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Fefferman–Graham ambient metrics of Patterson–Walker metrics

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Abstract

Given an n-dimensional manifold N with an affine connection D, we show that the associated Patterson–Walker metric g on T^*N admits a global and explicit Fefferman–Graham ambient metric. This provides a new and large class of conformal structures which are generically not conformally Einstein but for which the ambient metric exists to all orders and can be realised in a natural and explicit way. In particular, it follows that Patterson–Walker metrics have vanishing Fefferman–Graham obstruction tensors. As an application of the concrete ambient metric realisation we show in addition that Patterson–Walker metrics have vanishing Q-curvature. We further show that the relationship between the geometric constructions mentioned above is very close: the explicit Fefferman–Graham ambient metric is itself a Patterson–Walker metric.

1. *Introduction and main result*

Given a signature (p,q) conformal structure $[g]$ on an $m=p+q$ dimensional manifold M, it was shown in seminal work by Fefferman and Graham (see [**7**, **8**]) that under specific conditions the conformal structure can be encoded equivalently as a signature $(p+1,q+1)$ pseudo-Riemannian metric $(\mathbf{M},\mathbf{g})$ with vanishing Ricci curvature. This description has been fundamental in constructing and classifying conformal invariants (see, for example, [**3**, **7**]) and for constructing and studying conformally invariant differential operators (see [**11**, **12**]).

To build the Fefferman–Graham ambient metric for given local coordinates x on M, one first considers the ray bundle of metrics in the conformal class $[g]$, written as $\mathbb{R}_+\times\mathbb{R}^m$ with coordinates (t,x). The ambient space $\mathbf{M}$ is obtained by adding a new transversal coordinate $\rho\in\mathbb{R}$, and then an *ansatz* for the Fefferman–Graham ambient metric $\mathbf{g}$ is

$$\mathbf{g}=t^2g_{ij}(x,\rho)dx^i\odot dx^j+2\rho dt\odot dt+2tdt\odot d\rho, \tag{1}$$

where $g=g_{ij}(x,0)dx^idx^j$ is a representative metric in the conformal class. It is directly visible from the formula that $\mathbf{g}$ is homogeneous of degree 2 with respect to the *Euler field* $t\partial_t$ on $\mathbf{M}$.

To show existence of a Fefferman–Graham ambient metric $\mathbf{g}$ for given g, the *ansatz* (1) determines an iterative procedure to determine $g_{ij}(x,\rho)$ as a Taylor series in ρ satisfying $\mathrm{Ric}(\mathbf{g})=0$ to infinite order at $\rho=0$. For m odd, the existence (and a natural version of uniqueness) of $\mathbf{g}$ as an infinity-order series expansion in ρ is guaranteed for general $g_{ij}(x)$. For $m=2n$ even, the existence of an infinity-order jet for $g_{ij}(x,\rho)$ with $\mathrm{Ric}(\mathbf{g})=0$ asymptotically at $\rho=0$ is obstructed at order n, and is controlled by the vanishing of the conformally invariant Fefferman–Graham tensor $\mathcal{O}$. Moreover, existence does not in general guarantee uniqueness.

Results which provide global Fefferman–Graham ambient metrics, where $\mathbf{g}$ can then be constructed in a natural way from g and satisfies $\mathrm{Ric}(\mathbf{g})=0$ globally and not just asymptotically at $\rho=0$, are rare, in both even and odd dimensions. A special instance, where global ambient

