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Contact projective structures and chains

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Abstract Contact projective structures have been thoroughly studied by D. Fox. He associated to a contact projective structure a canonical projective structure on the same manifold. We interpret Fox's construction in terms of the equivalent parabolic (Cartan) geometries, showing that it is an analog of Fefferman's construction of a conformal structure associated to a CR structure. We show that, on the level of Cartan connections, this Fefferman-type construction is compatible with normality if and only if the initial structure has vanishing contact torsion. This leads to a geometric description of the paths that have to be added to the contact geodesics of a contact projective structure in order to obtain the subordinate projective structure. They are exactly the chains associated to the contact projective structure, which are analogs of the Chern–Moser chains in CR geometry. Finally, we analyze the consequences for the geometry of chains and prove that a chain–preserving contactomorphism must be a morphism of contact projective structures.

Keywords Projective structure · Contact projective structure · Path geometry · Fefferman construction · Chains · Cartan connection · Parabolic geometry

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1 Introduction

Classical projective structures can be viewed as describing the geometry of geodesics of affine connections, viewed as unparametrized curves (paths). The study of these structures was a very active part of differential geometry in the first decades of the 20th century. After some time of less activity, the interest in these geometries has been revived during the last years. Much of this recent interest is related to the fact that they form a simple instance of the large class of so-called parabolic geometries.

Among the parabolic geometries there is also a contact analog of classical projective structures, called contact projective structures. Such a structure is given by a contact structure and a family of paths in directions tangent to the contact distribution, which can be realized as geodesics of some affine connection. While basic ideas on these structures can be traced back to the classical era, they have been formally introduced and thoroughly studied by Fox in [10]. One of the main results in that article is that any contact projective structure can be canonically extended to a projective structure on the same manifold.

Studying Fox's canonical projective structure is the main purpose of this article. We first review some fundamental facts on projective and contact projective structures in Sect. 2. In Sect. 3, we give a geometric description of the paths in directions transverse to the contact distribution that have to be added to the given paths in contact directions in order to obtain the canonical projective structure. To describe these curves, recall that for CR manifolds of hypersurface type, there are the so-called Chern–Moser chains introduced in [9]. They form a family of canonical unparametrized curves available in all directions transverse to the contact distribution. The description of Chern–Moser chains via the canonical Cartan connection associated to a CR structure easily generalizes to all parabolic contact structures, see [7]. In this way, one obtains a family of chains associated to any contact projective structure, and these are the curves to be added in order to get the canonical projective structure, see Corollary 3.3.

This description is obtained via another result, which is of independent interest. In [10], the canonical projective structure was obtained via the so-called ambient descriptions or cone descriptions of contact projective and projective structures. We give a description in terms of the canonical Cartan connections, which shows that it is an analog of the Fefferman construction as described in [2], see Proposition 3.3. We also show that this Fefferman type construction produces not only the canonical projective structure but also its canonical Cartan connection if and only if the initial contact projective structure has vanishing contact torsion, see Theorem 3.2.

The fact that the chains of a contact projective structure can be realized as geodesics of an affine connection is in sharp contrast to the cases of CR structures and Lagrangean contact structures, see [8]. In the latter article, we have studied chains via the path geometry they determine. In Sect. 4 we discuss these issues in the contact projective case, where they are rather easy. In spite of the fact that the contact projective structure cannot be recovered from the path geometry of chains, we are able to prove that contactomorphisms which preserve chains actually are morphisms of contact projective structures, see Theorem 4.3.

2 Projective and contact projective structures

2.1 Projective structures

A *projective structure* on a smooth manifold M is given by a class of projectively equivalent linear connections $[\nabla]$ on TM. Two connections are called *projectively equivalent* if their difference tensor is of the form $A(\xi,\eta) = \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$ for some one-form $\Upsilon \in \Omega^1(M)$. Note that such a tensor is symmetric, so by definition, projectively equivalent connections have the same torsion. It is a well known classical result that two connections which have the same torsion are projectively equivalent if and only if they have the same geodesics up to parametrization.

This means that a projective structure on M is given by a class of linear connections on TM which have the same torsion and the same unparametrized geodesics. Since symmetrizing a connection does not change its geodesics, it is usually assumed the connections in the class are torsion-free, which is a natural normalization of the structure.

We can also interpret this description as saying that a projective structure on M is given by the smooth family of paths (unparametrized curves) formed by the geodesics. This is an example of a path geometry, i.e. a smooth family of paths with exactly one path through each point in each direction, see Sect. 4.1 for the precise definition. We will return to this point of view there.

The model projective structure is given by the real projective space $\mathbb{R}P^m = \mathcal{P}\mathbb{R}^{m+1}$ with the class of connections induced from the canonical flat connection on $\mathbb{R}^m$. The geodesics of these connections are the projective lines. The group of diffeomorphisms of $\mathbb{R}P^m$ which preserve this structure is $PGL(m+1,\mathbb{R})$, the quotient of $GL(m+1,\mathbb{R})$ by its center. For our purposes it is better to work with oriented projective structures. This means replacing $\mathbb{R}P^m$ by the sphere S^m, viewed as the space of rays in $\mathbb{R}^{m+1}$. Then the appropriate group is $\tilde{G} := SL(m+1,\mathbb{R})$, and the distinguished paths are the great circles on S^m. If m is even, then $\tilde{G}$ is isomorphic to $PGL(m+1,\mathbb{R})$, while for odd m it is a two-fold covering. In any case, $\tilde{G}$ acts transitively both on $\mathbb{R}^{m+1}\setminus\{0\}$ and on S^m. Let $\tilde{P} \subset \tilde{G}$ be the stabilizer of the ray generated by the first vector of the standard basis of $\mathbb{R}^{m+1}$ and let $\tilde{Q} \subset \tilde{P}$ be the stabilizer of the vector itself, so $S^m \cong \tilde{G}/\tilde{P}$ and $\mathbb{R}^{m+1}\setminus\{0\} \cong \tilde{G}/\tilde{Q}$. In terms of matrices, $\tilde{P}$ is represented by block matrices

$$\tilde{P} = \left\{ \begin{pmatrix} \det(A)^{-1} & Z \\ 0 & A \end{pmatrix} : A \in GL^+(m,\mathbb{R}),\, Z \in \mathbb{R}^{m*} \right\},$$

where $GL^+(m,\mathbb{R}) = \{A \in GL(m,\mathbb{R}) : \det(A) > 0\}$. The subgroup $\tilde{Q} \subset \tilde{P}$ is given by those matrices in $\tilde{P}$ for which $A \in SL(m,\mathbb{R})$. $\tilde{P}$ is a parabolic subgroup of the simple Lie group $\tilde{G}$ and the corresponding grading of the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{sl}(m+1,\mathbb{R})$ is given by the block decomposition

$$\begin{pmatrix} \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1 \\ \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_0 \end{pmatrix}$$

with blocks of sizes 1 and m. Hence $\tilde{\mathfrak{g}}_{-1} \cong \mathbb{R}^m$, $\tilde{\mathfrak{g}}_0 \cong \mathfrak{gl}(m,\mathbb{R})$, and $\tilde{\mathfrak{g}}_1 \cong \mathbb{R}^{m*}$. The Lie algebras of $\tilde{P}$ and $\tilde{Q}$ are $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ and $\tilde{\mathfrak{q}} = \tilde{\mathfrak{g}}_0^{ss} \oplus \tilde{\mathfrak{g}}_1$, respectively. Here $\tilde{\mathfrak{g}}_0^{ss}$ denotes the semisimple part of $\tilde{\mathfrak{g}}_0$, which is isomorphic to $\mathfrak{sl}(m,\mathbb{R})$.

