtration, there is an algebraic bracket induced by the Lie bracket of vector fields, which is called the *Levi bracket*. The parabolic geometry is *regular* if the Levi bracket corresponds to the bracket in $\mathfrak{g}$ under the identification above.

Parabolic contact geometry is a parabolic geometry whose underlying geometric structure consists of a contact distribution $H \subset TM$ and some additional structure on H . These correspond to contact gradings of simple Lie algebras as follows. The *contact grading* of a simple Lie algebra is a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_{-2}$ is one dimensional and the Lie bracket $[\cdot, \cdot] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ is non-degenerate. Let us consider regular parabolic geometry of type (G, P) such that the Lie algebra of G admits a contact grading. Then the corresponding filtration of the tangent bundle of M is just a distribution $H \subset TM$, which turns out to be contact, the Levi bracket $\mathcal{L} : H \times H \rightarrow TM/H$ is non-degenerate, and the reduction of $\text{gr}(TM) := (TM/H) \oplus H$ to the structure group G_0 gives rise to the additional structure on H .

For general parabolic geometry $(\mathcal{G} \rightarrow M, \omega)$, the curvature is often described by the so called *curvature function* $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, which is given by

$$\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) = d\omega(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) + [X, Y].$$

The Killing form on $\mathfrak{g}$ provides an identification $(\mathfrak{g}/\mathfrak{p})^*$ with $\mathfrak{p}_+$, hence the curvature function is viewed as having values in $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. The grading of $\mathfrak{g}$ induces grading also to this space, which brings the notion of homogeneity. In particular, parabolic

geometry is regular if and only if the curvature function has values in the part of positive homogeneity. Parabolic geometry is called *torsion free* if κ has values in $\Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{p}$; note that torsion free parabolic geometry is automatically regular. Next, parabolic geometry is called *normal* if $\partial^* \circ \kappa = 0$, where $\partial^* : \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g} \rightarrow \mathfrak{p}_+ \otimes \mathfrak{g}$ is the differential in the standard complex computing the homology $H_*(\mathfrak{p}_+, \mathfrak{g})$ of $\mathfrak{p}_+$ with coefficients in $\mathfrak{g}$. For a regular normal parabolic geometry, the *harmonic curvature* κ_H is the composition of κ with the natural projection $\ker(\partial^*) \rightarrow H_2(\mathfrak{p}_+, \mathfrak{g})$. By definition, κ_H is a section of $\mathcal{G} \times_P H_2(\mathfrak{p}_+, \mathfrak{g})$ and the point is it can be interpreted in terms of the underlying structure. For all details on parabolic geometries we primarily refer to [2].

2.2. Contact grading of $\mathfrak{so}(p+2, q+2)$. Consider the inner product on $\mathbb{R}^{n+4}$ given by the matrix

$$\begin{pmatrix} 0 & 0 & -\mathbb{I}_2 \\ 0 & \mathbb{I}_{p,q} & 0 \\ -\mathbb{I}_2 & 0 & 0 \end{pmatrix},$$

where $\mathbb{I}_{p,q} = \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix}$ and $\mathbb{I}_r$ is the unit matrix of rank r . According to this choice, the Lie algebra $\mathfrak{g} = \mathfrak{so}(p+2, q+2)$ has got the following form with blocks of sizes 2, n , and 2:

$$\begin{pmatrix} A & U & w\mathbb{J} \\ X & D & \mathbb{I}_{p,q}U^t \\ z\mathbb{J} & X^t\mathbb{I}_{p,q} & -A^t \end{pmatrix},$$

with $\mathbb{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where $z, w \in \mathbb{R}$, $D \in \mathfrak{so}(p, q)$, and X, A, U are real matrices of size $n \times 2, 2 \times 2, 2 \times n$, respectively. The contact grading of $\mathfrak{g}$ is read along the diagonals so that z parametrizes $\mathfrak{g}_{-2}$, X corresponds to $\mathfrak{g}_{-1}$, the pair (A, D) to $\mathfrak{g}_0$, U to $\mathfrak{g}_1$, and w to $\mathfrak{g}_2$. In particular, $X \in \mathfrak{g}_{-1}$ is understood as an element of $\mathbb{R}^{2*} \otimes \mathbb{R}^n$, the space of linear maps from $\mathbb{R}^2$ to $\mathbb{R}^n$.

If we write $X \in \mathfrak{g}_{-1}$ as the matrix (X_1, X_2) with columns $X_1, X_2 \in \mathbb{R}^n$, and similarly for Y , then the Lie bracket $[\cdot, \cdot] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ is explicitly given by

$$(1) \quad [X, Y] = (\langle X_1, Y_2 \rangle - \langle X_2, Y_1 \rangle) \cdot e,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^n$ corresponding to $\mathbb{I}_{p,q}$ and e is the generator of $\mathfrak{g}_{-2}$ corresponding to $z = 1$ in the description above. Indeed, this bracket is non-degenerate. Since $\mathfrak{g}_{-1}$ is identified with the space of linear maps from $\mathbb{R}^2$ to $\mathbb{R}^n$, an endomorphism of $\mathbb{R}^m$ acts also on $\mathfrak{g}_{-1}$ by the composition. In matrices, this is given by the left multiplication $X \mapsto CX$. From (1) it is obvious the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ is invariant under the action of endomorphisms on $\mathbb{R}^n$ preserving the inner product, i.e.

$$(2) \quad [CX, CY] = [X, Y]$$

for any $C \in O(p, q)$ and any $X, Y \in \mathfrak{g}_{-1}$. Similarly, an endomorphism A of $\mathbb{R}^2$ acts on $\mathfrak{g}_{-1}$ from the right by $X \mapsto XA$. An easy direct calculation shows that this is compatible with the bracket so that

$$(3) \quad [XA, YA] = \det A \cdot [X, Y]$$

for any $A \in \mathfrak{gl}(2, \mathbb{R})$ and any $X, Y \in \mathfrak{g}_{-1}$.