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metrics can at least be shown to exist, occurs for g real-analytic and m either being odd or m even with the obstruction tensor $\mathcal{O}$ of g vanishing. The simplest case of geometric origin for which one has global ambient metrics consists of locally conformally flat structures $(M,[g])$, where $(\mathbf{M},\mathbf{g})$ exists and is unique up to diffeomorphisms (see [8, Chapter 7]). Another well-known geometric case are conformal structures $(M,[g])$ which contain an Einstein metric g: If $\mathrm{Ric}(g)=2\lambda(m-1)g$, then $\mathbf{g}$ on $\mathbb{R}_+\times M\times\mathbb{R}$ can be written directly in terms of g as

$$\mathbf{g}=t^2(1+\lambda\rho)^2g+2\rho dt\odot dt+2tdt\odot d\rho. \tag{2}$$

In work by Thomas Leistner and Pawel Nurowski it was shown that the so-called *pp-waves* admit global ambient metrics in the odd-dimensional case and under specific assumptions in the even-dimensional case, see [14]. Concrete and explicit ambient metrics for specific examples of families of conformal structures induced by generic 2-distributions on 5-manifolds and generic 3-distributions on 6-manifolds have been constructed in [2, 15, 16, 18]. Recent progress in obtaining classes of conformal structures for which the Fefferman–Graham ambient metric equations become linear and have explicit solutions was obtained in [1].

The present article expands the class of metrics for which canonical ambient metrics exist globally to *Patterson–Walker metrics:* Given an affine connection D on an n-manifold N with $n\geqslant 2$, which is supposed to be torsion-free and to preserve a volume form, the *Patterson–Walker metric* g is a natural split-signature (n,n) metric on the total space of the co-tangent bundle T^*N, see [13] for historical background on Patterson–Walker metrics, references and a modern treatment. Our main result is as follows.

THEOREM 1. *Let D be a torsion-free affine connection on N which preserves a volume form. Denote local coordinates on N by x^A and the induced canonical fibre coordinates on T^*N by p_A. Let $\Gamma_A{}^C{}_B$ and Ric_{AB} denote the Christoffel symbols and the Ricci curvature of D, respectively. Let*

$$g=2\,\mathrm{d}x^A\odot\mathrm{d}p_A-2\,\Gamma_A{}^C{}_B\,p_C\,\mathrm{d}x^A\odot\mathrm{d}x^B \tag{3}$$

be the Patterson–Walker metric induced on T^*N by D. Then

$$\begin{aligned}\mathbf{g}&=2\rho dt\odot dt+2tdt\odot d\rho\\&\quad+t^2\Big(2dx^A\odot dp_A-2p_C\Gamma_A{}^C{}_Bdx^A\odot dx^B+\frac{2\rho}{n-1}\mathrm{Ric}_{AB}dx^A\odot dx^B\Big)\end{aligned} \tag{4}$$

is a globally Ricci-flat Fefferman–Graham ambient metric for the conformal class $[g]$.

For generic D, the resulting Patterson–Walker metric g is not conformally Einstein, see [13, Theorem 2]. In particular, Theorem 1 provides a large class of conformal structures which are not conformally Einstein but which admit globally Ricci-flat and explicit ambient metrics.

As an immediate consequence of the existence of the ambient metric for $(M,[g])$, the conformally invariant Fefferman–Graham obstruction tensor $\mathcal{O}$ associated to $[g]$ vanishes.

It is not difficult to check Ricci-flatness of (4) directly: Specifically, one employs [8, formula (3.17)], which is applicable to any ambient metric in the normal form (1). The computation is then based on the following key facts: The Ricci curvature of the Patterson–Walker metric g is up to a constant multiple just the pullback of the Ricci curvature of D and this tensor and its covariant derivative are totally isotropic, see [13]. We note that formula (4) for $\mathbf{g}$ says that the ambient metric is in fact linear in ρ and the iterative procedure determining the ambient metric stops after the first step.

A geometric proof of vanishing Ricci curvature is presented in the next section. This is based on a combination of the well-known Patterson–Walker and Thomas cone constructions, both of which we recall. It follows directly from the geometric construction outlined below that the Fefferman–Graham ambient metric of a Patterson–Walker metric is again a Patterson–Walker metric, namely, the one associated to the Thomas cone connection. This nice and interesting fact is elaborated in Theorem 2.