General oriented projective structures admit an equivalent description as Cartan geometries of type $(\tilde{G},\tilde{P})$. Projecting to the lower right block defines a homomorphism from $\tilde{P}$ onto $GL^+(m,\mathbb{R})$, so we can view the latter group as a quotient of $\tilde{P}$. Then with notation as above, the following holds, see e.g. [3]:

Theorem 2.1 *Let M be a smooth manifold of dimension ≥ 2 which is endowed with an oriented projective structure. Then the oriented linear frame bundle of M can be canonically extended to a principal $\tilde{P}$–bundle $\tilde{\mathcal{G}} \to M$, which can be endowed with a Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}},\tilde{\mathfrak{g}})$. The pair $(\tilde{\mathcal{G}},\tilde{\omega})$ is uniquely determined up to isomorphism if one in addition requires $\tilde{\omega}$ to be normal in the sense recalled in Sect. 3.2 below.*

The normalization condition on the Cartan connection in the theorem is the usual condition for parabolic geometries, namely that the curvature has values in the kernel of the Kostant codifferential.

The relation between the Cartan geometry $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ and the projective structure can be described in several ways. On the one hand, the quotient $\tilde{\mathcal{G}}_0 \to M$ of $\tilde{\mathcal{G}}$ by the connected subgroup with Lie algebra $\mathfrak{g}_1$ can be naturally identified with the oriented linear frame bundle of M. One can then take the component $\tilde{\omega}_0$ of $\tilde{\omega}$ in $\mathfrak{g}_0$ and pull it back via smooth sections $\sigma : \tilde{\mathcal{G}}_0 \to \tilde{G}$ satisfying an equivariancy condition. This leads to principal connections on $\tilde{\mathcal{G}}_0$ which exactly correspond to the connections in the projective class.

More easily, the unparametrized geodesics of the projective structure are given by projections to M of flow lines of the constant vector fields $\tilde{\omega}^{-1}(X) \in \mathfrak{X}(\tilde{\mathcal{G}})$ with $X \in \tilde{\mathfrak{g}}_{-1}$.

2.2 Contact projective structures

Recall that a contact manifold (M, H) is a smooth manifold M of odd dimension $2n + 1$ together with a smooth subbundle $H \subset TM$ of corank one which is maximally non-integrable. This means that the bilinear bundle map $\mathcal{L} : H \times H \to TM/H$ induced by the Lie bracket of vector fields, which is called the *Levi–bracket*, is non-degenerate. The contact analog of projective structures was formally introduced and thoroughly studied in [10]. Similarly to classical projective structures this contact analog can be described in several equivalent ways.

The simplest description is via the analog of path geometries, for which one only considers paths which are everywhere tangent to the contact distribution H. Then a *contact projective structure* can be defined as a smooth family of such contact paths with one path through each point in each direction in H, which are among the geodesics of some linear connection on TM. This is the definition used in [10]. While this implicitly also provides a definition as an equivalence class of linear connections on TM, more work is needed to obtain a nice description of this type.

Indeed, Theorem A of [10] provides a special class of such connections. One starts with a contact form $\theta \in \Omega^1(M)$, i.e. a one-form whose kernel in each point $x \in M$ is the contact distribution H_x. We will always assume that the contact structure in question admits global contact forms. This amounts to the fact that the line bundle TM/H (or equivalently its dual) admit global nonzero sections. Equivalently, these line bundles have to be orientable, and we further assume that an orientation has been chosen. Then we can talk about positive contact forms, and given one such form θ, any other is obtained by multiplication by a positive smooth function. Given a positive contact form θ, Theorem A of [10] shows the existence of a linear connection ∇ on TM which has the given paths among its geodesics, satisfies $\nabla\theta = 0$ and $\nabla d\theta = 0$ as well as normalization conditions on its torsion.

Since $\nabla\theta = 0$, the connection ∇ preserves the subbundle $H \subset TM$, and of course the geodesics in contact directions depend only on the restriction of ∇ to a linear connection on the vector bundle $H \to M$. Further, it turns out that all of ∇ is determined by the restriction of ∇ to an operator $\Gamma(H) \times \Gamma(H) \to \Gamma(H)$, a so-called *partial connection*. Finally, viewing the Levi–bracket $\mathcal{L}$ as a bundle map $\Lambda^2 H \to TM/H$, its kernel defines a corank one subbundle $\Lambda^2_0 H \subset \Lambda^2 H$. Since $\nabla d\theta = 0$, the linear connection on $\Lambda^2 H$ induced by ∇ preserves this subbundle, so ∇ (respectively, its restriction) is a (partial) *contact connection*. Now parallel to the projective case, one can define *contact projective equivalence* of partial contact connections, and characterize this in terms of the difference tensor. This leads to a formula in terms of a smooth section H^* which is similar to the one for projectively equivalent connections, see formula (2.8) of [10].

There is a significant difference to the case of projective structures, which concerns torsion. For linear connections on the tangent bundle, one can always remove the torsion without changing the geodesics. This is no more true in the contact setting. By Theorem 2.1 of [10], the restrictions of the torsions of all the representative connections ∇ associated to contact forms as above to $\Lambda^2 H^* \otimes H \subset \Lambda^2 T^*M \otimes TM$ coincide. This is called the *contact torsion* of the contact projective structure. (In the setting of partial contact connections, one has to further restrict to $(\Lambda^2_0 H)^* \otimes H$, but this needs only minor adaptions.)

The model contact projective structure is given by the space of rays in a symplectic vector space. Consider $\mathbb{R}^{2n+2}$ with the standard linear symplectic form Ω. Then Ω induces a contact structure on the space S^{2n+1} of rays, and the great circles tangent to the contact subbundle (which can locally be realized as geodesics for the standard flat connection on $\mathbb{R}^{2n+1}$) define a contact projective structure. One of the main results of [10] is the construction of a canonical projective structure from a contact projective structure. For the homogeneous model, it is obvious how to do this: One simply adds that great circles which are transverse to the contact distribution, to obtain the homogeneous model of oriented projective structures.

The contact projective structure on S^{2n+1} constructed above is evidently homogeneous under the symplectic group $G := Sp(2n+2, \mathbb{R})$. It is easy to see that the actions of elements of G are exactly those diffeomorphisms of S^{2n+1} which preserve both the contact structure and the projective structure. Generalizing this result to the curved case will be the main aim of Sect. 4. Now G acts transitively both on $\mathbb{R}^{2n+2} \setminus \{0\}$ and on S^{2n+1}, so as homogeneous spaces $S^{2n+1} \cong G/P$ and $\mathbb{R}^{2n+2} \setminus \{0\} \cong G/Q$, where P is the stabilizer of the ray generated by the first vector of the standard basis of $\mathbb{R}^{2n+2}$, and Q is the stabilizer of that vector. For the obvious inclusion $G \to \tilde{G}$ (with $m = 2n+1$) we get $P = G \cap \tilde{P}$ and $Q = G \cap \tilde{Q}$. As in 2.1, P is a parabolic subgroup in the simple Lie group G. To obtain a nice presentation of the Lie algebra $\mathfrak{g}$ of G, it is best to choose Ω to be represented by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{J} & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

with $\mathbb{J} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$ and $\mathbb{I}_n$ denoting identity matrix of rank n. Using this form, the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n+2, \mathbb{R})$ has the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & U & w \\ X & A & \mathbb{J}U^t \\ z & -X^t\mathbb{J} & -a \end{pmatrix} \right\},$$

with blocks of sizes 1, $2n$ and 1, $a, z, w \in \mathbb{R}$, $X \in \mathbb{R}^{2n}$, $U \in \mathbb{R}^{2n*}$, and $A \in \mathfrak{sp}(2n, \mathbb{R})$ (with respect to $\mathbb{J}$). We obtain a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_{-2}$ corresponding to z, $\mathfrak{g}_{-1}$ to X, $\mathfrak{g}_0$ to a and A, and so on. By construction, $\mathfrak{p}$ is formed by the matrices which are block upper triangular, i.e. satisfy $z = 0$ and $X = 0$, so $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The subalgebra $\mathfrak{q} \subset \mathfrak{p}$ corresponds to those matrices, which in addition satisfy $a = 0$. For the algebra $\tilde{\mathfrak{g}}$ from 2.1, we simply obtain all matrices of the same size, and the comparison with the description in 2.1 shows the various grading components and subalgebras.