Following the introductory section, let the group to the Lie algebra $\mathfrak{g} = \mathfrak{so}(p+2, q+2)$ be $G := PO(p+2, q+2)$. The parabolic subgroup $P \subset G$ with the Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is represented by block upper triangular matrices in $G = PO(p+2, q+2)$ with blocks of sizes 2, n , and 2. Obviously, P stabilizes the plane spanned by the first two vectors of the standard basis in $\mathbb{R}^{n+4}$ (which is

indeed isotropic). The subgroup $G_0 \subset P$ is represented by block diagonal matrices in P with blocks of the same size. Explicitly, G_0 is formed by the classes of matrices of the form

$$\begin{pmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & (B^{-1})^t \end{pmatrix},$$

where $B \in GL(2, \mathbb{R})$ and $C \in O(p, q)$. Each class has just two elements which differ by the sign, hence $G_0 \cong (GL(2, \mathbb{R}) \times O(p, q)) / \{\pm \text{id}\}$. Let us represent an element of G_0 by the pair (B, C) and an element of $\mathfrak{g}_-$ by (z, X) as above. Then a direct computation shows the adjoint action of G_0 on $\mathfrak{g}_-$ is explicitly given by

$$\text{Ad}(B, C)(z, X) = (\det B^{-1} \cdot z, CXC^{-1}),$$

which indeed does not depend on the representative matrix. This shows the restriction to $\mathfrak{g}_{-1} \cong \mathbb{R}^{2*} \otimes \mathbb{R}^n$ comes from the product of the standard representation of $GL(2, \mathbb{R})$ on $\mathbb{R}^2$ and of $O(p, q)$ on $\mathbb{R}^n$. Combining with (2) and (3), the action of G_0 on $\mathfrak{g}_-$ is compatible with the Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$, i.e. the bracket is indeed G_0 -equivariant.

2.3. Lie contact structures. *Lie contact structure of signature (p, q)* on a smooth manifold M of dimension $2n + 1$, $n = p + q$, consists of the following data:

- a contact distribution $H \subset TM$,
- two auxiliary vector bundles $L \rightarrow M$ and $R \rightarrow M$ of rank 2 and n ,
- a bundle metric of signature (p, q) on R , where $p + q = n$,
- an isomorphism $H \cong L^* \otimes R$,

such that the Levi bracket is invariant under the action of $O(R_x) \cong O(p, q)$ on H_x , for each $x \in M$, i.e. $\mathcal{L}(\xi, \eta) = \mathcal{L}(\gamma \circ \xi, \gamma \circ \eta)$ for any $\gamma \in O(R_x)$ and any $\xi, \eta \in L_x^* \otimes R_x$.

As we announced above, the Lie contact structure should coincide with the underlying structure corresponding to the parabolic geometry of type (G, P) , i.e. with the parabolic contact structure corresponding to the contact grading of $\mathfrak{g} = \mathfrak{so}(p + 2, q + 2)$. This is really the case and the equivalence is formulated as Proposition 4.2.5 in [2]:

Theorem. *Let $G = PO(p + 2, q + 2)$ and $P \subset G$ be the stabilizer of an isotropic plane. Then the category of regular normal parabolic geometries of type (G, P) is equivalent to the category of Lie contact structures of signature (p, q) .*

Remarks. (1) Note that the section 4.2.5 in [2] develops according to the choice $G = O(p + 2, q + 2)$. However, different choices of the Lie group to the given Lie algebra coincide generally up to a (usually finite) cover. In particular, the freedom in the choice does not affect the local description of the underlying structure.

(2) Although we define the Lie contact structure for general $n = p + q$, there are some specific features in low dimensions. For instance, the Lie contact structure of signature $(2, 0)$ (on 5-dimensional manifold) is basically equivalent to a non-degenerate CR structure with indefinite Levi form, which is discussed e.g. in section 5 in [7] in some detail. In our terms, the equivalence is provided by the Lie algebra isomorphism $\mathfrak{so}(4, 2) \cong \mathfrak{su}(2, 2)$ and the uniqueness of the contact grading. Similarly, for $\mathfrak{so}(3, 3) \cong \mathfrak{sl}(4, \mathbb{R})$, the Lie contact structure of signature $(1, 1)$ is equivalent to the Lagrangean contact structure (on a 5-manifold). The interpretation of $\mathfrak{so}(3, 2) \cong \mathfrak{sp}(4)$ is a bit different, since the Lie contact structure of signature $(1, 0)$ is rather trivial and usually excluded from the considerations. However, the structure corresponding to the contact grading of $\mathfrak{sp}(4)$ is the contact projective structure (on a 3-manifold). If dimension of the base manifold is greater than or equal to 7, the Lie contact structure starts to work with no specific issue, up to

the only exception in dimension 9, which is due to $\mathfrak{so}(6, 2) \cong \mathfrak{so}^*(8)$. An interested reader may investigate the equivalent structure behind that isomorphism.

2.4. Segre cone. For $H_x \cong L_x^* \otimes R_x$, there is a distinguished subset $\mathcal{C}_x \subset H_x$, the so called *Segre cone*, consisting of all the linear maps $L_x \rightarrow R_x$ of rank one. On the level of the Lie algebra $\mathfrak{g} = \mathfrak{so}(p+2, q+2)$, the cone $\mathcal{C}_x \subset H_x$ corresponds to the G_0 -invariant subset of $\mathfrak{g}_{-1} \cong \mathbb{R}^{2*} \otimes \mathbb{R}^n$ consisting of the elements of the form $f \otimes u$ for some $f \in \mathbb{R}^{2*}$ and $u \in \mathbb{R}^n$. It follows that $f \otimes \mathbb{R}^n$ is a maximal linear subspace in the cone (consisting of the linear maps $\mathbb{R}^2 \rightarrow \mathbb{R}^n$ with the common kernel $\ker f$). For arbitrary elements $X = f \otimes u_1$ and $Y = f \otimes u_2$ from $f \otimes \mathbb{R}^n$, the substitution into (1) yields that $[X, Y] = 0$, i.e. the subspace $f \otimes \mathbb{R}^n \subset \mathfrak{g}_{-1}$ is isotropic with respect to the Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$. Hence for general rank-one elements $f_1 \otimes u_1, f_2 \otimes u_2$ from $\mathbb{R}^{2*} \otimes \mathbb{R}^n = \mathfrak{g}_{-1}$, it follows that

$$[f_1 \otimes u_1, f_2 \otimes u_2] = |f_1, f_2| \langle u_1, u_2 \rangle \cdot e,$$

where $| \cdot, \cdot |$ denotes the standard exterior product on $\mathbb{R}^{2*}$, $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$ of signature (p, q) , and e is the generator of $\mathfrak{g}_{-2}$ as in 2.2. This shows the Lie bracket is actually of the form $\Lambda^2 \mathbb{R}^2 \otimes S^2 \mathbb{R}^{n*} \otimes \mathfrak{g}_{-2}$ and it determines the inner product on $\mathbb{R}^n$ provided we fix an identification $\Lambda^2 \mathbb{R}^2 \otimes \mathfrak{g}_{-2} \cong \mathbb{R}$. This is provided by the G_0 -invariant mapping $f_1 \wedge f_2 \otimes ze \mapsto z|f_1, f_2|$, which also yields a trivialization of the line bundle $\Lambda^2 L \otimes TM/H$ over any Lie contact manifold M .