2. *Geometric construction of the ambient metric*

The association $D \rightsquigarrow g$ generalises to a natural association from projective to conformal structures. Recall that two affine connections D, D' on N are called projectively related or equivalent if they have the same geodesics as unparameterised curves, which is the case if and only if there exists a 1-form $\Upsilon \in \Omega^1(N)$ with

$$D'_X Y = D_X Y + \Upsilon(X)Y + \Upsilon(Y)X \tag{5}$$

for all $X, Y \in \mathfrak{X}(N)$. It is sufficient to restrict ourselves to *special* connections in a projective class, that is, to those that preserve some volume form. For projective structures it is useful to employ suitably scaled *projective density bundles*, defined as $\mathcal{E}(w) := (\wedge^n TN)^{-w/(n+1)}$, for arbitrary *weight* w. Then a section $s : N \to \mathcal{E}_+(1)$ corresponds to a choice of a special affine connection D in the projective equivalence class $[D]$, and any $s' = e^f s$ corresponds to D' projectively related to D via (5) with $\Upsilon = df$. We define $M := T^*N \otimes \mathcal{E}(2)$ the (projectively) weighted co-tangent bundle of N. Then, as was shown in [**13**], two projectively related affine connections D, D' on N induce two conformally related metrics g, g' on M, and we therefore have a natural association $(N, [D]) \rightsquigarrow (M, [g])$.

The cone $\mathcal{C} := \mathcal{E}_+(1)$ carries the canonical and well-known Ricci-flat *Thomas cone connection* ∇ associated to the projective class $[D]$, see [**5**, **17**]. (The specific weight 1, which is different from the one in [**5**], is more convenient for our computations.) We will need a local formula for ∇: Let $s : N \to \mathcal{C}$ be the scale corresponding to an affine connection $D \in [D]$, providing a trivialisation $\mathcal{C} \cong \mathbb{R}_+ \times N$ via $(x^0, x) \mapsto s(x)x^0$. In this trivialisation the Thomas cone connection is given by

$$\nabla_X Y = D_X Y - \frac{1}{n-1}\mathrm{Ric}(X, Y)Z, \quad \nabla_X Z = X, \tag{6}$$

where $X, Y \in \mathfrak{X}(N)$ and $Z = x^0 \partial_{x^0}$ is the Euler field on $\mathcal{C}$. It is in fact easy to see directly from formula (6) that the thus defined affine connection ∇ on the Thomas cone $\mathcal{C}$ is independent of the choice of scale and it is Ricci-flat.

Employing a local coordinate patch on N which induces coordinates x^A, y_A on the co-tangent bundle T^*N and coordinates x^0, x^A, y_A, y_0 on $T^*\mathcal{C} \cong \mathbb{R}_+ \times T^*N \times \mathbb{R}$, the Patterson–Walker metric $\mathbf{g}$ associated to ∇ is

$$\begin{aligned} \mathbf{g} = 2dx^A \odot dy_A + 2dx^0 \odot dy_0 - \frac{4}{x^0} y_B dx^0 \odot dx^B \\ - 2y_C \Gamma_A{}^C{}_B dx^A \odot dx^B + 2\frac{x^0 y_0}{n-1}\mathrm{Ric}_{AB} dx^A \odot dx^B. \end{aligned} \tag{7}$$

Ricci-flatness of $\mathbf{g}$ follows directly from Ricci-flatness of ∇, see [**13**, Theorem 2]. Via the change of coordinates $t = x^0,\ \rho = y_0/x^0,\ p_A = y_A/(x^0)^2$ the metric $\mathbf{g}$ transforms to (4), which is the form of a Fefferman–Graham ambient metric (1). In particular, this shows Theorem 1.