In Theorem C of [10], the author proves existence of a canonical Cartan connection associated to a contact projective structure, which reads as follows:

Theorem 2.2 *Let (M, H) be a contact manifold which admits a global contact form and is endowed with a contact projective structure. Then there exists a principal P–bundle $p : \mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that $H = Tp(\omega^{-1}(\mathfrak{g}_{-1} \oplus \mathfrak{p}))$ and* *the contact geodesics are projections to M of flow lines of constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$ with $X \in \mathfrak{g}_{-1}$. The pair $(\mathcal{G}, \omega)$ is uniquely determined up to isomorphism provided that one in addition requires ω to be normal in the sense discussed in* Sect. 3.2 *below.*

Remark 2.2 (1) The normalization condition in the theorem is a generalization of the uniform normalization condition for parabolic geometries. As we shall discuss in more detail in Sect. 3.2 below, it reduces to the standard condition if and only if the projective contact structure has vanishing contact torsion, see Proposition 4.1 of [10]. In this special case, the theorem follows from general results on parabolic geometries, see [6].

(2) The description of the relation between the Cartan geometry and the underlying contact projective structure in the theorem is different from the original one in [10]. The characterization in [10] uses the ambient connection to be discussed in Sect. 2.3 below. Section 4.3 of [10] discusses the characterization of contact geodesics via the development of curves (induced by the Cartan connection ω) into the homogeneous model $G/P = S^{2n+1}$. Contact geodesics on M are exactly the curves which develop to the contact geodesics in the model. In [7] it is shown how the description in terms of development is equivalent to being a projection of an integral curve of a certain type of constant vector fields (of the Cartan connection ω). Then it suffices to observe that the contact geodesics in the model G/P are precisely the orbits of one-parameter subgroups of G generated by elements of $\mathfrak{g}_{-1}$.

(3) There is another distinguished family of curves in the model space. As in (2), they may be either characterized via development or as projections of integral curves of constant vector fields, but this time with generator in $\mathfrak{g}_{-2}$. In view of the similarity to the concept in CR geometry introduced by Chern–Moser, these are called *chains*. In particular, a chain is uniquely determined by its initial direction as an unparametrized curve. For the homogeneous model S^{2n+1}, the chains are exactly those great circles which are transverse to the contact distribution.

2.3 Ambient descriptions

The basis of the construction of a projective structure subordinate to a contact projective structure is the so-called ambient description or cone description of projective and contact projective structures. In the projective case, this goes back to the work of Tracy Thomas in the 1930s, in the contact projective case it is due to Fox. In [10] the ambient connection is constructed first (in Theorem B) and then used to construct a Cartan connection. Here we take the opposite point of view, and use the Cartan connection to construct the ambient connection.

The starting point for the ambient description of both types of structure is a principal bundle $\mathcal{L} \to M$ with structure group $\mathbb{R}_+$, namely the frame bundle of the bundle of $(\frac{-1}{m+1})$–densities. In the contact projective case, it is easy to see that one may also view this density bundle as a square root of the bundle of positive contact forms. In the (oriented) projective case, $\mathcal{L}$ can be constructed from the Cartan bundle $\tilde{\mathcal{G}} \to M$ via a homomorphism $\tilde{P} \to \mathbb{R}_+$ with kernel $\tilde{Q} \subset \tilde{P}$. Hence $\mathcal{L} \cong \tilde{\mathcal{G}}/\tilde{Q}$, and $\tilde{\mathcal{G}} \to \mathcal{L}$ is a principal bundle with structure group $\tilde{Q}$, on which $\tilde{\omega}$ is a Cartan connection. In particular, $T\mathcal{L} \cong \tilde{\mathcal{G}} \times_{\tilde{Q}} (\tilde{\mathfrak{g}}/\tilde{\mathfrak{q}})$ with the action of $\tilde{Q}$ coming from the adjoint representation. In the contact projective case, there is a completely analogous description in terms of the canonical Cartan geometry $(\mathcal{G} \to M, \omega)$ induced by the contact projective structure. In particular, $T\mathcal{L} \cong \mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$ in the contact projective case.

Proposition 2.3 *Consider $\tilde{G} := SL(m+1, \mathbb{R})$ and let $\tilde{Q} \subset \tilde{G}$ be the stabilizer of the first vector in the standard basis of $\mathbb{R}^{m+1}$. Then, as a representation of $\tilde{Q}$, $\tilde{\mathfrak{g}}/\tilde{\mathfrak{q}}$ is isomorphic to the restriction to $\tilde{Q}$ of the standard representation $\mathbb{R}^{m+1}$ of $\tilde{G}$.*

If m is odd, say $m = 2n+1$, then the analogous statement holds for $G = Sp(2n+2, \mathbb{R})$ and the stabilizer $Q \subset G$ of the first basis vector.

Proof The Lie subalgebra $\tilde{\mathfrak{q}} \subset \tilde{\mathfrak{g}}$ consists of all matrices for which all entries in the first column are zero. To describe the $\tilde{Q}$-representation $\tilde{\mathfrak{g}}/\tilde{\mathfrak{q}}$, we may thus simply look at the action of the adjoint representation of $\tilde{Q}$ to the first column of matrices. Since $\tilde{G}$ is a matrix group, the adjoint representation is given by conjugation. By definition, the first column of any matrix in $\tilde{Q}$ equals the first unit vector, so multiplying any matrix from the right by an element of $\tilde{Q}$ leaves the first column unchanged. But this implies that for $A \in \tilde{Q}$ and $X \in \tilde{\mathfrak{g}}$, the first column of AXA^{-1} equals the first column of AX, which implies the claim for $\tilde{Q}$. But then the same statement is true for any subgroup of $\tilde{Q}$, hence in particular for $Q = \tilde{Q} \cap G$ in the case of odd m. □

Using this, we may view the bundle $T\mathcal{L} \to \mathcal{L}$ as the associated bundle $\tilde{\mathcal{G}} \times_{\tilde{Q}} \mathbb{R}^{m+1}$, respectively, $\mathcal{G} \times_Q \mathbb{R}^{2n+2}$, and since the inducing representations are restrictions of representations of $\tilde{G}$, respectively G, we can invoke the general construction of [4]. This shows that the Cartan connection $\tilde{\omega}$, respectively, ω induces a linear connection on $T\mathcal{L}$.

Theorem 2.3 (1) *The linear connection on $T\mathcal{L}$ induced by a projective structure on M as described above coincides with the ambient connection from Theorem 3.1 of* [10].

(2) *The linear connection on $T\mathcal{L}$ induced by a contact projective structure on M as described above coincides with the ambient connection from Theorem B of* [10].

Proof There are various ways to prove this, which all boil down to rather straightforward verifications. On the one hand, one may simply verify that the linear connections we have constructed satisfy the properties listed in the theorems of [10], and then invoke the uniqueness parts of these theorems.