Proposition. *A Lie contact structure on a smooth manifold M of dimension $2n+1$ is equivalent to the following data:*

- a contact distribution $H \subset TM$,
- two auxiliary vector bundles $L \rightarrow M$ and $R \rightarrow M$ of rank 2 and n ,
- an isomorphism $H \cong L^* \otimes R$,
- an isomorphism $\Lambda^2 L^* \cong TM/H$,

such that for $\varphi \in L_x^*$ the subspace $\varphi \otimes R_x \subset H_x$ is isotropic with respect to the Levi bracket, i.e. $\mathcal{L}(\varphi \otimes v_1, \varphi \otimes v_2) = 0$ for any $v_1, v_2 \in R_x$.

Proof. According to the discussion above, we only need to construct a Lie contact structure on M from the later data. In general, the Levi bracket is a section of the bundle $\Lambda^2 H^* \otimes TM/H$, which decomposes according to the isomorphism $H \cong L^* \otimes R$ as $(\Lambda^2 L \otimes S^2 R^* \otimes TM/H) \oplus (S^2 L \otimes \Lambda^2 R^* \otimes TM/H)$. Since we assume $\varphi \otimes R$ is isotropic with respect to $\mathcal{L}$, for any φ , the Levi bracket $\mathcal{L}$ factorizes through the first summand only. Since we assume a fixed identification of line bundles $\Lambda^2 L^*$ and TM/H , i.e. a trivialization of $\Lambda^2 L \otimes TM/H$, the Levi bracket determines a (non-degenerate) bundle metric on R , which accomplishes the Lie contact structure on M . In particular, if $O(R_x)$ is the group of orthogonal transformations of R_x according to this inner product, then the Levi bracket is by construction invariant under the action of $O(R_x)$, over each $x \in M$. $\square$

2.5. Split quaternions. The algebra of *split quaternions* $\mathbb{H}_s$ is the four-dimensional real algebra admitting a basis $(1, i, j, k)$ such that $i^2 = j^2 = 1$ and $k = ij = -ji$ (consequently $k^2 = -1$, $kj = -jk = i$, etc.). As an associative algebra, $\mathbb{H}_s$ is isomorphic to the space of 2×2 real matrices so that the norm on $\mathbb{H}_s$ corresponds to the determinant. This correspondence is explicitly given by

$$(4) \quad 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In particular, the norm squared of i, j , and k , equals $-1, -1$, and 1 , respectively. For general imaginary quaternion $q = ai + bj + ck$, the norm squared is $|q|^2 = -a^2 - b^2 + c^2$, which can also be given by $q^2 = -|q|^2$.

The *split-quaternionic structure* on a vector bundle H over M is defined as a 3-dimensional subbundle $\mathcal{Q} \subset \text{End } H$ admitting a basis (I, J, K) such that $I^2 = J^2 = -\text{id}$ and $K = I \circ J = -J \circ I$. Note that, in contrast to the true quaternionic structures, split-quaternionic structure may exist on vector bundles of any even rank.

Lemma. *Let $H \rightarrow M$ be a vector bundle of rank $2n$.*

- (1) *A split-quaternionic structure $\mathcal{Q}$ on H is equivalent to an isomorphism $H \cong L^* \otimes R$, where $L \rightarrow M$ and $R \rightarrow M$ are vector bundles of rank 2 and n , respectively.*
- (2) *In particular, a non-zero element ξ in $H_x \cong L_x^* \otimes R_x$ has rank one if and only if there is $A \in \mathcal{Q}_x$ such that $A(\xi) = \xi$ (consequently, A is a product structure).*
- (3) *Moreover, the maximal linear subspaces contained in the cone of rank-one elements in H_x are the +1-eigenspaces of the product structures in $\mathcal{Q}_x$.*

Proof. (1) Let $H \cong L^* \otimes R$. Since $L_x \cong \mathbb{R}^2$ and $R_x \cong \mathbb{R}^n$, for each $x \in M$, the natural action of $\mathbb{H}_s$ on $\mathbb{R}^2$ extends, by the composition, to the action on $\mathbb{R}^{2*} \otimes \mathbb{R}^n$. Denoting by I, J , and K the endomorphisms of $H_x \cong \mathbb{R}^{2*} \otimes \mathbb{R}^n$ corresponding to the elements i, j , and k from (4), we obtain a split-quaternionic structure $\mathcal{Q}_x := \langle I, J, K \rangle$.

Conversely, let a split-quaternionic structure $\mathcal{Q} = \langle I, J, K \rangle$ be given. For each $x \in M$, the subspace $H_x \subset T_x M$ decomposes to the eigenspaces $H_x^+ \oplus H_x^-$ with respect to the product structure I . Let us define the auxiliary vector bundles over M to be $R := H^+ \subset H$ and $L := \langle I, J \rangle^*$, the dual of $\langle I, J \rangle \subset \mathcal{Q}$. (Obviously, R and L have got rank n and 2, respectively.) We claim that $L^* \otimes R \cong H$ under the mapping $\varphi \otimes X \mapsto \varphi(X)$: By definition, $\varphi \in L^*$ is a linear combination of the endomorphisms I, J and the restriction of I to $R = H^+$ is the identity. Since $IJ = -JI$, it follows that J swaps the subspaces H^+ and H^- . Hence any $\xi \in H_x$ can be uniquely written as $\xi = v_1 + Jv_2$, for $v_1, v_2 \in R_x$, which is the image of $I \otimes v_1 + J \otimes v_2$ under the mapping above. Because it is a linear map between the spaces of the same dimension, the claim follows.

(2) As above, let us interpret $\xi \in H_x$ as a linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^n$ and $A \in \mathcal{Q}_x$ as an endomorphism $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Hence $A(\xi)$ is interpreted as the composition $\xi \circ A$. Easily, $\xi \circ A = \xi$ for some A if and only if A is the identity or $\ker \xi$ is non-trivial and invariant under A and there further is a non-zero vector fixed by A . In other words, the later condition means that A has two real eigenvalues 1 and λ so that the eigenspace corresponding to λ coincides with $\ker \xi$. Considering an endomorphism $A = aI + bJ + cK$, the eigenvalues of A turn out to be the solutions of $\lambda^2 + \det A = 0$. Hence 1 is an eigenvalue of A if and only if $\det A = -1$. Consequently, the second eigenvalue is -1 and A is a skew reflection. In terms of split quaternions, $|A|^2 = -1$ and A corresponds to a product structure on H_x . Note that for any one-dimensional subspace ℓ in $\mathbb{R}^2$ there is a skew reflection $A = aI + bJ + cK$ whose eigenspace to -1 is ℓ . Altogether, since A is never the identity, the equivalence follows.