We conclude this section by summarising the construction:

THEOREM 2. *Given a projective structure $(N,[D])$ on an n-dimensional manifold N, the geometric constructions indicated in the following diagram commute:*

$$\begin{array}{ccc} (\mathcal{C},\nabla) & \rightsquigarrow & (\mathbf{M},\mathbf{g}) \\ \uparrow & & \uparrow \\ (N,[D]) & \rightsquigarrow & (M,[g]) \end{array}$$

In particular, the induced conformal structure $[g]$ admits a globally Ricci-flat Fefferman–Graham ambient metric $\mathbf{g}$ which is itself a Patterson–Walker metric.

REMARK. As a Patterson–Walker metric, $(\mathbf{M},\mathbf{g})$ carries a naturally induced homothety $\mathbf{k}$ of degree 2, which takes the form $2p_A\partial_{p_A}+2\rho\partial_\rho$. According to [**13**, Lemma 5.1] the infinitesimal affine symmetry Z of ∇ lifts to a Killing field, which one computes as $t\partial_t-2p_A\partial_{p_A}-2\rho\partial_\rho$. In particular it follows that the Euler field $t\partial_t$ of the Fefferman–Graham ambient metric $\mathbf{g}$ can be written as the sum of this Killing field and the homothety $\mathbf{k}$. The tangent bundle $T\mathbf{M}$ carries the maximally isotropic $(n+1)$-dimensional subspace spanned by $\{\partial_{p_A},\partial_\rho\}$ which is preserved by $\boldsymbol{\nabla}$. This subspace can be equivalently described by a $\boldsymbol{\nabla}$-parallel pure spinor $\mathbf{s}$ on $\mathbf{M}$. The ambient Killing field $\mathbf{k}$ and the ambient parallel pure spinor $\mathbf{s}$ correspond to a homothety k of g and a parallel pure spinor χ on M that belong to the characterising objects of the Patterson–Walker metric g, see [**13**, Theorem 1].

3. *Vanishing 'Q-curvature'*

The Q-curvature Q_g of a given metric g is a Riemannian scalar invariant with a particularly simple transformation law with respect to conformal change of metric. It has been introduced by Branson in [**4**] and has been the subject of intense research in recent years, see, for example, [**6**] for an overview. Computation of Q-curvature is notoriously difficult, see, for example, [**10**]. An explicit form of the Fefferman–Graham ambient metric $\mathbf{g}$ for a given metric g allows a computation of Q_g. Using the fact that our $\mathbf{g}$ is actually a Patterson–Walker metric, this computation is particularly simple.

THEOREM 3. *The Patterson–Walker metric g associated to a volume-preserving, torsion-free affine connection D has vanishing Q-curvature Q_g.*

Proof. We follow the computation method for Q_g from [**9**]: For this it is necessary to compute $-\boldsymbol{\Delta}^n\log(t)$, where $\boldsymbol{\Delta}$ is the ambient Laplacian on $\mathbf{M}=\mathbb{R}_+\times T^*N\times\mathbb{R}$ associated to $\mathbf{g}$ and $t:\mathbf{M}\to\mathbb{R}_+$ is the first coordinate. Restricting $-\boldsymbol{\Delta}^n\log(t)$ to the cone $\mathbb{R}_+\times T^*N\times\{0\}$ and evaluating at $t=1$ yields Q_g. To show that the Q-curvature vanishes for g, it is in particular sufficient to show that $\boldsymbol{\Delta}\log(t)=0$.

However, the function $t:\mathbf{M}\to\mathbb{R}_+$ is horizontal since it is just the pullback of the coordinate function $x^0:\mathcal{C}\to\mathbb{R}_+$ on the Thomas cone $\mathcal{C}\cong\mathbb{R}_+\times N$ via the canonical projection $T^*\mathcal{C}\to\mathcal{C}$. It follows from the explicit formula for the Christoffel symbols of a Patterson–Walker metric that $\boldsymbol{\Delta}$ vanishes on any horizontal function, see [**13**, Section 2.1]. Thus in particular $\boldsymbol{\Delta}\log(t)=0$, and then also $Q_g=0$. □

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