Even easier, one may follow the construction of the Cartan connection in [10] backwards. First, one lifts the principal $\mathbb{R}_+$–action on $\mathcal{L}$ to a free right action by vector bundle homomorphisms on $T\mathcal{L}$ in such a way that the orbit space $T\mathcal{L}/\mathbb{R}_+$ (which evidently is a vector bundle over $\mathcal{L}/\mathbb{R}_+ = M$) can be identified with the standard tractor bundle of the structure in question. Then one shows that the ambient connection induces a tractor connection on that bundle, which by the general methods of [4] gives rise to a Cartan connection. This corresponds to the fact that $\tilde{P}/\tilde{Q} \cong \mathbb{R}_+$ acts on $\tilde{\mathcal{G}}/\tilde{Q} = \mathcal{L}$ with orbit space $\tilde{\mathcal{G}}/\tilde{P} = M$ (and the analogous statement for P/Q). Now one immediately verifies that the lift of the action is exactly defined in such a way that the linear connection on $T\mathcal{L}$ induces the usual tractor connection on the tractor bundles, which completes the proof. □

3 The subordinate projective structure

Having collected the background, we can now move to proving the first main results of this article. We show that the construction of a projective structure subordinate to a contact projective structure in [10] can be interpreted as a generalized Fefferman construction. This interpretation leads to immediate payoff, since it implies a geometric description of the subordinate projective structure in terms of chains.

3.1 The Fefferman–type construction

The scheme for generalized Fefferman constructions is by now fairly familiar, see [2], where also the application to contact projective structures was suggested. As before, consider $G = Sp(2n+2,\mathbb{R})$ and $\tilde{G} := SL(2n+2,\mathbb{R})$, let $\psi : G \hookrightarrow \tilde{G}$ be the obvious inclusion. Then put $i := \psi|_P : P \to \tilde{P}$ and $\alpha := \psi' : \mathfrak{g} \to \tilde{\mathfrak{g}}$. Now suppose that we have given a contact manifold (M, H) of dimension $m = 2n+1$, which is endowed with a contact projective structure, and let $(\mathcal{G} \to M, \omega)$ be the canonical Cartan geometry of type (G, P) determined by this structure as in 2.2. Then the homomorphism $i : P \to \tilde{P}$ defines a left action of P on $\tilde{P}$, so we can form the associated bundle $\tilde{\mathcal{G}} := \mathcal{G} \times_P \tilde{P} \to M$. This clearly is a principal bundle with structure group $\tilde{P}$, and we have a natural map $j : \mathcal{G} \to \tilde{\mathcal{G}}$ induced by mapping $u \in \mathcal{G}$ to the class of (u, e). It is easy to prove (compare with 3.1 and 3.2 of [8]) that there is a unique Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ such that $j^*\tilde{\omega} = \alpha \circ \omega$.

This construction actually defines a functor from Cartan geometries of type (G, P) to Cartan geometries of type $(\tilde{G}, \tilde{P})$, both defined on the same manifold. (The fact that we obtain a geometry on the same manifold is due to the fact that $\tilde{P} \cap G$ is already a parabolic subgroup of G. For other Fefferman–type constructions, this is not the case. Then one has to pass to a parabolic subgroup containing this intersection, and the new geometry will be defined on the total space of a natural bundle.) Since any Cartan geometry of type $(\tilde{G}, \tilde{P})$ on a manifold M gives rise to an underlying projective structure, we obtain a functor mapping contact projective structures to projective structures on the same manifold. It is not clear, however, whether the Cartan connection $\tilde{\omega}$ is normal and hence coincides with the canonical Cartan connection associated to the projective structure in general. This is a familiar phenomenon for generalized Fefferman constructions.

Before we discuss the question of normality of $\tilde{\omega}$, we give a geometric description of the projective structure produced by the generalized Fefferman construction.

Proposition 3.1 *Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) and let $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ be the Cartan geometry of type $(\tilde{G}, \tilde{P})$ obtained by the generalized Fefferman construction.*

Then the paths of the projective structure determined by $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ are the projections of the flow lines of the constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$ generated by elements $X \in \mathfrak{g}_{-1} \cup \mathfrak{g}_{-2}$.

Proof It is well known that the paths of the projective structure can be obtained as the projections of the flow lines of the constant vector fields $\tilde{\omega}^{-1}(\tilde{X})$ for all elements $\tilde{X} \in \tilde{\mathfrak{g}}_{-1}$. Moreover, it is well known that there is exactly one such path through each point of M in each direction. As we have seen above, viewing $\mathfrak{g}$ as a subalgebra of $\tilde{\mathfrak{g}}$, the Cartan connection $\tilde{\omega}$ is characterized by $j^*\tilde{\omega} = \omega$. In particular, for $X \in \mathfrak{g} \subset \tilde{\mathfrak{g}}$, the constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$ and $\tilde{\omega}^{-1}(X) \in \mathfrak{X}(\tilde{\mathcal{G}})$ are j-related. Hence their flows are j-related and in particular have the same projection to M.

From the description of the Lie algebras $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ in 2.1 and 2.2, we first see that $\mathfrak{g}_{-2} \subset \tilde{\mathfrak{g}}_{-1}$. Hence for $X \in \mathfrak{g}_{-2}$, the projection of the flow line of $\omega^{-1}(X)$ is among the paths of the projective structure. The tangent directions of these paths exhaust all directions which are transverse to the contact distribution.

On the other hand, $\mathfrak{g}_{-1} \subset \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0$ (with a nontrivial component in $\tilde{\mathfrak{g}}_0$ for any nonzero element of $\mathfrak{g}_{-1}$). For $X \in \mathfrak{g}_{-1}$ let $\tilde{X}$ be the $\tilde{\mathfrak{g}}_{-1}$-component of X (i.e. the matrix with the same first column as X and all other columns zero), and put $\tilde{A} = X - \tilde{X} \in \tilde{\mathfrak{g}}_0$. From the explicit presentations of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ one immediately verifies that $[\tilde{A}, \tilde{X}] = 0$, and hence $\mathrm{Ad}(\exp(-t\tilde{A}))(\tilde{X}) = \tilde{X}$ for all t. Now for $u \in \mathcal{G}$, let $\tilde{c}(t)$ be the flow line of the constant vector field $\tilde{\omega}^{-1}(\tilde{X})$. Then the curve $c(t) := \tilde{c}(t) \cdot \exp(t\tilde{A})$ has the same projection to M

as $\tilde{c}(t)$, so this projection is among the paths determined by the projective structure. But denoting by r the principal right action and by $\zeta_{\tilde{A}}$ the fundamental vector field generated by $\tilde{A}$, one computes that

$$c'(t) = Tr^{\exp(-t\tilde{A})} \cdot \tilde{c}(t) + \zeta_{\tilde{A}}(c(t)),$$

and so $\tilde{\omega}(c'(t)) = \mathrm{Ad}(\exp(-t\tilde{A}))(\tilde{X}) + \tilde{A} = X$ for all t. This shows that the flow line of $\omega^{-1}(X)$ is also among the paths of the induced projective structure. Since the tangents of such paths exhaust all directions in the contact distribution, this completes the proof. □

Remark 3.1 (1) A nice alternative argument for the last part of the proof is as follows: Since $[\tilde{A}, \tilde{X}] = 0$, we get $\exp(tX) = \exp(t\tilde{X})\exp(t\tilde{A})$, and hence the exponential curves generated by X and $\tilde{X}$ have the same projection to $\tilde{G}/\tilde{P}$. Via development, this implies the same result for the flow lines of the constant vector fields.

(2) A regular Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) as in the proposition gives rise to a contact projective structure on M. The distinguished paths (in contact directions) of this structure are the flow lines of the vector fields $\omega^{-1}(X)$ for $X \in \mathfrak{g}_{-1}$. The proposition in particular says, that these are among the paths of the projective structure obtained via the generalized Fefferman construction. Hence the projective structure obtained from the generalized Fefferman construction is *subordinate* to the initial contact projective structures in the sense of Definition 3.1 of [10].