(3) For the last statement, note that the maximal linear subspaces in the Segre cone $\mathcal{C}_x$ have dimension n and all of them are parametrized by one-dimensional subspaces in $\mathbb{R}^2$ as follows. For any line $\ell \subset \mathbb{R}^2$, let us consider the set W_ℓ of all linear maps $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ so that $\ker \xi = \ell$. Indeed, $W_\ell \subset H_x$ is a linear subspace of dimension n whose each element has got rank one. As before, there is a skew reflection A in $\mathbb{R}^2$ so that $A|_\ell = -\text{id}$. Hence $\xi \circ A = \xi$ for any $\xi \in W_\ell$, i.e. W_ℓ is the eigenspace to 1 of the corresponding product structure on H_x . The converse is evident as well. $\square$

For any $A \in \mathcal{Q}_x$, $A \circ A$ is a multiple of the identity, hence there is a natural inner product in each fibre $\mathcal{Q}_x$; the corresponding norm is defined by $A \circ A = -|A|^2 \text{id}$, the signature is $(1, 2)$. Let us denote by $\mathcal{S}$ the line bundle over M , whose fiber consists of all real multiples of this inner product on $\mathcal{Q}_x$. For a Lie contact manifold M ,

the corresponding split-quaternionic structure $\mathcal{Q}$ on $H \subset TM$ has to be compatible with the Levi bracket, which is expressed by (3) on the level of Lie algebra. Since $\det A$ corresponds to $|A|^2$ under the above identifications, we conclude with the following description of the Lie contact structure.

Proposition. *A Lie contact structure on a smooth manifold M of dimension $2n+1$ is equivalent to the following data:*

- a contact distribution $H \subset TM$,
- a split-quaternionic structure $\mathcal{Q} = \langle I, J, K \rangle \subset \text{End } H$,
- an identification $\mathcal{S} \cong TM/H$,

such that the Levi bracket is compatible as $\mathcal{L}(A(\xi), A(\eta)) = |A|^2 \cdot \mathcal{L}(\xi, \eta)$, for any $\xi, \eta \in H_x$ and $A \in \mathcal{Q}_x$ in each $x \in M$.

Proof. According to the previous discussion, we just construct a Lie contact structure from the later data. By the lemma above, the split-quaternionic structure $\mathcal{Q}$ provides an identification $H \cong L^* \otimes R$ where L and R are appropriate vector bundles. (Namely, $R_x \subset H_x$ is defined as the $+1$ -eigenspace of the product structure I and L_x as the dual of $\langle I, J \rangle$.) Now, for $\varphi \in L_x^*$, let us consider the subset $W := \varphi \otimes R_x \subset H_x$. This is a linear n -dimensional subspace contained in the Segre cone of H_x and there is a product structure $A \in \mathcal{Q}_x$ such that $A|_W = \text{id}$. Hence for any $\xi, \eta \in W$, the compatibility $\mathcal{L}(A(\xi), A(\eta)) = |A|^2 \cdot \mathcal{L}(\xi, \eta)$ yields $\mathcal{L}(\xi, \eta) = -\mathcal{L}(\xi, \eta)$ and so $\mathcal{L}(\xi, \eta) = 0$. In other words, $\varphi \otimes R_x$ is an isotropic subspace in H_x for any φ .

According to Proposition 2.4, in order to determine a Lie contact structure on M we further need to identify the line bundles $\Lambda^2 L^*$ and TM/H over M . However, this is provided by the identification $\mathcal{S} \cong TM/H$ assumed above and the identification $\mathcal{S} \cong \Lambda^2 L^*$ which follows: By definition, L_x^* is identified with $\langle I, J \rangle \subset \mathcal{Q}_x$. Hence the correspondence between the volume forms on $\langle I, J \rangle$ and the multiples of the natural inner product on $\mathcal{Q}_x$ is obvious. $\square$

3. Chains

3.1. Path geometry of chains. For general parabolic contact geometry $(\mathcal{G} \rightarrow M, \omega)$ of type (G, P) , the *chains* are defined as projections of flow lines of constant vector fields on $\mathcal{G}$ corresponding to non-zero elements of $\mathfrak{g}_{-2}$, where $\mathfrak{g}_{-2}$ is the 1-dimensional subspace from the contact grading of $\mathfrak{g}$ as above. In the homogeneous model G/P , the chains are the curves of type $t \mapsto g \exp(tX)P$, for $g \in G$ and $X \in \mathfrak{g}_{-2}$. As unparametrized curves, chains are uniquely determined by a tangent direction in a point, [3, section 4]. By definition, chains are defined only for directions which are transverse to the contact distribution $H \subset TM$. In classical terms, the family of chains defines a path geometry on M restricted, however, only to the directions transverse to H .

A *path geometry* on M is a smooth family of unparametrized curves (paths) on M with the property that for any point $x \in M$ and any direction $\ell \subset T_x M$ there is unique path C from the family such that $T_x C = \ell$. It turns out that path geometry on M can be equivalently described as a parabolic geometry over $\mathcal{PTM}$ of type $(\tilde{G}, \tilde{P})$, where $\tilde{G} = PGL(m+1, \mathbb{R})$, $m = \dim M$, and $\tilde{P}$ is the parabolic subgroup as follows. Let us consider the grading of $\tilde{\mathfrak{g}} = \mathfrak{sl}(m+1, \mathbb{R})$ which is schematically described by the block decomposition with blocks of sizes 1, 1, and $m-1$:

$$\begin{pmatrix} \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1^E & \tilde{\mathfrak{g}}_2 \\ \tilde{\mathfrak{g}}_{-1}^E & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1^V \\ \tilde{\mathfrak{g}}_{-2} & \tilde{\mathfrak{g}}_{-1}^V & \tilde{\mathfrak{g}}_0 \end{pmatrix}.$$

Then $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$ is a parabolic subalgebra of $\tilde{\mathfrak{g}}$ and $\tilde{P} \subset \tilde{G}$ is the subgroup represented by block upper triangular matrices so that its Lie algebra is $\tilde{\mathfrak{p}}$. As usual, the parabolic geometry associated to the path geometry on M is uniquely determined (up to isomorphism) provided we consider it is regular and normal.

Denote by $\tilde{M} = \mathcal{P}_0 TM$ the open subset of $\mathcal{P}TM$ consisting of all lines in TM which are transverse to the contact distribution $H \subset TM$. Now the family of chains on M gives rise to a parabolic geometry $(\tilde{\mathcal{G}} \rightarrow \tilde{M}, \tilde{\omega})$ of type $(\tilde{G}, \tilde{P})$ which we call the *path geometry of chains* and which we are going to describe in a direct way extending somehow the Cartan geometry $(\mathcal{G} \rightarrow M, \omega)$. For this reason it is crucial to observe that $\tilde{M}$ is naturally isomorphic to the quotient bundle $\mathcal{G}/Q$, where $Q \subset P$ is the stabilizer of the 1-dimensional subspace $\mathfrak{g}_{-2} \subset \mathfrak{g}_-$ under the action induced from the adjoint representation. Evidently, the Lie algebra of Q is $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$. Hence the couple $(\mathcal{G} \rightarrow \tilde{M}, \omega)$ forms a Cartan (but not parabolic) geometry of type (G, Q) .