3.2 Normality

As mentioned above, there are few general results on the compatibility of Fefferman type constructions with normality of Cartan connections, except for the fact that the result of a generalized Fefferman construction is locally flat if and only if the original geometry is locally flat. To discuss normality in our case, let us first recall the normalization condition used for parabolic geometries. Consider a semisimple Lie algebra $\mathfrak{g}$ with a parabolic subalgebra $\mathfrak{p}$ and the corresponding grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ (with $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$). Then the Killing form induces a duality between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$, which is equivariant for the natural action of any parabolic subgroup $P \subset G$ with Lie algebra $\mathfrak{p}$. Now there is a standard complex for computing the Lie algebra homology of $\mathfrak{p}_+$ with coefficients in $\mathfrak{g}$. The differential in this complex is often denoted by ∂^* and referred to as the *Kostant codifferential* since it can also be obtained by dualizing a Lie algebra cohomology differential. For the normalization condition, we need the map

$$\partial^* : \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} \to \mathfrak{p}_+ \otimes \mathfrak{g},$$

which on decomposable elements is given by

$$\partial^*(Z \wedge W \otimes A) = -W \otimes [Z, A] + Z \otimes [W, A] - [Z, W] \otimes A.$$

Now the curvature of a Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) can be described by the curvature function $\kappa : \mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$, which is characterized by

$$\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) = d\omega(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) + [X, Y].$$

As noted above, the target space of κ can be identified with $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$, and the geometry is called *normal* if $\partial^* \circ \kappa = 0$.

This is the normalization condition used for projective structures in Theorem 2.1. For contact projective structures, this normalization condition is not general enough, however.

The reason for this can be seen from one of the nice properties of the normalization condition given by the Kostant codifferential. Namely, there is an operator $\Box : \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} \to \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ called the *Kostant Laplacian*. This is not equivariant for the action of the parabolic subgroup P but only for its Levi component, a subgroup $G_0 \subset P$ with Lie algebra $\mathfrak{g}_0$. Now due to the gradings on $\mathfrak{p}_+$ and $\mathfrak{g}$, the space $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ is naturally graded, so one may split the curvature function κ into homogeneous components with respect to these gradings. One has to assume throughout that the geometry is regular, so all homogeneous components of degree less or equal to zero vanish identically. If this is the case, then it turns out that the lowest nonzero homogeneous component of κ always has values in $\ker(\Box)$. This is extremely useful, since $\ker(\Box)$ can be computed explicitly as a G_0–representation (which is the main step towards the proof of Kostant's version of the Bott–Borel–Weil theorem in [11]).

For the parabolic pairs $(\mathfrak{g}, \mathfrak{p})$ and $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}})$ considered in Sect. 2, the description of $\ker(\Box)$ is particularly easy. In each case, this is an irreducible representation of G_0, contained in one homogeneity. The result is listed in the tables below, and $\ker(\Box)$ is always the component of highest weight in the indicated subrepresentation.

$(\mathfrak{g}, \mathfrak{p}), n = 1$

homog.	contained in
3	$\mathfrak{g}_1 \wedge \mathfrak{g}_2 \otimes \mathfrak{g}_0$

$(\mathfrak{g}, \mathfrak{p}), n > 1$

homog.	contained in
2	$\mathfrak{g}_1 \wedge \mathfrak{g}_1 \otimes \mathfrak{g}_0$

$(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}}), m = 2$

homog.	contained in
3	$\tilde{\mathfrak{g}}_1 \wedge \tilde{\mathfrak{g}}_1 \otimes \tilde{\mathfrak{g}}_1$

$(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}}), m > 2$

homog.	contained in
2	$\tilde{\mathfrak{g}}_1 \wedge \tilde{\mathfrak{g}}_1 \otimes \tilde{\mathfrak{g}}_0$

In particular, we see that in all cases the maps in $\ker(\Box)$ have values in $\mathfrak{p} \subset \mathfrak{g}$, respectively, in $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$. Since the same is evidently true for all maps of higher homogeneous degree, we see that in both cases the curvature function of a regular normal parabolic geometry always has values in $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{p}$, i.e. such geometries are always torsion free. Now it is easy to see that a Cartan geometry of type (G, P) is torsion free if and only if the induced contact projective structure has vanishing contact torsion.

Therefore, to deal with contact projective structures with non-vanishing contact torsion, one has to generalize the normalization condition, and this has been done in [10]. In Definition 4.1 of that article, the author explicitly describes a P–submodule $\mathcal{K} \subset \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, which consists of maps of positive homogeneity and contains $\ker(\partial^*)$. The normalization condition used in Theorem 2.2 then is that the curvature function has values in $\mathcal{K}$.

Having the necessary background at hand, we can now clarify compatibility of the generalized Fefferman construction with normality.

Theorem 3.2 *Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) whose curvature function has values in the P–mdoule $\mathcal{K}$ from above and let $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ be the result of the generalized Fefferman construction from 3.1. Then $\tilde{\omega}$ is normal if and only if ω is torsion free. Moreover, $\tilde{\omega}$ is locally flat if and only if ω is locally flat.*

Proof The description of the generalized Fefferman construction in 3.1 immediately leads to the relation between the curvatures of the two geometries. Let us denote by $\tilde{\kappa}$ and κ the curvature functions of $\tilde{\omega}$ and ω. Noting that $\alpha = \psi' : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is a homomorphism of Lie algebras, we obtain (compare with Proposition 3.3 of [8])

$$\tilde{\kappa}(j(u))(\tilde{X} + \tilde{\mathfrak{p}}, \tilde{Y} + \tilde{\mathfrak{p}}) = \alpha(\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p})), \quad (1)$$

for all $u \in \mathcal{G}$, $\tilde{X}, \tilde{Y} \in \tilde{\mathfrak{g}}$ and $X, Y \in \mathfrak{g}$ such that $\alpha(X) + \tilde{\mathfrak{p}} = \tilde{X} + \tilde{\mathfrak{p}}$ and likewise for Y and $\tilde{Y}$. Note that for given $\tilde{X}$, we can always find an element X with this property, since α induces a linear isomorphism $\mathfrak{g}/\mathfrak{p} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Note also that by equivariancy (1) uniquely determines $\tilde{\kappa}$. Since α is injective, we see that $\tilde{\kappa}$ vanishes identically if and only if κ vanishes identically, so the statement about local flatness follows readily.

Second, $\tilde{\omega}$ by definition is torsion free if and only if $\tilde{\kappa}$ has values in $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$, and since $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$ is $\tilde{P}$–invariant this is equivalent to the same statement for $\tilde{\kappa} \circ j$. Since $\alpha^{-1}(\tilde{\mathfrak{p}}) = \mathfrak{p}$ by construction, the latter statement via (1) is equivalent to κ having values in $\Lambda^2(\mathfrak{g}/\mathfrak{p}) \otimes \mathfrak{p}$ and hence to torsion freeness of ω. As we have seen above, normal Cartan connections of type $(\tilde{G}, \tilde{P})$ are always torsion free, so we see that normality of $\tilde{\omega}$ implies torsion freeness of ω.