3.2. Induced Cartan connection. Starting with the homogeneous model G/P of the parabolic contact geometry, let $(\tilde{\mathcal{G}} \rightarrow \tilde{G}/\tilde{P}, \tilde{\omega})$ be the regular normal parabolic geometry of type $(\tilde{G}, \tilde{P})$ corresponding to the path geometry of chains. Since $\tilde{G}/\tilde{P} = \mathcal{P}_0 T(G/P)$ is isomorphic to the homogeneous space G/Q , the Cartan geometry $(\tilde{\mathcal{G}} \rightarrow G/Q, \tilde{\omega})$ is homogeneous under the action of the group G . Hence it is known there is a pair (i, α) of maps, consisting of a Lie group homomorphism $i : Q \rightarrow \tilde{P}$ and a linear map $\alpha : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$, so that $\tilde{\mathcal{G}} \cong G \times_Q \tilde{P}$ and $j^* \tilde{\omega} = \alpha \circ \mu$, where j is the canonical inclusion $\mathcal{G} \hookrightarrow G \times_Q \tilde{P}$ and μ is the Maurer–Cartan form on G . In particular, the pair (i, α) has to satisfy the following conditions:

- (1) $\alpha \circ \text{Ad}(h) = \text{Ad}(i(h)) \circ \alpha$ for all $h \in Q$,
- (2) the restriction $\alpha|_{\mathfrak{q}}$ coincides with $i' : \mathfrak{q} \rightarrow \tilde{\mathfrak{p}}$, the derivative of i ,
- (3) the map $\underline{\alpha} : \mathfrak{g}/\mathfrak{q} \rightarrow \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ induced by α is a linear isomorphism.

On the other hand, any pair of maps (i, α) which are compatible in the above sense gives rise to a functor from Cartan geometries of type (G, Q) to Cartan geometries of type $(\tilde{G}, \tilde{P})$. There is an easy control over the natural equivalence of functors associated to different pairs, see section 3 in [4].

If κ is the curvature function of the Cartan geometry $(\mathcal{G} \rightarrow M, \omega)$, then the curvature function $\tilde{\kappa}$ of the Cartan geometry induced by (i, α) is completely determined by κ and the contribution of α . More specifically, this is given by

$$(5) \quad \tilde{\kappa} \circ j = \alpha \circ \kappa \circ \underline{\alpha}^{-1} + \Psi_\alpha,$$

where $\Psi_\alpha : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \rightarrow \tilde{\mathfrak{g}}$ is defined as follows. Consider a bilinear map $\mathfrak{g} \times \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ defined by

$$(X, Y) \rightarrow [\alpha(X), \alpha(Y)] - \alpha([X, Y]).$$

The map is obviously skew symmetric and, due to the compatibility conditions on (i, α) , it factorizes to a well-defined Q -equivalent map $\mathfrak{g}/\mathfrak{q} \times \mathfrak{g}/\mathfrak{q} \rightarrow \tilde{\mathfrak{g}}$. The map Ψ_α is then obtained according to the isomorphism $\underline{\alpha}^{-1} : \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \rightarrow \mathfrak{g}/\mathfrak{q}$. Note that Ψ_α vanishes if and only if α is a homomorphism of Lie algebras.

3.3. The pair. Here we describe the pair (i, α) for Lie contact structures explicitly. Considering the dimension of the base manifold M is $m = 2n + 1$, we have to consider $G = PO(p + 2, q + 2)$, $p + q = n$, and $\tilde{G} = PGL(2n + 2, \mathbb{R})$. According to the definition in 3.1, the subgroup $Q \subset P$ is represented by the block matrices of the form

$$\begin{pmatrix} B & 0 & wC\mathbb{J} \\ 0 & C & 0 \\ 0 & 0 & (B^{-1})^t \end{pmatrix},$$

where $\mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $w \in \mathbb{R}$, $B \in GL(2, \mathbb{R})$, and $C \in O(p, q)$. If we denote $\beta := \det B$ and substitute $B = \begin{pmatrix} p & r \\ s & q \end{pmatrix}$, then the two maps $i : Q \rightarrow \tilde{P}$ and $\alpha : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ are defined explicitly by

$$i \begin{pmatrix} p & r & 0 & -rw & pw \\ s & q & 0 & -qw & sw \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & \frac{q}{\beta} & -\frac{s}{\beta} \\ 0 & 0 & 0 & -\frac{r}{\beta} & \frac{p}{\beta} \end{pmatrix} := \begin{pmatrix} \sqrt{|\beta|} & -w\sqrt{|\beta|} & 0 & 0 \\ 0 & \frac{1}{\sqrt{|\beta|}} & 0 & 0 \\ 0 & 0 & \frac{q}{\sqrt{|\beta|}}C & -\frac{s}{\sqrt{|\beta|}}C \\ 0 & 0 & -\frac{r}{\sqrt{|\beta|}}C & \frac{p}{\sqrt{|\beta|}}C \end{pmatrix},$$

$$\alpha \begin{pmatrix} a & b & U_1 & 0 & w \\ c & d & U_2 & -w & 0 \\ X_1 & X_2 & D & \mathbb{I}_{p,q}U_1^t & \mathbb{I}_{p,q}U_2^t \\ 0 & z & X_1^t\mathbb{I}_{p,q} & -a & -c \\ -z & 0 & X_2^t\mathbb{I}_{p,q} & -b & -d \end{pmatrix} := \begin{pmatrix} \frac{a+d}{2} & -w & \frac{1}{2}U & \frac{1}{2}V \\ z & -\frac{a+d}{2} & -\frac{1}{2}Y^t\mathbb{I}_{p,q} & \frac{1}{2}X^t\mathbb{I}_{p,q} \\ X_1 & -\mathbb{I}_{p,q}U_2^t & D + \frac{d-a}{2}\mathbb{I}_n & -c\mathbb{I}_n \\ X_2 & \mathbb{I}_{p,q}U_1^t & -b\mathbb{I}_n & D + \frac{a-d}{2}\mathbb{I}_n \end{pmatrix}.$$

Proposition. *The map $i : Q \rightarrow \tilde{P}$ is an injective Lie group homomorphism, $\alpha : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ is linear, and the pair (i, α) satisfies the conditions (1)–(3) from 3.2.*

Hence the pair (i, α) gives rise to an extension functor from Cartan geometries of type (G, Q) to Cartan geometries of type $(\tilde{G}, \tilde{P})$.

Proof. The map i is obviously well defined, i.e. the image of an element of $Q \subset PO(p+2, q+2)$ does not depend on the representative matrix. Further, i is smooth and injective and a direct computation shows this is a homomorphism of Lie groups. The map α is linear and the compatibility of the pair (i, α) follows as follows: (1) involves a little tedious but very straightforward checking, (2) is an easy exercise, (3) follows from $\mathfrak{g}/\mathfrak{q} \cong \mathfrak{g}_- \oplus \mathfrak{g}_1$ and $\tilde{\mathfrak{g}}_- \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$, restricting α to $\mathfrak{g}_- \oplus \mathfrak{g}_1$. $\square$

3.4. The properties. Here we analyze the map $\Psi_\alpha : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \rightarrow \tilde{\mathfrak{g}}$ from 3.2. For this reason we need a bit of notation:

As a linear space, $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ can be identified with $\tilde{\mathfrak{g}}_- = \tilde{\mathfrak{g}}_-^E \oplus \tilde{\mathfrak{g}}_-^V \oplus \tilde{\mathfrak{g}}_{-2}$. Using brackets in $\tilde{\mathfrak{g}}$, we may identify $\tilde{\mathfrak{g}}_-^V$ with $\tilde{\mathfrak{g}}_1^E \otimes \tilde{\mathfrak{g}}_{-2}$ if necessary. We will further view $\tilde{\mathfrak{g}}_{-2}$ as $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and correspondingly write $X \in \tilde{\mathfrak{g}}_{-2}$ as $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ for $X_1, X_2 \in \mathbb{R}^n$. Next, the semi-simple part $\tilde{\mathfrak{g}}_0^{ss}$ of $\tilde{\mathfrak{g}}_0$ is isomorphic to $\mathfrak{sl}(2n, \mathbb{R})$ and the restriction of the adjoint representation of $\tilde{\mathfrak{g}}_0^{ss}$ on $\tilde{\mathfrak{g}}_{-2}$ coincides with the standard representation of $\mathfrak{sl}(2n, \mathbb{R})$ on $\mathbb{R}^{2n}$. Finally, by $\langle \cdot, \cdot \rangle$ we denote the standard inner product of signature (p, q) on $\mathbb{R}^n$ as above, i.e. $\langle X_1, X_2 \rangle = X_1^t \mathbb{I}_{p,q} X_2$ for any $X_1, X_2 \in \mathbb{R}^n$.

Lemma. *Viewing the map Ψ_α from 3.2 as an element of $(\tilde{\mathfrak{g}}_-)^* \wedge (\tilde{\mathfrak{g}}_-)^* \otimes \tilde{\mathfrak{g}}$, then it lies in the subspace $(\tilde{\mathfrak{g}}_-^V)^* \wedge (\tilde{\mathfrak{g}}_{-2})^* \otimes \tilde{\mathfrak{g}}_0^{ss}$. Denoting by W_0 a non-zero element of $\tilde{\mathfrak{g}}_1^E$, then the trilinear map $\tilde{\mathfrak{g}}_{-2} \times \tilde{\mathfrak{g}}_{-2} \times \tilde{\mathfrak{g}}_{-2} \rightarrow \tilde{\mathfrak{g}}_{-2}$ defined by $(X, Y, Z) \mapsto [\Psi_\alpha(X, [Y, W_0]), Z]$ is (up to a non-zero multiple) the complete symmetrization of the map*

$$(X, Y, Z) \mapsto \begin{pmatrix} \langle X_1, Y_2 \rangle Z_1 - \langle X_1, Y_1 \rangle Z_2 \\ \langle X_2, Y_2 \rangle Z_1 - \langle X_1, Y_2 \rangle Z_2 \end{pmatrix}.$$

Proof. For $X, Y \in \tilde{\mathfrak{g}}_-$, let $\hat{X}, \hat{Y}$ be the unique elements in $\mathfrak{g}_- \oplus \mathfrak{g}_1$ so that $\alpha(\hat{X})$ and $\alpha(\hat{Y})$ is congruent to X and Y modulo $\tilde{\mathfrak{p}}$, respectively. By the definition, one computes directly $\Psi_\alpha(X, Y) = [\alpha(\hat{X}), \alpha(\hat{Y})] - \alpha([\hat{X}, \hat{Y}])$ which has always values in the lower right $2n \times 2n$ block of $\tilde{\mathfrak{g}}$ with vanishing trace, i.e. in the semi-simple part of $\tilde{\mathfrak{g}}_0$. Further, Ψ_α vanishes whenever both the entries X, Y lie in $\tilde{\mathfrak{g}}_-^V$ or in $\tilde{\mathfrak{g}}_{-2}$ or any of them lies in $\tilde{\mathfrak{g}}_-^E$. Hence the map Ψ_α is indeed of the form $\tilde{\mathfrak{g}}_-^V \wedge \tilde{\mathfrak{g}}_{-2} \rightarrow \tilde{\mathfrak{g}}_0^{ss}$.

Considering $X, Y \in \tilde{\mathfrak{g}}_{-2}$, the element $[Y, W_0]$ lies in $\tilde{\mathfrak{g}}_{-1}^V$ and the non-zero $2n \times 2n$ block of $\Psi_\alpha(X, [Y, W_0])$ looks explicitly like

$$\begin{pmatrix} \frac{1}{2}(R_{12} + \text{tr } R_{12} \cdot \mathbb{I}_n) - R_{21} & \frac{1}{2}(R_{11} - \text{tr } R_{11} \cdot \mathbb{I}_n) \\ \frac{1}{2}(R_{22} - \text{tr } R_{22} \cdot \mathbb{I}_n) & R_{12} - \frac{1}{2}(R_{21} + \text{tr } R_{21} \cdot \mathbb{I}_n) \end{pmatrix},$$

where $R_{11} = (X_1 Y_1^t + Y_1 X_1^t) \mathbb{I}_{p,q}$, $R_{22} = (X_2 Y_2^t + Y_2 X_2^t) \mathbb{I}_{p,q}$, $R_{12} = (X_1 Y_2^t + Y_1 X_2^t) \mathbb{I}_{p,q}$, and $R_{21} = R_{12}^t$. The value of $[\Psi_\alpha(X, [Y, W_0]), Z]$ is then obtained by applying the above matrix to the vector $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ from $\tilde{\mathfrak{g}}_{-2} = \mathbb{R}^n \times \mathbb{R}^n$. The result turns out to be the cyclic sum of

$$\frac{1}{2}(\langle Y_1, X_2 \rangle + \langle Y_2, X_1 \rangle) \begin{pmatrix} Z_1 \\ -Z_2 \end{pmatrix} + \begin{pmatrix} -\langle X_1, Y_1 \rangle Z_2 \\ \langle X_2, Y_2 \rangle Z_1 \end{pmatrix},$$

which is the complete symmetrization of

$$\langle X_1, Y_2 \rangle \begin{pmatrix} Z_1 \\ -Z_2 \end{pmatrix} + \begin{pmatrix} -\langle X_1, Y_1 \rangle Z_2 \\ \langle X_2, Y_2 \rangle Z_1 \end{pmatrix}$$

up to a non-zero multiple. $\square$

In 3.2 we motivated the definition of the pair (i, α) . In order to justify the choice we made in 3.3, we have to check that starting with the homogeneous model, the associated Cartan geometry determined by (i, α) is regular and normal, i.e. it is the canonical Cartan geometry describing the path geometry of chains on G/P . This is provided by the following Theorem.