Let us conversely assume that ω is torsion free and that its curvature function has values in $\mathcal{K}$. Then by Proposition 4.1 of [10] the curvature function $\tilde{\kappa}$ has values in $\ker(\partial^*)$, so we may apply general results for parabolic geometries. The isomorphism $\underline{\alpha} : \mathfrak{g}/\mathfrak{p} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ induced by α is equivariant over the inclusion $P \hookrightarrow \tilde{P}$, so the same is true for $\varphi := (\underline{\alpha}^{-1})^* : (\mathfrak{g}/\mathfrak{p})^* \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$. Hence also the map $\Phi := \Lambda^2\varphi \otimes \alpha$ is equivariant in the same sense, and in terms of this map we can write (1) as $\tilde{\kappa} \circ j = \Phi \circ \kappa$. Now let $\tilde{\partial}^*$ be the Kostant codifferential associated to $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}})$. Then equivariancy of Φ implies that $\Phi^{-1}(\ker(\tilde{\partial}^*)) \subset \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is a P–submodule. Clearly, normality of $\tilde{\omega}$ is equivalent to the fact that κ has values in this P–submodule. By Corollary 3.2 of [1], this is equivalent to the fact that the harmonic part κ_H of the curvature function has values in it.

As discussed above, $\kappa_H(u)$ has values in $\ker(\Box) \subset \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, which is an irreducible representation of G_0. Denoting by $w \in \ker(\Box)$ a highest weight vector in this representation, it is therefore, sufficient to prove that $\Phi(w) \in \ker(\tilde{\partial}^*)$. This can be verified by a simple direct computation. Alternatively, it is easy to verify that the G_0–representation $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ in which $\tilde{\partial}^*$ has values does not contain an irreducible component isomorphic to $\ker(\Box)$. □

Remark 3.2 The proof of this theorem is significantly easier then the proofs of normality for the classical Fefferman construction (see [5]) or other generalized Fefferman constructions. This is due to the fact that $G \cap \tilde{P} = P$ and hence $\mathfrak{g} \cap \tilde{\mathfrak{p}} = \mathfrak{p}$ in our case. The second property directly shows that torsion freeness of ω implies torsion freeness of $\tilde{\omega}$, which otherwise needs more involved proofs. On the other hand, the first property implies equivariancy of the map Φ, which together with the general results obtained using BGG sequences allow a reduction of the problem to harmonic curvature.

3.3 Comparing to the construction by Fox

For a contact projective structure on a contact manifold (M, H), there is the canonical Cartan geometry $(\mathcal{G} \to M, \omega)$ from Theorem 2.2. Applying to this geometry the generalized Fefferman construction from 3.1, we obtain a canonical projective structure on M, which is subordinate to the contact projective structure in the sense of Remark 3.1(2). Our final aim in this section is to prove that the result coincides with the subordinate projective structure constructed in Sect. 3.3 of [10].

Fox's construction is based on the ambient description of contact projective and projective structures as discussed in 2.3. There we already noticed that the spaces on which the ambient connection is defined are the same for both types of structures. Moreover, the ambient connection associated to a contact projective structure in Theorem B of [10] satisfies all the properties of a projective ambient connection from Theorem 3.1 of [10], except for torsion freeness. Symmetrizing the contact projective ambient connection, one obtains a torsion free connection, which then is the canonical connection associated to a projective structure. This is the canonical projective structure as defined by Fox. Notice that the ambient connection associated to a contact projective structure is torsion free if and only if the structure has vanishing contact torsion. This in turn is equivalent to the fact that this ambient connection coincides with the ambient connection of the canonical subordinate projective structure defined by Fox, which is the analog of Theorem 3.2 in this setting.

Proposition 3.3 *For a contact projective structure on a contact manifold* (M, H)*, the subordinate projective structure obtained via the generalized Fefferman construction coincides with the subordinate projective structure constructed in Sect. 3.3 of* [10].

Proof Let $(\mathcal{G} \to M, \omega)$ be the Cartan geometry associated to the contact projective structure as in Theorem 2.2 and let $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ be the result of the generalized Fefferman construction from 3.1. We want to show that, via the procedure from 2.3, these two Cartan geometries lead to the same ambient connection. From property 4 of an ambient connection in Theorem 3.1 of [10] one easily concludes that the paths of a projective structure can be realized as projections of geodesics of the ambient connection. Since symmetrizing the projective ambient connection does not change its geodesics, this will complete the proof.

To compute the ambient connections, recall that we can realize the space $\mathcal{L}$ on which the ambient connection is defined as $\mathcal{G}/Q$ or $\tilde{\mathcal{G}}/\tilde{Q}$. Further, $\mathfrak{g}/\mathfrak{q} \cong \mathbb{R}^{2n+2}$ as a representation of Q and $\tilde{\mathfrak{g}}/\tilde{\mathfrak{q}} \cong \mathbb{R}^{2n+2}$ as a representation of $\tilde{Q}$. These identifications are compatible with the isomorphism $\mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{q}}$ induced by the inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$. Using $T\mathcal{L} \cong \mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$, vector fields on $\mathcal{L}$ are in bijective correspondence with Q–equivariant smooth functions $\mathcal{G} \to \mathbb{R}^{2n+2}$. Likewise, $T\mathcal{L} \cong \tilde{\mathcal{G}} \times_{\tilde{Q}} (\tilde{\mathfrak{g}}/\tilde{\mathfrak{q}})$ identifies such vector fields with $\tilde{Q}$–equivariant smooth functions $\tilde{\mathcal{G}} \to \mathbb{R}^{2n+2}$. The correspondence between functions and vector fields is given by taking preimages under the Cartan connections, and then projecting to the base. For the canonical inclusion $j : \mathcal{G} \to \tilde{\mathcal{G}}$, we by definition have $j^*\tilde{\omega} = \omega$ (identifying $\mathfrak{g}$ with a subset of $\tilde{\mathfrak{g}}$). This immediately implies that if $\tilde{f} : \tilde{\mathcal{G}} \to \mathbb{R}^{2n+2}$ is the equivariant function corresponding to $\eta \in \mathfrak{X}(\mathcal{L})$, then the equivariant function $f : \mathcal{G} \to \mathbb{R}^{2n+2}$ corresponding to η is simply given by $f = \tilde{f} \circ j$.

As we know from 2.3, the ambient connections are special cases of tractor connections, so their actions are described in terms of equivariant functions in the proof of Theorem 2.7 in [4]. We first look at the contact projective ambient connection. Given another vector field $\xi \in \mathfrak{X}(\mathcal{L})$, we first have to choose a lift $\hat{\xi} \in \mathfrak{X}(\mathcal{G})$. Then the function $\mathcal{G} \to \mathbb{R}^{2n+2}$ corresponding to the covariant derivative of η in direction ξ is given by $\hat{\xi} \cdot f + \omega(\hat{\xi}) \circ f$. (In the first summand the vector field differentiates the function, while in the second, $\omega(\hat{\xi})$ acts algebraically on the values of f.) Now we can extend $Tj \circ \hat{\xi}$ to a lift $\tilde{\xi} \in \mathfrak{X}(\tilde{\mathcal{G}})$ of ξ. From $j^*\tilde{\omega} = \omega$ we conclude that $\tilde{\omega}(\tilde{\xi}) \circ j = \omega(\hat{\xi})$, and by construction $\hat{\xi} \cdot f = (\tilde{\xi} \cdot \tilde{f}) \circ j$. But this says that the function describing the covariant derivative with respect to the projective ambient connection is just the equivariant extension of the function describing the covariant derivative with respect to the contact projective ambient connection. This shows that the two connections actually coincide, which completes the proof. □

Of course, the nice geometric interpretation of the subordinate projective structure provided by the generalized Fefferman construction in Proposition 3.1 now carries over to the construction by Fox.