3.5. Theorem. *Let $\dim M \geq 7$ and $(\mathcal{G} \rightarrow M, \omega)$ be a regular normal parabolic geometry of type (G, P) and let $(\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P} \rightarrow \mathcal{P}_0 TM, \tilde{\omega}_\alpha)$ be the parabolic geometry obtained using the extension functor associated to the pair (i, α) defined in 3.3. Then this geometry is regular and normal if and only if $(\mathcal{G} \rightarrow M, \omega)$ is locally flat. In that case the induced Cartan geometry is non-flat and torsion free.*

Proof. If $(\mathcal{G} \rightarrow M, \omega)$ is locally flat parabolic geometry of type (G, P) then by (5) the curvature of the induced Cartan geometry is determined only by Ψ_α . Since Ψ_α is non-trivial, the induced Cartan geometry is non-flat and since the values of Ψ_α are in $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{p}}$, it is torsion free and hence regular. In order to prove the normality, one has to show that $\partial^* \Psi_\alpha = 0$. This can be proved either by a direct computation or by a more conceptual argument as in the proof of Theorem 3.6 in [4].

Conversely, let us suppose the induced Cartan geometry $(\tilde{\mathcal{G}} \rightarrow \mathcal{P}_0 TM, \tilde{\omega}_\alpha)$ is regular and normal. Then, analyzing the harmonic curvature of $\tilde{\omega}_\alpha$ as in the first part of the proof of Theorem 3.8 in [4], it easily follows the Cartan geometry $(\mathcal{G} \rightarrow M, \omega)$ is necessarily torsion free. If $\dim M \geq 7$, different from 9, there are two harmonic curvature components of ω ; if $\dim M = 9$, one of the two components above is further split into two parts. In any case, all the components are of homogeneity one, i.e. all are torsions. Hence for $\dim M \geq 7$, torsion free and normal parabolic geometry of type (G, P) is flat, i.e. locally isomorphic to the homogeneous model. $\square$

4. Applications

The applications in this section are based on the normality of the induced Cartan geometry. Hence according to Theorem 3.5 we consider the Lie contact structure is locally flat.

4.1. Theorem. *Let M be a contact manifold of dimension ≥ 7 with locally flat Lie contact structure. Then there is no linear connection on TM which has the chains among its geodesics.*

Proof. Since we deal with locally flat Lie contact structures, the induced path geometry of chains is regular and normal by Theorem 3.5 and its curvature is completely determined by the map Ψ_α . From lemma 3.4 we know that Ψ_α is of homogeneity three, hence it must coincide with the unique harmonic curvature component which is there in this homogeneity for generalized path geometries, see e.g. the summary in section 3.7 in [4]. However by the second part of Theorem 4.7 in [1], the vanishing of this component is equivalent to the fact the path geometry comes from a projective structure on M . Since $\Psi_\alpha \neq 0$, the claim follows. $\square$

In other words, even working with the locally flat Lie contact structures, the family of chains forms a rather complicated system of curves. In terms of second order ODE's, the chain equation is never locally equivalent either to the trivial equation or to the geodesic equation. In particular, considering the homogeneous model, the chain equation provides an example of non-trivial torsion free second order ODE with a reasonably large automorphism group. More specifically, the automorphism group obviously contains $G = PO(p+2, q+2)$, i.e. it has dimension at least $\frac{n^2+5n+6}{2}$, provided $n = p+q$ as before. We will see in the next section the dimension actually equals to the dimension of G .

4.2. The reconstruction. As we know from 3.4, the harmonic curvature $\tilde{\kappa}_H$ of the induced Cartan geometry is a section of the bundle associated to $(\tilde{\mathfrak{g}}_{-1}^V)^* \otimes (\tilde{\mathfrak{g}}_{-2})^* \otimes \tilde{\mathfrak{g}}_0^{ss}$. We are going to interpret this quantity geometrically, which will allow us to reconstruct the Lie contact structure from the path geometry of chains.

Let $(\mathcal{G} \rightarrow M, \omega)$ be the parabolic geometry corresponding to a locally flat Lie contact structure on M . Let $\tilde{M} = \mathcal{P}_0 TM$ and let $(\tilde{\mathcal{G}} \rightarrow \tilde{M}, \tilde{\omega})$ be the parabolic geometry induced by the path geometry of chains. Let us denote by $\tilde{E}$ and $\tilde{V}$ the subbundles in $T\tilde{M}$ corresponding to the subspaces $\tilde{\mathfrak{g}}_{-1}^E$ and $\tilde{\mathfrak{g}}_{-1}^V$ in $\tilde{\mathfrak{g}}_{-1}$, respectively. Let us further denote $\tilde{F} := T\tilde{M}/(\tilde{E} \oplus \tilde{V})$; as an associated bundle over $\tilde{M}$ this corresponds to $\tilde{\mathfrak{g}}_{-2} \cong \tilde{\mathfrak{g}}_{-1}^E / \tilde{\mathfrak{g}}_{-1}^V$. As before, we can replace the space $(\tilde{\mathfrak{g}}_{-1}^V)^* \otimes (\tilde{\mathfrak{g}}_{-2})^* \otimes \tilde{\mathfrak{g}}_0^{ss}$ by $\tilde{\mathfrak{g}}_{-1}^E \otimes (\otimes^3 (\tilde{\mathfrak{g}}_{-2})^*) \otimes \tilde{\mathfrak{g}}_{-2}$; the corresponding associated bundle is $\tilde{E} \otimes (\otimes^3 \tilde{F}^*) \otimes \tilde{F}$. Altogether, since $\tilde{E} \subset T\tilde{M}$ is a line bundle, we can view the harmonic curvature $\tilde{\kappa}_H$ as a section of the bundle $\otimes^3 \tilde{F}^* \otimes \tilde{F} \rightarrow \tilde{M}$ determined up to a non-zero multiple. (More specifically, lemma 3.4 shows it is actually completely symmetric and trace free.)