Corollary 3.3 *In the language of paths, the canonical subordinate projective structure defined in* [10] *is obtained by adding the chains of a contact projective structure to the contact geodesics.*

4 The path geometry of chains

The chains in a contact projective structure determine a generalized path geometry. For Lagrangean contact structures and partially integrable almost CR structures, this path geometry and its relation to the parabolic geometry associated to the original structure has been discussed in [8]. For contact projective structures, this relation is much easier, since the path geometry of chains is obtained as a restriction of the path geometry induced by the subordinate projective structure. This is a simple instance of the general construction of correspondence spaces from [1]. Therefore, the analogs of the results form [8] on the path geometry of chains are rather easy to deduce. Still this path geometry turns out to be very useful, since it allows us to prove that a contactomorphism between two contact projective structures which maps chains to chains actually is a morphism of the contact projective structures.

4.1 Generalized path geometries

As we have briefly mentioned in Sect. 2.1, path geometries can be viewed as smooth families of curves on a manifold M with exactly one curve through each point in each direction. More formally, let M be a smooth manifold of dimension m, and let $N := \mathcal{P}TM$ be the projectivized tangent bundle of M. This is a smooth fiber bundle over M with fiber the projective space $\mathbb{R}P^{m-1}$. In particular, there is a canonical projection $\pi : N \to M$ and we have the vertical subbundle $VN \subset TN$. Next, by definition a point in N is a line $\ell \subset T_xM$, where $x = \pi(\ell)$. This leads to a smooth subbundle $\mathcal{H} \subset TN$, called the *tautological subbundle*. By definition, a tangent vector $\xi \in T_\ell N$ lies in the subspace $\mathcal{H}_\ell$ if and only if $T_\ell\pi \cdot \xi \in \ell \subset T_xM$. By construction, $\mathcal{H} \subset TN$ is a smooth subbundle of rank m, which contains the vertical subbundle V that has rank $m - 1$.

Now one defines a *path geometry* on M as a smooth line subbundle $E \subset \mathcal{H} \subset TN$, such that $\mathcal{H} = E \oplus V$. As a line bundle, E is integrable and hence determines a foliation of $N = \mathcal{P}TM$ by 1-dimensional submanifolds. Since $E \cap V = \{0\}$, a local integral manifold for E always projects to a local 1-dimensional submanifold of M. Hence we really obtain a family of paths in M. Moreover, taking the integral submanifold through $\ell \in N$, the projection evidently passes through $x = \pi(\ell)$ with tangent space $\ell \subset T_xM$. Hence we see that in this family there is exactly one path through each point in each direction. It should be mentioned, that a path geometry can be also interpreted as describing the geometry of systems of second order ODE's, see e.g. [1,2].

In the spirit of filtered manifolds, one may go one step further, drop the requirement that one deals with a projectivized tangent bundle and just keep the configuration of subbundles with certain (non-)integrability properties: Consider an arbitrary smooth manifold N of dimension $2m - 1$ and two subbundles $E, V \subset TN$ of rank 1 and $m - 1$, such that $E \cap V = \{0\}$. Putting $\mathcal{H} := E \oplus V$, the Lie bracket of vector fields induces a skew–symmetric bundle map $\mathcal{H} \times \mathcal{H} \to TN/\mathcal{H}$. Now the pair (E, V) is said to define a *generalized path geometry* on N if and only if this bundle map vanishes on $V \times V$ and induces an isomorphism $E \otimes V \to TN/\mathcal{H}$. It is easy to see that this is always satisfied if E and V come from a path geometry, see [1]. Further it turns out that for $m \neq 3$, the subbundle V in a generalized path geometry is always involutive, and then the given geometry is locally isomorphic to a path geometry on a local leaf space for the corresponding foliation.

Any generalized path geometry on a manifold N of dimension $2m - 1$ induces a canonical normal parabolic geometry of type $(\hat{G}, \hat{P})$, where $\hat{G} = \tilde{G} = SL(m + 1, \mathbb{R})$ and $\hat{P}$ is the subgroup of all elements which stabilize both the line spanned by the first vector in the standard basis and the plane spanned by the first two vectors in the standard basis of $\mathbb{R}^{m+1}$.

On the level of Lie algebras, we obtain a decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{-2} \oplus \hat{\mathfrak{g}}_{-1} \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_2$ such that $\hat{\mathfrak{p}} = \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_2$ as well as decompositions $\hat{\mathfrak{g}}_{\pm 1} = \hat{\mathfrak{g}}^E_{\pm 1} \oplus \hat{\mathfrak{g}}^V_{\pm 1}$ according to the following block decomposition with blocks of size 1, 1, and $m-1$:

$$\begin{pmatrix} \hat{\mathfrak{g}}_0 & \hat{\mathfrak{g}}^E_1 & \hat{\mathfrak{g}}_2 \\ \hat{\mathfrak{g}}^E_{-1} & \hat{\mathfrak{g}}_0 & \hat{\mathfrak{g}}^V_1 \\ \hat{\mathfrak{g}}_{-2} & \hat{\mathfrak{g}}^V_{-1} & \hat{\mathfrak{g}}_0 \end{pmatrix}$$

The subspaces $\hat{\mathfrak{g}}^E_{-1}$ and $\hat{\mathfrak{g}}^V_{-1}$ give rise to $\hat{P}$–invariant subspaces in $\hat{\mathfrak{g}}/\hat{\mathfrak{p}}$ and the relation between the parabolic geometry and the generalized path geometry is given by the fact these two subspaces induce the subbundles E and V of TN, which define the generalized path geometry. Requiring the parabolic geometry to be regular and to satisfy the normalization condition discussed in 3.2, it is uniquely determined up to isomorphism. One obtains an equivalence of categories between generalized path geometries and regular normal parabolic geometries in this way.

4.2 The path geometry of chains

Let (M, H) be a contact manifold of dimension $2n+1$ endowed with a contact projective structure, and let $(\mathcal{G} \to M, \omega)$ be the associated canonical Cartan geometry of type (G, P) as in Theorem 2.2. The chains of the contact projective structure can be described as follows: Consider the one-dimensional subspace $\mathfrak{g}_{-2} \subset \mathfrak{g}$ and the corresponding rank one subbundle $\omega^{-1}(\mathfrak{g}_{-2}) \subset T\mathcal{G}$. This is involutive and since the vertical subbundle corresponds to $\mathfrak{p} \subset \mathfrak{g}$, local integral submanifolds project to local one-dimensional immersed submanifolds in M. Alternatively, the chains can be viewed as the projections of the flow lines of the constant vector fields $\omega^{-1}(X)$ with $X \in \mathfrak{g}_{-2}$. This concept generalizes to all parabolic contact structures. In that setting, it was shown in Sect. 4 of [7] that chains are available through each point in M tangent to each line $\ell \in T_xM$ which is not contained in $H_x \subset T_xM$ and, as an unparametrized curve, a chain is uniquely determined by its tangent in one point.

This nicely fits into the picture of generalized path geometries. The subset $\mathcal{P}_0TM \subset \mathcal{P}TM$ of lines not contained in the contact distribution evidently is open, and it is a fiber bundle over M with fiber the complement of a hyperplane in $\mathbb{R}P^{2n}$. It is also clear that the chains give rise to a generalized path geometry on $\mathcal{P}_0TM$. A description of this geometry in terms of $(\mathcal{G} \to M, \omega)$ can be found in Sect. 2.4 of [8]. In our situation, there is, however, a simple way to describe the parabolic geometry corresponding to the path geometry of chains, at least in the case of vanishing contact torsion. Namely, we know that the chains are actually among the paths of the canonical subordinate projective structure associated to the contact projective structure.