In order to express $\tilde{\kappa}_H$ in terms of the underlying Lie contact structure on M , let us fix $x \in M$ and $\ell \in \pi^{-1}(x)$, where $\pi : \tilde{M} \rightarrow M$ is the natural projection. (By definition, ℓ is a line in $T_x M$ which is transverse to the contact distribution $H_x \subset T_x M$.) For each $\xi \in T_x M$, let $\tilde{\xi} \in T_\ell \tilde{M}$ be any lift and consider its class in $\tilde{F}_\ell = T_\ell \tilde{M} / (\tilde{E}_\ell \oplus \tilde{V}_\ell)$. Since $\tilde{V} \subset T\tilde{M}$ is the vertical subbundle of the projection π , this class is independent of the choice of the lift. From the explicit description of the tangent map of the projection π it easily follows that its appropriate restriction yields a linear isomorphism $H_x \cong \tilde{F}_\ell$. Altogether, for a fixed x and ℓ , the $\tilde{\kappa}_H$ gives rise to an element of $\otimes^3 H_x^* \otimes H_x$, denoted by S_x , which is determined up to a non-zero multiple. Its explicit description and the geometrical meaning are as follows.

Lemma. *Let $(M, H = L^* \otimes R)$ be a locally flat Lie contact manifold and let $\mathcal{Q} = \langle I, J, K \rangle$ be the corresponding split-quaternionic structure on H as in 2.5. Let S_x be the element of $\otimes^3 H_x^* \otimes H_x$ constructed from the harmonic curvature of the associated path geometry of chains as above.*

(1) *Then, up to a non-zero multiple, S is the cyclic sum of the mapping*

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}(\xi, I\eta)I\zeta + \mathcal{L}(\xi, J\eta)J\zeta - \mathcal{L}(\xi, K\eta)K\zeta,$$

which is independent of the choice of basis of $\mathcal{Q}$.

(2) A non-zero element ξ in $H_x = L_x^* \otimes R_x$ is of rank one if and only if $S(\xi, \xi, \xi) = 0$ or there is $\eta \in H_x$ such that $S(\xi, \xi, \eta)$ is a non-zero multiple ξ .

Although the Levi bracket $\mathcal{L}$ has values in TM/H , we view it here as a real valued bilinear form, because of the non-zero multiple freedom.

Proof. (1) Since $\tilde{\kappa}_H$ is given by Ψ_α , we just need to compare the map above with the one described in lemma 3.4. On the level of Lie algebra $\mathfrak{g}$, the Levi bracket $\mathcal{L}$ corresponds to $[\cdot, \cdot] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$, which is explicitly described by (1) in 2.2. The operators I, J , and K correspond to the multiplication by matrices (4) on $\mathfrak{g}_{-1} = \mathbb{R}^{2*} \otimes \mathbb{R}^n$ from the right. Representing $\xi \in H_x$ by a matrix (X_1, X_2) with columns $X_1, X_2 \in \mathbb{R}^n$, the images $I\xi, J\xi$ and $K\xi$ corresponds to $(X_1, -X_2), (X_2, X_1)$, and $(-X_2, X_1)$, respectively. Representing also η and ζ by (Y_1, Y_2) and (Z_1, Z_2) , the expressions $\mathcal{L}(\xi, I\eta), \mathcal{L}(\xi, J\eta)$, and $\mathcal{L}(\xi, K\eta)$, correspond then to $-\langle X_1, Y_2 \rangle - \langle X_2, Y_1 \rangle, \langle X_1, Y_1 \rangle - \langle X_2, Y_2 \rangle$, and $\langle X_1, Y_1 \rangle + \langle X_2, Y_2 \rangle$, respectively. Hence the direct substitution yields that the two mappings correspond each other up to a non-zero multiple. The independence of the choice of basis of $\mathcal{Q}$ follows by a straightforward checking.

(2) In the above terms, ξ is of rank one if and only if the corresponding vectors X_1 and X_2 are linearly dependent. Using this it is then easy to check that $S(\xi, \xi, \xi) = 0$ and, moreover, that $S(\xi, \xi, \eta)$ equals to a multiple of ξ , for any η .

For the converse statement, let us distinguish the two cases: Firstly, suppose $\mathcal{L}(\xi, I\xi), \mathcal{L}(\xi, J\xi)$, and $\mathcal{L}(\xi, K\xi)$ does not vanish simultaneously. Since the assumption $S(\xi, \xi, \xi) = 0$ is equivalent to $\mathcal{L}(\xi, I\xi)I\xi + \mathcal{L}(\xi, J\xi)J\xi - \mathcal{L}(\xi, K\xi)K\xi = 0$, it follows that if two of the summands vanish then ξ will be 0. Hence if $\xi \neq 0$ and $S(\xi, \xi, \xi) = 0$ then at most one of the three summands vanishes and this yields that $\xi = A\xi$ for a specific element $A \in \langle I, J, K \rangle$. Hence ξ has got rank one by lemma 2.5.

Secondly, suppose that $\mathcal{L}(\xi, I\xi) = \mathcal{L}(\xi, J\xi) = \mathcal{L}(\xi, K\xi) = 0$. Then $S(\xi, \xi, \xi)$ vanishes trivially, however, it also turns out that $S(\xi, \xi, \eta)$ is a non-zero multiple of $\mathcal{L}(\xi, I\eta)I\xi + \mathcal{L}(\xi, J\eta)J\xi - \mathcal{L}(\xi, K\eta)K\xi$, for any η . Hence the assumption there is η so that $S(\xi, \xi, \eta) = \xi$ yields that there is $A \in \langle I, J, K \rangle$ so that $A\xi = \xi$. The rest follows again by lemma 2.5. $\square$

According to the development in 2.4, to recover the Lie contact structure on M it is enough to determine the rank-one elements in H_x . The above lemma provides this in terms of S_x , hence we conclude by the following interesting result:

Theorem. *Let M be a locally flat Lie contact manifold. Then the Lie contact structure can be reconstructed from the harmonic curvature of the regular normal parabolic geometry associated to the path geometry of chains. Consequently, a contact diffeomorphism on M which maps chains to chains is an automorphism of the Lie contact structure.*

Proof. A contact diffeomorphism f on M lifts to a diffeomorphism $\tilde{f}$ on $\tilde{M} = \mathcal{P}_0 TM$. The assumption that f preserves chains yields that $\tilde{f}$ is an automorphism of the associated path geometry of chains. In particular, $\tilde{f}$ is compatible with the harmonic curvature $\tilde{\kappa}_H$, which by assumption corresponds to the mapping S above. Hence f is compatible with S and the rest follows. $\square$

4.3. Final remarks. As we noted in remark 2.3, the Lie contact structures in dimension 5 and 3 are basically equivalent to other parabolic contact structures. The procedure of previous sections is in effect independent of the dimension, however, the difference in these cases is that there are harmonic curvature components with higher homogeneity. Hence the torsion freeness condition does not imply the

local flatness as in 3.5 and the results are more general. Consult the corresponding sections in [4] and [5] for details.

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