Applying the generalized Fefferman construction from 3.1 to $(\mathcal{G} \to M, \omega)$, we obtain a Cartan geometry $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$, which induces the subordinate projective structure on M. To get the associated (generalized) path geometry, one applies the correspondence space construction from [1]: By construction, the subgroup $\hat{P} \subset \hat{G} = \tilde{G}$ is contained in $\tilde{P}$. Hence one can form $N := \tilde{\mathcal{G}}/\hat{P}$, which can be identified with the total space of the fiber bundle $\tilde{\mathcal{G}} \times_{\tilde{P}} (\tilde{P}/\hat{P})$ over M. One immediately verifies that $\tilde{\mathcal{G}} \times_{\tilde{P}} (\tilde{P}/\hat{P}) \cong \mathcal{P}TM$. By construction $(\tilde{\mathcal{G}} \to N, \tilde{\omega})$ is a Cartan geometry of type $(\hat{G}, \hat{P})$. In the case of vanishing contact torsion, $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ is torsion free and normal. One easily verifies that torsion freeness implies that the parabolic geometry $(\tilde{\mathcal{G}} \to N, \tilde{\omega})$ is regular and by Proposition 2.4 of [1] it is normal, too. Hence it is the canonical parabolic geometry associated to the underlying path geometry, whose paths are the geodesics of the connections in the projective class, see Sect. 4.7 of [1].

Of course, the path geometry of chains can be recovered from this as the restriction to the open subset $\mathcal{P}_0TM \subset \mathcal{P}TM = N$. Having made these observations, the first part of the following result is obvious, while the second essentially follows from the general theory of correspondence spaces.

Proposition 4.2 *Let (M, H) be a contact manifold endowed with a contact projective structure.*

(1) *There is a linear connection on TM that has chains among its geodesics.*
(2) *If the given contact projective structure has vanishing contact torsion, then the associated path geometry is torsion free if and only if it is locally flat, which is equivalent to local flatness of the initial contact projective structure.*

Proof (2) In Theorem 3.2 we have observed that local flatness of $(\mathcal{G} \to M, \omega)$ is equivalent to local flatness of $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$. Since $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ and $(\tilde{\mathcal{G}} \to N, \tilde{\omega})$ share the same curvature function, it is equivalent to local flatness of the latter geometry, too. Since $\mathcal{P}_0TM \subset N$ is a dense open subset, we get the equivalence to local flatness of the path geometry of chains. Finally, it has been proved in Theorem 4.7 of [1] that for path geometries induced by projective structures torsion freeness implies local flatness. □

Remark 4.2 We have pointed out part (1) of this proposition only because it is in sharp contrast with the case of other parabolic contact structures. In [8] it is shown that for integrable Lagrangean contact structures as well as CR structures of hypersurface type, the chains can *never* be obtained as geodesics of a linear connection. Also part (2) is significantly different for those structures. While local flatness of the initial structure is equivalent to torsion freeness of the path geometry of chains, these path geometries are always non-flat for integrable Lagrangean contact and CR structures.

4.3 Chain preserving contactomorphisms

As we have mentioned in the remark above, for the parabolic contact structures studied in [8] the path geometry of chains is always non-flat. It is proved there, that the parabolic contact structure can essentially be recovered from the curvature of the path geometry of chains. This leads to a conceptual proof of the fact that contactomorphisms which map chains to chains (as unparametrized curves) are homomorphisms (or anti-homomorphisms in an appropriate sense) of the underlying parabolic contact structure.

For contact projective structures, the situation is different of course, since for a locally flat contact projective structure, also the path geometry of chains is locally flat. Still we can show that, assuming vanishing contact torsion, contactomorphisms which map chains to chains are morphisms of contact projective structures.

Theorem 4.3 *For $i = 1, 2$ let (M_i, H_i) be a contact manifolds endowed with contact projective structures with vanishing contact torsion. Let $f : M_1 \to M_2$ be a contact diffeomorphism which maps chains to chains. Then f is an isomorphism of contact projective structures.*

Proof Put $N_i := \mathcal{P}TM_i$ and let $(p_i : \tilde{\mathcal{G}}_i \to N_i, \tilde{\omega}_i)$ be the (regular normal) parabolic geometries associated to the path geometries determined by the subordinate projective structures. Consider $\mathcal{P}_0TM_i \subset N_i$ and the restricted parabolic geometries $(p_i^{-1}(\mathcal{P}_0TM_i) \to \mathcal{P}_0TM_i, \tilde{\omega}_i)$, which describe the path geometries of chains. By assumption, the contactomorphism f induces a morphism of these path geometries, so it lifts to a morphism Ψ between the Cartan geometries.

Our aim is to extend Ψ to a morphism between the Cartan geometries $(\tilde{\mathcal{G}}_i \to N_i, \tilde{\omega}_i)$. Choose a local smooth section σ of the principal bundle $\tilde{p}_1 : \tilde{\mathcal{G}}_1 \to M_1$, which has values in $p_1^{-1}(\mathcal{P}_0 TM_1)$, and let $U \subset M_1$ be its domain of definition. Then there is a unique $\tilde{P}$–equivariant map $\Psi_\sigma : \tilde{p}_1^{-1}(U) \to \tilde{\mathcal{G}}_2$ such that $\Psi_\sigma(\sigma(x)) = \Psi(\sigma(x))$ for any $x \in U$. We claim that Ψ_σ is an extension of Ψ to $\tilde{p}_1^{-1}(U)$. To prove this take, a point $x \in U$ and consider

$$A_x := \{u \in p_1^{-1}(x) : \Psi(u) = \Psi_\sigma(u)\} \subseteq \tilde{p}_1^{-1}(x).$$

By definition $\sigma(x) \in A_x$, so this set is non-empty. Further, since both Ψ and Ψ_σ are equivariant for the principal right action of $\hat{P} \subset \tilde{P}$, the set A_x is $\hat{P}$–invariant, and it is closed by definition.

For $A \in \tilde{\mathfrak{p}}$, the fundamental vector fields $\zeta_A^i \in \mathfrak{X}(\tilde{\mathcal{G}}_i)$ are given by $\tilde{\omega}_i^{-1}(A)$. Since $\Psi^*\tilde{\omega}_2 = \tilde{\omega}_1$, we conclude that $T\Psi \circ \zeta_A^1 = \zeta_A^2 \circ \Psi$, so Ψ also intertwines the flows of these vector fields, whenever they are defined. Otherwise put, for any $u \in p_1^{-1}(\mathcal{P}_0 TM_1)$ there is a neighbourhood V of $e \in \tilde{P}$ such that $\Psi(ug) = \Psi(u)g$ for all $g \in V$. Since Ψ_σ is $\tilde{P}$–equivariant by definition, this implies that for any $u \in A_x$ a neighborhood of u is contained in A_x, so A_x is open. Since we have noted above that A_x is $\hat{P}$-equivariant, we can prove that $A_x = p_1^{-1}(x)$ and hence our claim by showing that the image of A_x surjects onto $\mathcal{P}_0 T_x M_1 \subset \mathcal{P}TM_1$. But the projection to $\mathcal{P}_0 T_x M_1$ is a surjective submersion and hence an open mapping. Since both A_x and its complement are open, also the image of A_x in $\mathcal{P}_0 T_x M_1$ is open and closed. Since $\mathcal{P}_0 T_x M_1$ is the complement of a hyperplane in projective space and hence connected, the proof of the claim is complete.

By construction, $\Psi_\sigma : \tilde{p}^{-1}(U) \to \tilde{\mathcal{G}}_2$ covers $f|_U : U \to M_2$, so we can view it as a morphism between the Cartan geometries $(\tilde{p}_1^{-1}(U), \tilde{\omega}_1)$ and $(\tilde{p}_2^{-1}(f(U)), \tilde{\omega}_2)$. But this exactly means that $f|_U$ is a morphism between the subordinate projective structures, so in particular it locally preserves the contact geodesics. Hence locally and thus globally f is a morphism of contact projective structures. □

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